Summation-Integral-Phillips for a Sequence of λ-Bernstein Type Operators

Asmaa J. Barak[1,*], Zahra Abdullah Shawi[2], Mayada Marid Abdul Hussain[3,*]

[1] Ministry of Education,Haifa Preparatory School for Girl, Basrah, Iraq

[2] Ministry of Education, General Directorate of Education,AL-Qurna Preparatory School for Girls

[3] Department of Mathematics, Southern Technical University, Basra Technical Institute

Email: Asmaa J.Barak@gmail.com,

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Abstract: This paper introduces a new hybrid operator based on combining the Phillips concept with a sequence of lambda-Bernstein operators. This operator represents a qualitative improvement over classical Bernstein-Durrmeyer operators, which faced significant limitations in controlling the behavior of functions at critical points such as the zero point and suffered from a significantly slow rate of convergence. The developed operator overcomes these challenges, achieving a substantial improvement in the quality and accuracy of convergence. To demonstrate the effectiveness of this operator, the study proves a set of basic theoretical results. First, the paper proves the regular convergence theorem for the operator. This is followed by establishing the error estimation theorem using a continuum measure, which in turn confirms the achievement of first-order convergence. Finally, the study presents a precise Voronovskaya-type asymptotic formula that reveals the detailed behavior of the operator's approximation rate when studying functions regularly.

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1. Introduction

One of the challenges facing researchers in the field of approximation is their attempt to solve the problems that plague classical operators by introducing modified operators that improve the convergence rate at the boundary and other properties.

Classical Bernstein- Durrmeyer operators represented an important development of their Bernsteinian counterparts. They partially solved the problem of slow convergence by incorporating the idea of integration, which allowed them to operate on the average of the function rather than its discrete values.

However, these operators still suffered from limitations in flexibility and precise control over convergence behavior at critical boundary points such as x = 0.

λ -Bernstein operators provided an elegant solution to this problem by introducing a control parameter λ into their fundamentals, giving them great flexibility in tuning and improving convergence performance at the boundary. However, their "additive" nature may limit their maximum effectiveness in some contexts.

In this study, we present a modification of the classical Bernstein-Durrmeyer operator [1] that achieves faster convergence at critical points

such as $x = 0$, takes into account the boundary conditions of the function (f) due to the extra term$(1 - x)^\eta$, allows tuning the approximation behavior using the parameter λ, and reduces fluctuations at $x = 0$ if (f) is discontinuous there. Although Bernstein-Durrmeyer operator is a modification of Bernstein, compare to the operator we introduce, it suffers from slow convergence even for analytic functions, poor performance at the boundary, and an inability to adapt to the properties of the function f. We note here [2,3]. In order to present$\dot{W}_{\eta,\lambda}(f(d); x)$and understand how it is built, we review this historical introduction

Weierstrass's Approximation Theory [4] garner much interest from mathematicians, each trying to prove it with various sequences. One notable contributor was Russian mathematician Bernstein [5], who introduce a positive linear operator sequence in 1912. The definition of polynomials is:

$$H_\eta(g; x) = \sum_{k=0}^{\eta} \hbar_{\eta,k}(x) f\left(\frac{k}{\eta}\right), \quad \text{where}$$

$$\hbar_{\eta,k}(x) = \binom{\eta}{k} x^k (1 - x)^{\eta-1}, x \in [0,1].$$

And Bernstein-Durrmeyer Operators define as:

$$\tilde{\tilde{\omega}}_{t,\eta,k}(x)$$
$$= (\eta + 1) \sum_{k=0}^{\eta} \hbar_{\eta,k}(x) \int_{k=0}^{\eta} \hbar_{\eta,k}(x) f(d) d(d),$$

Many researchers based their work on Bernstein polynomials, and the results were different types of Bernstein operators [6–10].

Among these researchers are Lon and Zeng (2010) [11], who take the Bezier curve and surfaces to present the Bernstein polynomial in a new shape.:

$$\nabla_{\eta,\lambda}(f; x) = \sum_{k=0}^{n} \hbar_{\eta,k}(\lambda; x) f\left(\frac{k}{\eta}\right),$$

$$\hbar_{\eta,k}(\lambda; x) = \hbar_{\eta,0}(x) - \frac{\lambda}{\eta + 1} \hbar_{\eta+1,\eta}(x) \; ; k = 0$$

$$\hbar_{\eta,k}(\lambda; x) = \hbar_{\eta,k}(x)$$
$$+ \lambda\left(\frac{\eta - 2k + 1}{\eta^2 - 1} \hbar_{\eta+1,k}(x)\right.$$
$$\left. - \frac{\eta - 2k - 1}{\eta^2 - 1} \hbar_{\eta+1,k+1}(x)\right), 1$$
$$\leq k \leq \eta - 1,$$

$$\hbar_{\eta,k}(\lambda; x) = \hbar_{\eta,\eta}(x) - \frac{\lambda}{\eta + 1} \hbar_{\eta+1,\eta}(x) (x) \; ; k$$
$$= \eta,$$

where $\lambda \in [-1,1]$. In 2018, Lain and Zhou [12] study uniform convergence properties and derive aVoronovaskaja-type asymptotic formula for the sequence $\nabla_{\eta,\lambda}(f; x)$ in classical approximation theory.

