Comparison of Some Estimation Methods for Reliability Function of Generalized Rayleigh Distribution

طرق تقدير الانكماش لدالة المعولية لتوزيع رايلي العام باستخدام المحاكاة

Associate Professor. Isaam Kamel[1*],
[1]Department of Mathematics, College of Sciences, University of Anbar, Iraq
isam_kml@uoanbar.edu.iq
Assistant Lecturer. Watheq Nadhim Daham[2*]
[2]Postgraduate Studies affairs Dept, University Headquarter, University of Anbar, Iraq
Watheq.n.daham@uoanbar.edu.iq

أ.م.د. عصام كامل احمد **م.م. واثق ناظم دهام**

كلية العلوم / جامعة الانبار **رئاسة جامعة الانبار**

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Abstract

In this paper we derivate mathematical formula of the reliability of $Rs = p[\max(x_{z+1}, x_{z+2}) < \min(x_1, x_2, \dots, x_z)]$ when $x_1, x_2, \dots, x_z$ are strengths subject to one of the stresses x_{z+1}, x_{z+2} assuming that $x_1, x_2, \dots, x_z, x_{z+1}, x_{z+2}$ follow Independent generalized Rayleigh distributions. It was estimated of Rs is given for the distribution, by using (***the following methods***) maximum likelihood (ML), shrinkage estimation (SH) (three type), least square (LS) and Bayes method (B). Also make a comparison among results of the estimation methods of reliability function by mean square error (MSE).

Keywords *:Rayleigh distribution , maximum likelihood, shrinkage estimation, least square, reliability system and Bayes method, reliability system.*

المستخلص

في هذا البحث تم اشتقاق الصيغة الرياضية لدالة المعولية لـــ $Rs = p[\max(x_{z+1}, x_{z+2}) < \min(x_1, x_2, \dots, x_z)]$ عندما تكون $x_1, x_2, \dots, x_z$ تمثل نقاط قوة تخضع لإحدى الضغوطات x_{z+1}, x_{z+2} بافتراض أن $x_1, x_2, \dots, x_z, x_{z+1}, x_{z+2}$ تتبع توزيع رالي العام المستقل. تم تقدير Rs المعطاة للتوزيع، باستخدام مقدر الامكان الاعظم (ML) ، مقدر الانكماش (استخدام ثلاث انواع) ، مقدر المربعات الصغرى (ls) والطريقة البيزية (B). وكذلك عمل مقارنة بين نتائج طرق تقدير دالة الموثوقية بواسطة متوسط الخطأ التربيعي(MSE).

الكلمات المفتاحية *: توزيع رالي ، مقدر الامكان الاعظم (ML) ، مقدر الانكماش، المعولية، مقدر المربعات الصغرى (ls) والطريقة البيزية(B).*

1. Introduction

The generalized Rayleigh distribution denoted by (GRD) is very important in various life Applications, agriculture, biology, engineering and other sciences surplus and Padgett in(1998-2001) introduced what is called burr distribution including generalized Rayleigh distribution as special case (burr type x distribution).

The GRD has been studied extensively by kudu and ragab (2005), and GRD is particular case of the generalized Weibull distribution generalized Weibull distribution as[10],[11] and [13]

$F(x) = (1 - e^{-(\lambda x)^{\beta}})^{\propto}$; $x > 0$ and $\propto, \beta, \lambda > 0$

When $= 2$, then this distribution reduce to generalized Rayleigh distribution .

The estimation of parameters of GRD has been discussed in the literature In (2014) Rao studied the reliability system in "stress – strength model" for generalized Rayleigh distribution through simulation [12].

In (2015), Abbas and May Mona, estimated the shape parameter of "generalized Rayleigh distribution" using single stage Bayesian – shrinkage estimator[1]. In this paper, the estimation of $Rs = p[\max(x_{z+1}, x_{z+2}) < \min(x_1, x_2, \dots, x_z)]$ is considered when $(x_1, x_2, \dots, x_z)$ are strengths to one of the stresses (x_{z+1}, x_{z+2}) assuming that $(x_1, x_2, \dots, x_z, x_{z+1}, x_{z+2})$ follow independent generalized Rayleigh distribution using different estimation methods and make a comparison using simulation.

2- system reliability

In this paper, the estimation of $Rs = p[\max(x_{z+1}, x_{z+2}) < \min(x_1, x_2, \dots, x_z)]$ is considered when $(x_1, x_2, \dots, x_z)$ are strengths subject to one of the stresses (x_{z+1}, x_{z+2}) .
Let $x_1, x_2, \dots, x_z, x_{z+1}, x_{z+2}$ be independent and follow "generalized Rayleigh distribution" with shape parameter βi ($i = 1, 2, \dots, z, z+1, z+2$) and common scale parameter λ .