λ -Bernstein operators have received extensive attention in scientific research, where their properties have been discussed and various forms have been developed. For more details on these operators and their developments, we recommend referring to [13–18].

In this study, which was separated into four parts, we first used convergence conditions for Korovkin[2] to show that Summation-Integral-Phillips for Sequence of λ-Bernstein Type Operators converges uniformly to f(x). An estimate of the rate of convergence is given in

Theorem 2 using the modulus of continuity and Hölder's inequality, which allows measuring the speed at which the operator approximates the function. Theorem 3: Holder's condition and Taylor's expansion are used to analyze the accuracy of the approximation, with the application of integration of the residues to estimate the upper bounds of the error. Theorem 4 extends the results using Taylor expansion and the Cauchy-Schwartz inequality to estimate the error when approximating more regular functions such as differentiable functions.

Now, we present the formula of Summation-Integral-Phillips for Sequence of λ-Bernstein Type Operators,

$$\dot{W}_{\eta,\lambda}(\ (\text{d}); x)$$
$$= (\eta$$
$$+ 1) \sum_{k=1}^{\eta} \text{b}_{\eta,k}(\lambda; x) \int_{0}^{1} \text{b}_{\eta,k-1}(\lambda; \text{d}) f(\text{d}) d\text{d}$$
$$+ f(0)(1 - \lambda x)(1 - x)^{\eta}. \qquad (1)$$

2. Preliminary Result

Lemma 2.1. Let Summation-Integral-Phillips for Sequence of λ-Bernstein Operators, satisfied the following properties

i.$\dot{W}_{\eta,\lambda}(1; x) = 1$;

ii.$\dot{W}_{\eta,\lambda}(\text{d}; x) = \left(\frac{1}{(\eta+2)} + \frac{4\lambda}{(\eta-1)(\eta+2)(\eta+3)}\right)\left[\eta x + \frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta-1)}\lambda\right] - \frac{\lambda}{(\eta+2)(\eta+3)} + f(0)(1-\lambda x)(1-x)^{\eta}$;

iii.$\dot{W}_{\eta,\lambda}(\text{d}^2; x) = \left(\frac{1}{(\eta+2)(\eta+3)} + \frac{6\lambda}{(\eta-1)(\eta+2)(\eta+3)(\eta+4)}\right)\left[\eta^2 x^2 + \eta x(1-x) + \lambda\left\{\frac{2\eta x - 4\eta x + 2\eta x^{\eta+1}}{(\eta-1)} + \frac{x^{\eta+1}+(1-x)^{\eta+1}-1}{(\eta-1)}\right\}\right] +$

$$\left(\frac{1}{(\eta+2)(\eta+3)} + \frac{(-2\eta+8)\lambda}{(\eta-1)(\eta+2)(\eta+4)(\eta+3)}\right)\left[\eta x + \frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta-1)}\lambda\right] - \frac{2\lambda}{(\eta+2)(\eta+3)(\eta+4)} +$$
$$f(0)(1-\lambda x)(1-x)^{\eta};$$

proof. By (1) not difficulty to show (i), hence we get it. Now we will prove the second property.

$$\dot{W}_{\eta,\lambda}(\text{d}; x) = \sum_{k=1}^{\eta} \text{b}_{\eta,k}(\lambda; x)\left\{\frac{k}{(\eta+2)}\right.$$
$$+ \lambda\left\{\left[\frac{\eta+1}{\eta-1}\right.\right.$$
$$\left.- \frac{2k}{\eta-1}\right]\frac{k}{(\eta+2)(\eta+3)}$$
$$- \left[1 - \frac{2k}{\eta-1}\right]\left(\frac{k}{(\eta+2)(\eta+3)}\right.$$
$$\left.\left.\left.+ \frac{1}{(\eta+2)(\eta+3)}\right)\right\}\right\}$$

$$= \sum_{k=1}^{\eta} k\, \text{b}_{\eta,k}(\lambda; x)\left\{\frac{1}{(\eta+2)}\right.$$
$$\left.+ \frac{4\lambda}{(\eta+2)(\eta+3)(\eta-1)}\right\}$$

By using Lemma 2.1 in [11] with applied Korovkin conditions and doing some simple calculations, which show that the result be x when η → ∞. Hence, the proof of this property is computed, and we get

$$\dot{W}_{\eta,\lambda}(\text{d}; x) = \left(\frac{1}{(\eta+2)} + \frac{4\lambda}{(\eta-1)(\eta+3)}\right)\left[\eta x\right.$$
$$\left.+ \frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta-1)}\lambda\right]$$
$$- \frac{\lambda}{(\eta+2)(\eta+3)};$$

Finally, we calculate the third property,

$$\dot{W}_{\eta,\lambda}(d;x)$$

$$= \sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x)\left\{\frac{k^2+k}{(\eta+2)(\eta+3)}\right.$$

$$+ \lambda\left\{\left[\frac{\eta+1}{\eta-1} - \frac{2k}{\eta-1}\right]\frac{k^2+k}{(\eta+2)(\eta+3)(\eta+4)}\right.$$

$$- \left[1 - \frac{2k}{\eta-1}\right]\left(\frac{k^2}{(\eta+2)(\eta+3)(\eta+4)}\right.$$

$$+ \frac{3k}{(\eta+2)(\eta+3)(\eta+4)}$$

$$\left.\left.\left.+ \frac{2}{(\eta+2)(\eta+3)(\eta+4)}\right)\right\}\right\}$$

$$= \sum_{k=1}^{\eta} k^2 b_{\eta,k}(\lambda;x)\left\{\frac{1}{(\eta+2)(\eta+3)}\right.$$

$$\left.+ \frac{6\lambda}{(\eta+2)(\eta+3)(\eta+4)(\eta-1)}\right\}$$

$$+ \sum_{k=1}^{\eta} k b_{\eta,k}(\lambda;x)\left\{\frac{1}{(\eta+2)(\eta+3)}\right.$$

$$\left.+ \frac{-2\eta\lambda+8\lambda}{(\eta+2)(\eta+3)(\eta+4)(\eta-1)}\right\}$$

$$- \frac{2\lambda}{(\eta+2)(\eta+3)(\eta+4)};$$

Using the same steps as in proving the second property, this property can be easily proved and its result shown to be equal to x^2 when $\eta \to \infty$. Thus, Theorem.1 is shown.

Next, we introduce the definition of the m-th order moment of Summation-Integral-Phillips for the sequence of λ-Bernstein Operators.

Definition.2.1. Here, the definition of the moment $d_{\eta,m,\lambda}(x)$ for the sequence $\dot{W}_{\eta,\lambda}(f(d);x)$

$$d_{\eta,m,\lambda}(x)$$

$$= (\eta$$

$$+ 1)\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x)\int_0^1 b_{\eta,k-1}(\lambda;d)(d-x)^m dd$$

$$+ (1-\lambda x)(-x)^m(1-x)^{\eta}.$$

Lemma.2.2. $d_{\eta,m,\lambda}(x)$ verify the following:

i. $d_{\eta,0,\lambda}(x) = 1$;

ii. $d_{\eta,1,\lambda}(x) = \frac{-2x}{\eta+2} + \frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta+2)(\eta-1)}\lambda +$

$\frac{4\lambda}{(\eta-1)(\eta+2)(\eta+3)}\left[\eta x + \frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta-1)}\lambda\right] -$

$\frac{\lambda}{(\eta+2)(\eta+3)}$

iii. $d_{\eta,2,\lambda}(x) = \frac{2\eta x+6x^2-2\eta x^2}{(\eta+2)(\eta+3)} +$

$\frac{\lambda}{(\eta+2)(\eta+3)}\left(\frac{2\eta x-4\eta x+2\eta x^{\eta+1}}{(\eta-1)} + \frac{x^{\eta+1}+(1-x)^{\eta+1}-1}{(\eta-1)}\right) +$

$\frac{\lambda}{(\eta+2)(\eta+3)}\left(\frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta-1)}\right) +$

$\frac{6\lambda}{(\eta-1)(\eta+2)(\eta+3)(\eta+4)}\left[\eta^2x^2 + \eta x(1-x) +\right.$

$\lambda\left\{\frac{2\eta x-4\eta x+2\eta x^{\eta+1}}{(\eta-1)} + \frac{x^{\eta+1}+(1-x)^{\eta+1}-1}{(\eta-1)}\right\}\bigg] +$

$\frac{(-2\eta+8)\lambda}{(\eta-1)(\eta+2)(\eta+4)(\eta+3)}\left[\eta x + \frac{1-2x+x^{\eta+1}-(1-x)^{\eta+1}}{(\eta-1)}\lambda\right] -$

$\frac{2\lambda}{(\eta+2)(\eta+3)(\eta+4)}$;

Proof. In a direct and easy way, the first property can be proven; therefore, we immediately go to prove ii and iii

$$d_{\eta,1,\lambda}(x) = \dot{W}_{\eta,\lambda}((d-x);x)$$
$$= \dot{W}_{\eta,\lambda}(d;x) - x\dot{W}_{\eta,\lambda}(1;x)$$

$$= \left(\frac{1}{(\eta + 2)} + \frac{4\lambda}{(\eta - 1)(\eta + 2)(\eta + 3)}\right)\Bigg[\eta\chi$$

$$+ \frac{1 - 2\chi + \chi^{\eta+1} - (1-\chi)^{\eta+1}}{(\eta - 1)}\lambda\Bigg]$$

$$- \frac{\lambda}{(\eta + 2)(\eta + 3)} - \chi$$

By simplifying the terms and performing the mathematical operations, we get the result of ii.

$$\mathfrak{q}_{\eta,2,\lambda}(\chi) = \dot{W}_{\eta,\lambda}((\mathfrak{d} - \chi)^2; \chi)$$

$$= \dot{W}_{\eta,\lambda}(\mathfrak{d}^2 - 2\chi\mathfrak{d} + \chi^2; \chi)$$

$$= \dot{W}_{\eta,\lambda}(\mathfrak{d}^2; \chi) - 2\chi\dot{W}_{\eta,\lambda}(\mathfrak{d}; \chi)$$

$$+ \chi^2\dot{W}_{\eta,\lambda}(1; \chi)$$

$$\left(\frac{1}{(\eta + 2)(\eta + 3)}\right.$$

$$\left. + \frac{6\lambda}{(\eta - 1)(\eta + 2)(\eta + 3)(\eta + 4)}\right)\Bigg[\eta^2\chi^2$$

$$+ \eta\chi(1 - \chi)$$

$$+ \lambda\left\{\frac{2\eta\chi - 4\eta\chi + 2\eta\chi^{\eta+1}}{(\eta - 1)}\right.$$

$$\left. + \frac{\chi^{\eta+1} + (1-\chi)^{\eta+1} - 1}{(\eta - 1)}\right\}\Bigg]$$

$$+ \left(\frac{1}{(\eta + 2)(\eta + 3)}\right.$$

$$\left. + \frac{(-2\eta + 8)\lambda}{(\eta - 1)(\eta + 2)(\eta + 4)(\eta + 3)}\right)\Bigg[\eta\chi$$

$$+ \frac{1 - 2\chi + \chi^{\eta+1} - (1-\chi)^{\eta+1}}{(\eta - 1)}\lambda\Bigg]$$

$$- \frac{2\lambda}{(\eta + 2)(\eta + 3)(\eta + 4)}$$

$$- 2\chi\left(\frac{1}{(\eta + 2)} + \frac{4\lambda}{(\eta - 1)(\eta + 2)(\eta + 3)}\right)\Bigg[\eta\chi$$

$$+ \frac{1 - 2\chi + \chi^{\eta+1} - (1-\chi)^{\eta+1}}{(\eta - 1)}\lambda\Bigg]$$

$$- \frac{\lambda}{(\eta + 2)(\eta + 3)} + \chi^2.$$

We collect the terms on the variable x to get a function that helps prove the property; we obtain

$$\mathfrak{q}_{\eta,2,\lambda}(\chi)$$

$$= \frac{2\eta\chi + 6\chi^2 - 2\eta\chi^2}{(\eta + 2)(\eta + 3)}$$

$$+ \frac{\lambda}{(\eta + 2)(\eta + 3)}\left(\frac{2\eta\chi - 4\eta\chi + 2\eta\chi^{\eta+1}}{(\eta - 1)}\right.$$

$$\left. + \frac{\chi^{\eta+1} + (1-\chi)^{\eta+1} - 1}{(\eta - 1)}\right)$$

$$+ \frac{\lambda}{(\eta + 2)(\eta + 3)}\left(\frac{1 - 2\chi + \chi^{\eta+1} - (1-\chi)^{\eta+1}}{(\eta - 1)}\right)$$

$$+ \frac{6\lambda}{(\eta - 1)(\eta + 2)(\eta + 3)(\eta + 4)}\Bigg[\eta^2\chi^2$$

$$+ \eta\chi(1 - \chi)$$

$$+ \lambda\left\{\frac{2\eta\chi - 4\eta\chi + 2\eta\chi^{\eta+1}}{(\eta - 1)}\right.$$

$$\left. + \frac{\chi^{\eta+1} + (1-\chi)^{\eta+1} - 1}{(\eta - 1)}\right\}\Bigg]$$

$$+ \frac{(-2\eta + 8)\lambda}{(\eta - 1)(\eta + 2)(\eta + 4)(\eta + 3)}\Bigg[\eta\chi$$

$$+ \frac{1 - 2\chi + \chi^{\eta+1} - (1-\chi)^{\eta+1}}{(\eta - 1)}\lambda\Bigg]$$

$$- \frac{2\lambda}{(\eta + 2)(\eta + 3)(\eta + 4)}.$$

Lemma.3.2. For Summation-Integral-Phillips for a sequence of λ-Bernstein operators the central moment is given by

$$\left|\dot{W}_{\eta,\lambda}((\mathfrak{d} - \chi); \chi) \le \mathfrak{s}(\eta, \lambda)\right|, \text{ where } \mathfrak{s}(\eta; \lambda) = \frac{2}{\eta+1} + \frac{2|\lambda|}{(\eta+2)(\eta+1)} \text{ for } \eta \ge 2,$$

$$\left|\dot{W}_{\eta,\lambda}((\mathfrak{d} - \chi)^2; \chi) \le \mathfrak{r}(\eta, \lambda)\right|, \text{where } \mathfrak{r}(\eta; \lambda) = \frac{6}{(\eta+2)(\eta+3)} + \frac{(4\eta+5)|\lambda|}{(\eta+2)(\eta+3)(\eta-1)} +$$

$$\frac{(2\eta+10)|\lambda|}{(\eta+2)(\eta+3)(\eta-1)(\eta+4)} +$$

$$\frac{|-2\eta+8|(\eta+\frac{2|\lambda|}{\eta-1}|\lambda|}{(\eta+2)(\eta+3)(\eta-1)(\eta+4)} \text{ for } \eta \geq 2.$$

Definition 2.2 (Hölder Continuity**):** A function $f \in C[0,1]$ is said to belong to the Hölder **class** $\text{Lip}_{\varepsilon}(L)$ if there exist constants $F > 0$ and $\varepsilon \in (0,1]$ such that:

$$\ell(f, \kappa) \leq F \kappa^{\varepsilon} \text{ for all } \kappa > 0,$$

where $\ell(f, \kappa F)$ denotes the modulus of continuity of f.

3. Main Result

Theorem.3.1. Let $f(x) \in C[0,1]$, then

$\dot{W}_{\eta,\lambda}(f; x)$ converge uniformly to $f(x)$ as η
$$\to \infty.$$

Proof. By using Lemma 1 and applying Korovkin conditions [2], we obtain the proof.

Theorem.2.3. For $g \in C[0,1]$, we get

$$\left|\dot{W}_{\eta,\lambda}(g; x) - g(x)\right| \leq \frac{3}{2}\mathfrak{z}\left(g; \sqrt[3]{\mathfrak{r}(\eta; \lambda)}\right).$$

Proof. By using the definition of the continuity coefficient.

$$|g(d) - g(x)| \leq \mathfrak{z}(g, \vartheta)(1 + \frac{|d - x|^3}{\vartheta^3}),$$

Applying the operators $\dot{W}_{\eta,\lambda}$ on both sides of the inequality, we have

$$\left| \dot{W}_{\eta,\lambda}(g; x) - g(x)\right| \leq \dot{W}_{\eta,\lambda}(|g(d) - g(x)|; x) \leq \mathfrak{z}(g, \vartheta)(1 + \frac{1}{\vartheta^3} \dot{W}_{\eta,\lambda}(|d - x|^3; x).$$

By using Hölder inequality,

$$\dot{W}_{\eta,\lambda}(|d - x|^3; x) \leq \dot{W}_{\eta,\lambda}(|d - x|^2; x)^{\frac{3}{2}}$$

$$= \left(\mathfrak{r}(\eta; \lambda)\right)^{\frac{3}{2}}, \text{ choose } \vartheta$$

$$= \sqrt[3]{\mathfrak{r}(\eta; \lambda)}.$$

We get

$$\left| \dot{W}_{\eta,\lambda}(g; x) - g(x)\right|$$

$$\leq \mathfrak{z}(g, \vartheta)\left(1 + \frac{\left(\mathfrak{r}(\eta; \lambda)\right)^{\frac{3}{2}}}{\vartheta^3}\right)$$

$$\leq \mathfrak{z}\left(g, \sqrt[3]{\mathfrak{r}(\eta; \lambda)}\right)\left(1 + \frac{\left(\mathfrak{r}(\eta; \lambda)\right)^{\frac{3}{2}}}{\mathfrak{r}(\eta; \lambda)}\right),$$

$1 + \left(\mathfrak{r}(\eta; \lambda)\right)^{\frac{1}{2}} \leq \frac{3}{2}$, because $\left(\mathfrak{r}(\eta; \lambda)\right)^{\frac{1}{2}}$ is sufficient large for η, by combination the steps we arrive to the result,

$$\left|\dot{W}_{\eta,\lambda}(g; x) - g(x)\right| \leq \frac{3}{2}\mathfrak{z}\left(g; \sqrt[3]{\mathfrak{r}(\eta; \lambda)}\right),$$

and the proof is completed.

Theorem.3.3. Let $g \in C[0,1]$, then

$$\left|\dot{W}_{\eta,\lambda}(g; x) - g(x)\right|$$
$$\leq \mathfrak{s}(\eta; \lambda)\left|g'(x)\right|$$
$$+ 4\tilde{V}\left(\mathfrak{r}(\eta; \lambda)\right)^{\frac{1+i}{2}}.$$

Proof. Since g' satisfies the Hölder `s condition, that means

$\mathfrak{z}(g', \vartheta) \leq \tilde{V}\vartheta^{i}$ for some $0 < i \leq 1$, and $\tilde{V} > 0$.

Now, by Taylor's expansion of $g(d)$ around the point x and using integration residue, we get

$$g_\flat(d) = g_\flat(x) + g_\flat^{`}(x)(d - x)$$
$$+ \int_x^d \left(g_\flat^{`}(\zeta) - g_\flat^{`}(x) \right) d\zeta.$$

Applying $\dot{W}_{\eta,\lambda}$ on both sides and using the linearity of our operators, we obtain

$$\dot{W}_{\eta,\lambda}(g_\flat; x) = g_\flat(x) + g_\flat^{`}(x)\dot{W}_{\eta,\lambda}((d - x); x)$$
$$+ \dot{W}_{\eta,\lambda} \left(\int_x^d \left(g_\flat^{`}(\zeta) \right. \right.$$
$$\left. \left. - g_\flat^{`}(x) \right) d\zeta; x \right).$$

Since $\left| \dot{W}_{\eta,\lambda}\left((d - x); x \right) \right| \leq$
$\varsigma(\eta; \lambda)$, where $\varsigma(\eta; \lambda)$ is introduced in Lemma 3.

Estimate the integral by using Hölder`s condition for $g_\flat^{`}(x)$;

$$\left| \int_x^d \left(g_\flat^{`}(\zeta) - g_\flat^{`}(x) \right) d\zeta \right| \leq \mathfrak{z}(\quad g_\flat^{`}, |d - x|)|d - x| \leq \tilde{V}|d - x|^{1+i}.$$

Thus,

$$\left| \dot{W}_{\eta,\lambda} \left(\int_x^d \left(g_\flat^{`}(\zeta) - g_\flat^{`}(x) \right) d\zeta; x \right) \right|$$
$$\leq \tilde{V}\dot{W}_{\eta,\lambda}(|d - x|^{1+i}; x).$$

Apply Lyapunov inequality,

$$\dot{W}_{\eta,\lambda}(|d - x|^{1+i}; x) \leq \dot{W}_{\eta,\lambda}((d - x)^2; x)^{\frac{1+i}{2}}$$
$$= \left(\mathfrak{r}(\eta; \lambda) \right)^{\frac{1+i}{2}}$$

$$\left| \dot{W}_{\eta,\lambda}((g_\flat; x) - g_\flat(x) \right|$$
$$\leq \varsigma(\eta; \lambda)\left| g_\flat^{`}(x) \right| + \tilde{V}\left(\mathfrak{r}(\eta; \lambda) \right)^{\frac{1+i}{2}}.$$

Implies 4 $\tilde{V}$ instead of $\tilde{V}$ for comparison; after that, we have the result. By this, the proof is finished.

Theorem.4.3. If $f(x) \in C[0,1], \lambda \in [-1,1]$, $x \in (0,1)$ and $f''(x)$ exist, we get

$$\lim_{\eta \to \infty} \eta(\dot{W}_{\eta,\lambda}(f; x) - f(x)) = \rho_1 f^{`}(x) + \rho_2 f^{``}(x).$$

Where ρ_1, ρ_2 are coefficients with x.

Proof. By Taylor`s expansion, we have

$$f(d) = f(x) + f^{`}(x)(d - x) + \frac{f^{``}(x)}{2}(d - x)^2$$
$$+ y(d, x)(d - x)^2$$

$$= f(x)\dot{W}_{\eta,\lambda}(1; x) + f^{`}(x)\dot{W}_{\eta,\lambda}((d - x); x)$$
$$+ \frac{f^{``}(x)}{2}\dot{W}_{\eta,\lambda}((d - x)^2; x)$$
$$+ \dot{W}_{\eta,\lambda}(y(d, x)(d - x)^2; x);$$

$$= f(x) + f^{`}(x) \, d_{\eta,1,\lambda}(x) + \frac{f^{``}(x)}{2} \, d_{\eta,2,\lambda}(x)$$
$$+ \dot{W}_{\eta,\lambda}(y(d, x)(d - x)^2; x);$$

$$\lim_{\eta \to \infty} \eta(\dot{W}_{\eta,\lambda}(f; x - f(x)) = -2x f^{`}(x)$$
$$+ x(1 - x)f^{``}(x)$$
$$+ \lim_{\eta \to \infty} \eta \dot{W}_{\eta,\lambda}(y(d, x)(d - x)^2; x)$$

$$\left| \eta \dot{W}_{\eta,\lambda}(y(d, x)(d - x)^2; x) \right| \leq$$

$$\eta(\eta + 1)\left\{ \sum_{k=1}^{\eta} \mathfrak{b}_{\eta,k}(\lambda; x) \int_0^1 \mathfrak{b}_{\eta,k-1}(\lambda; d)dd \right.$$
$$+ |y(d, x)(d - x)^2| \, dd$$
$$+ y(d, x)(-x)^2(1$$
$$\left. - \lambda x)(1 - x)^\eta \right\};$$

$$= \eta(\eta$$
$$+ 1)\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|<\varsigma} b_{\eta,k-1}(\lambda;d) |\mathrm{y}(d,x)(d$$
$$- x)^2| \, dd$$
$$+ \eta(\eta$$
$$+ 1)\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|\geq\varsigma} b_{\eta,k-1}(\lambda;d) |\mathrm{y}(d,x)(d$$
$$- x)^2| \, dd$$
$$+ \eta(\eta+1)\mathrm{y}(0,x)(-x)^2(1-\lambda x)(1-x)^{\eta}$$
$$:= \Theta_1 + \Theta_2 + \Theta_3.$$

$\Theta_3 \to \infty$ as $\eta \to \infty$ and $\mathrm{y}(d,x) \to 0$ as $d \to x$ for given $\epsilon > 0, \exists \varsigma > 0$ such that $|d - x| < \varsigma$

then, $\mathrm{y}(d,x) < \epsilon$

$$\Theta_1$$
$$< \eta(\eta$$
$$+ 1)\epsilon \sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|<\varsigma} b_{\eta,k-1}(\lambda;d)(d$$
$$- x)^2 \, dd$$
$$= \eta(\eta$$
$$+ 1)\epsilon \sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|<\varsigma} b_{\eta,k-1}(\lambda;d)(d$$
$$- x)^2 \, dd + \eta\epsilon(-x)^2(1-\lambda x)(1-x)^{\eta}$$
$$- \eta\epsilon(-x)^2(1-\lambda x)(1-x)^{\eta};$$
$$\leq \eta\epsilon \ d_{\eta,2,\lambda}(x) - \eta\epsilon(-x)^2(1-\lambda x)(1-x)^{\eta}$$
$$= \epsilon O(1)$$
$$- \eta\epsilon(-x)^2(1-\lambda x)(1-x)^{\eta};$$

$\Theta_1 \to 0$ as $\eta \to \infty$ and ϵ is arbitrary

$$\Theta_2$$
$$< \eta(\eta$$
$$+ 1)\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|\geq\varsigma} b_{\eta,k-1}(\lambda;d) |\mathrm{y}(d,x)(d$$
$$- x)^2| \, dd,$$

Now define $C_\delta[0,1]$ by

$$C_\delta[0,1] = \{f \in C[0,1]: |f(d)| \leq Cd^\delta\}, C > 0, \delta$$
$$> 0$$

$$\Theta_2$$
$$= \sup \left| \eta(\eta \right.$$
$$\left. + 1)\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|\geq\varsigma} b_{\eta,k-1}(\lambda;d) \, Cd^\delta \, dd \right|$$
$$\leq C\eta(\eta$$
$$+ 1)\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \int_{|d-x|\geq\varsigma} b_{\eta,k-1}(\lambda;d) \sum_{j=0}^{\infty} \frac{d^j}{dx^j}\frac{d^\delta}{j}\Big|_{d=x} |d$$
$$- x|^j \, dd$$

Applying the Cauchy Schwarz- inequality, we have

$$\leq \eta(\eta$$
$$+ 1)\sum_{j=0}^{\infty}\sum_{k=1}^{\eta} b_{\eta,k}(\lambda;x) \left[\int_{|d-x|\geq\varsigma} b_{\eta,k-1}(\lambda;d)\right]^{\frac{1}{2}} \left[\int_{|d-x|\geq\varsigma} b_{\eta,k-1}(\lambda;d)\right.$$
$$\left. - x|^{2j} \, dd\right]^{\frac{1}{2}}$$

Θ_2

$$\leq C\eta \sum_{j=0}^{\infty} \left[\sum_{k=1}^{\eta} \mathcal{b}_{\eta,k}(\lambda; x) \right]^{\frac{1}{2}} \left[(\eta \right.$$

$$+ 1) \sum_{k=1}^{\eta} \mathcal{b}_{\eta,k}(\lambda; x) \int_{|d-x|\geq \varsigma} \mathcal{b}_{\eta,k-1}(\lambda; d)|d$$

$$\left. - x|^{2j} dd \right]^{\frac{1}{2}}$$

since $\sum_{k=1}^{\eta} \mathcal{b}_{\eta,k}(\lambda; x) + (1-x)^{\eta} - (1-x)^{\eta}$

$$= \sum_{k=0}^{\eta} \mathcal{b}_{\eta,k}(\lambda; x) - (1-x)^{\eta}$$

Θ_2

$$\leq C\eta \sum_{j=0}^{\infty} (1 - (1-x)^{\eta})^{\frac{1}{2}} \left[(\eta \right.$$

$$+ 1) \sum_{k=1}^{\eta} \mathcal{b}_{\eta,k}(\lambda; x) \int_{|d-x|\geq \varsigma} \mathcal{b}_{\eta,k-1}(\lambda; d)|d$$

$$\left. - x|^{2j} dd \right]^{\frac{1}{2}}$$

Now $(\eta + 1) \sum_{k=1}^{\eta} \mathcal{b}_{\eta,k}(\lambda; x) \int_{|d-x|\geq \varsigma} \mathcal{b}_{\eta,k-1}(\lambda; d)|d$

$$- x|^{2j} dd$$
$$+ (-x)^{2j}(1-\lambda x)(1-x)^{\eta}$$
$$- (-x)^{2j}(1-\lambda x)(1-x)^{\eta}$$

$$= d_{\eta,2j,\lambda}(x) - (-x)^{2j}(1-\lambda x)(1-x)^{\eta}$$
$$= O(\eta^{-j}) + O(\eta^{-g}), g > 0$$

$$\Theta_2 \leq C\eta \sum_{j=0}^{\infty} [O(\eta^{-j})]^{\frac{1}{2}} [O(\eta^{-g})]^{\frac{1}{2}}$$

$$\Theta_2 \leq C \sum_{j=0}^{\infty} O\left(\eta^{\frac{-j+2}{2}}\right) O\left(\eta^{\frac{-g}{2}}\right) = O(\eta^{-2})$$

$$= O(1).$$

$\Theta_2 \rightarrow 0$ as $\eta \rightarrow \infty$. Thus, Theorem 1 has been shown.

4. Numerical Examples

In this part of the study, we will present multiple numerical examples to illustrate how the operator $\dot{W}_{\eta,\lambda}(f; x)$ converges to the function f (x) by adjusting both the parameter λ and η.

Example 1 For $\lambda = -1; \eta = 10,20,30, f(x) = e^x \sin(x), \ x \in [0,1]$ .We will get the first figure which shows that the operator $\dot{W}_{\eta,\lambda}(f; x)$ converges to the function f (x) better when η increases its value.
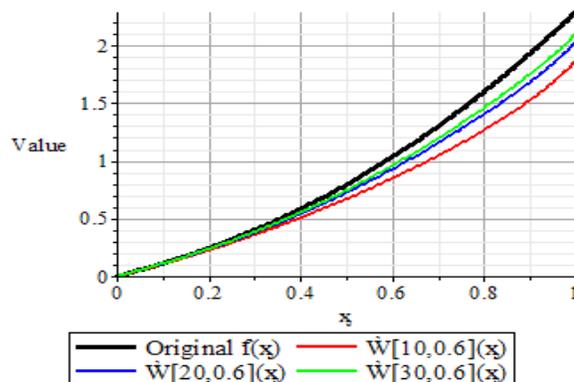

Fig. (1): The graphs of $\dot{W}_{\eta,-1}(f; x)$ with η=10,20,30.

Example 2 For $\lambda = 0.6; \eta = 10,20,30, f(x) = e^x \sin(x), \ x \in [0,1]$ .We will get the second figure which shows that the operator $\dot{W}_{\eta,\lambda}(f; x)$

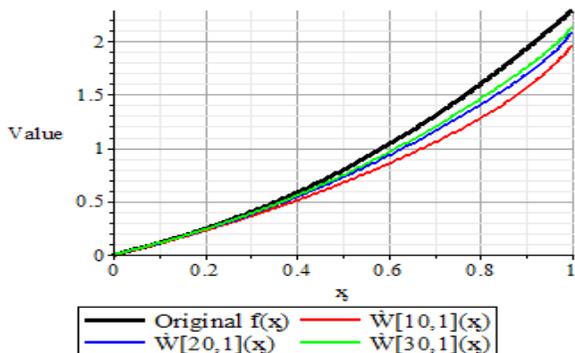converges to the function f (x) better when ꞑ increases its value. In this case the converge got much better.

Fig. (2): The graphs of $\dot{W}_{\text{ꞑ},0.6}(f;x)$ with ꞑ=10,20,30.

Example 3 For $\lambda = 1; \text{ꞑ} = 10,20,30, f(x) = e^x \sin(x),\ x \in [0,1]$ .We will get the third figure which shows that the operator $\dot{W}_{\text{ꞑ},\lambda}(f;x)$ converges to the function f (x) better when ꞑ increases its value and in this example, we notice that the approximation became better when we took this value.
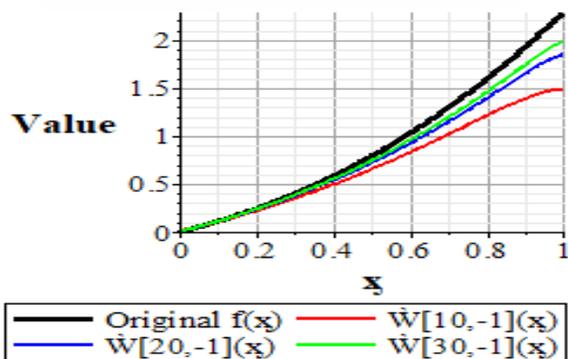
Fig. (3): The graphs of $\dot{W}_{\text{ꞑ},1}(f;x)$ with ꞑ=10,20,30.

The table 1 explains the errors of the approximation of $\dot{W}_{\text{ꞑ},\lambda}(f;x)$ to f(x) when ꞑ, λ are taken different values.

Table 1: The errors of the approximation.

$\left\| f - \dot{W}_{\text{ꞑ},\lambda}(f;x) \right\|_{\infty}$			
λ	ꞑ=10	ꞑ=20	ꞑ=30
-1	0.042444	0.019579	0.012631
0.6	0.035334	0.017913	0.012017
1	0.035153	0.017881	0.011975

Figure 2 shows good convergence with $\lambda = 0.6$, while Figure 3 shows the best convergence when using $\lambda = 1$, highlighting the role of the λ parameter in improving the accuracy of the approximation process. Increasing λ to 1 result in an additional improvement in approximation accuracy compared to $\lambda = 0.6$. Table 1 shows quantitatively confirms this conclusion, showing that the error $\left\| f - \dot{W}_{\text{ꞑ},\lambda}(f;x) \right\|_{\infty}$ is 0.017881 when $\lambda = 1$ and ꞑ = 20, while it is 0.017913 when $\lambda = 0.6$ and ꞑ = 20.

5. Conclusion

In this research, the classical operator $\tilde{\omega}_{\text{c},\text{ꞑ},k}(x)$ and the modified operator $\dot{W}_{\text{ꞑ},\lambda}(f;x)$ were analyzed and we proved that the operator presented in the research improved the convergence rate through the parameter λ and the additional term $f(0)(1 - \lambda x)(1 - x)^{\text{ꞑ}}$. and gave higher accuracy at the limits, especially at $x = 0$, and better flexibility thanks to the possibility of adjusting the parameter λ according to the characteristics of f. We have given numerical examples to show that the parameter λ and ꞑ increases the convergence order if it is set correctly.

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