The pdf and cdf of x_i are respecfively given by

$$f(x, \beta, \lambda) = 2\,\beta\,\lambda^2\,x\,e^{-(\lambda x)^2}(1 - e^{-(\lambda x)^2})^{\beta} \quad (1)$$

$$F(x, \beta, \lambda) = (1 - e^{-(\lambda x)^2})^{\beta} \quad \text{For } x > 0;\ \beta, \lambda > 0 \quad (2)$$

Then the distribution function of $k = \max(x_{z+1}, x_{z+2})$ is given by

$$H(K) = p(K < k) = \prod_{i=z+1}^{z+2} p(x_i < k) = p(x_{z+1} < k) * p(x_{z+2} < k)$$

$$=(1 - e^{-(\lambda x)^2})^{\beta_{z+1}} * (1 - e^{-(\lambda x)^2})^{\beta_{z+2}} = (1 - e^{-(\lambda x)^2})^{\sum_{i=z+1}^{z+2}\beta_i} \quad (3)$$

Also, the distribution function of $\min(x_1, x_2, \dots, x_z)$ is given by

$$w(k) = \min(x_1, x_2, \dots, x_z) = \prod_{i=1}^{z} p(x_i > k) = p(x_2 > k) * p(x_2 > k) \dots p(x_k > k)$$

$$= (1 - (1 - e^{-(\lambda k)^2})^{\beta_1}) * (1 - (1 - e^{-(\lambda k)^2})^{\beta_2}) \dots (1 - (1 - e^{-(\lambda k)^2})^{\beta_z}$$

$$= \prod_{i=1}^{z}(1 - (1 - e^{-(\lambda k)^2})^{\beta_i}) \quad (4)$$

In series system, the system reliability is given by

$$Rs = p[\max\,(x_{z+1}, x_{z+2}) < min(x_1, x_2, \dots, x_z)]$$

$$= \int_0^{\infty} w(k) * dH(k)$$

$$\int_0^\infty \prod_{i=1}^{z} (1-(1-e^{-(\lambda k)^2})^{\beta_i}) * d\left((1-e^{-(\lambda x)^2})^{\sum_{i=z+1}^{z+2}\beta_i}\right)$$

$$= \int_0^\infty \prod_{i=1}^{z} (1-(1-e^{-(\lambda k)^2})^{\beta_i})$$

$$* \sum_{i=z+1}^{z+2} \beta_i (1-e^{-(\lambda k)^2})^{\sum_{i=z+1}^{z+2}\beta_i - 1} * e^{-(\lambda k)^2} * 2\lambda^2 * k * dk$$

$$= 2\lambda^2 \sum_{i=z+1}^{z+2} \beta_i * \int_0^\infty \prod_{i=1}^{z} (1-(1-e^{-(\lambda k)^2})^{\beta_i}) * (1-e^{-(\lambda k)^2})^{\sum_{i=z+1}^{z+2}\beta_i - 1} * e^{-(\lambda k)^2} k * dk$$

$$let\ y = 1 - e^{-(\lambda k)^2} \rightarrow k = \frac{\sqrt{-\ln(1-y)}}{\lambda}$$

$$dk = \frac{dy}{2\lambda\sqrt{-\ln(1-y)}\ (1-y)}$$

So, $\prod_{i=1}^{z}(1-(1-e^{-(\lambda k)^2})^{\beta_i}) = \prod_{i=1}^{z}(1-y^{\beta_i})$

$$= 1 - \sum_{m=1}^{z} (-1)^{m-1} \sum_{i1=i2\ldots iz=1}^{z} y^{\sum_{i=1}^{m}\beta_{ij}}$$

There fore

$$Rs = \sum_{i=z+1}^{z+2} \beta_i \int_0^1 (1 - \sum_{m=1}^{z} (-1)^{m-1} * \sum_{i1=i2\ldots iz=1}^{z} y^{\sum_{i=1}^{m}\beta_{ij}} * y^{\sum_{i=z+1}^{z+2}\beta_i - 1} dy$$

$$= \sum_{i=z+1}^{z+2} \beta_i \int_0^1 y^{\sum_{i=z+1}^{z+2}\beta_i - 1} dy - \sum_{i=z+1}^{z+2} \beta_i \sum_{m=1}^{z} (-1)^{m-1} * \sum_{i1=i2\ldots iz=1}^{z} \int_0^1 y^{\sum_{j=1}^{m}\beta_{ij} + \sum_{i=z+1}^{z+2}\beta_i - 1} * dy$$

So, we have, Rs on the form

$$Rs = 1 - \sum_{i=z+1}^{z+2} \beta_i \sum_{m=1}^{z} (-1)^{m-1} * \sum_{i1=i2\ldots iz=1}^{z} \int_0^1 y^{\sum_{j=1}^{m}\beta_{ij} + \sum_{i=z+1}^{z+2}\beta_i - 1} * dy \quad (5)$$

3- maximum likelihood Estimators (MLE)

Let $x_{i1}, x_{i2}, \ldots, x_{iz}$ $(i = 1, 2, \ldots, n)$ be random sample on strengths of (n) systems following generalized Rayleigh distribution with shape parameter $\beta_i (i = 1, 2, \ldots, z)$, scale parameter λ and $x_{i_{z+1}}, x_{i_{z+2}}$ $(i = 1, 2, \ldots, n)$ be arandom sample on stresses corresponding to (n) system, that follows a generalized Rayleigh distribution with shape parameter $\beta_i (i = z+1, z+2)$ and common scale parameter λ

Then the likelihood function of x_{ij} is given by

$$L(x_{ij}; \beta_i, \lambda)$$

$$= \prod_{j=1}^{m} \prod_{i=1}^{z+2} (x_{ij}\ \beta_i\ \lambda^2\ e^{-(\lambda x_{ij})^2}) * (1-e^{-(\lambda x_{ij})^2})^{\beta_i - 1}$$

$$= \prod_{i=1}^{z+2} \beta_i^n \lambda^{2n(z+2)} * \prod_{i=1}^{n}\prod_{i=1}^{z+2} x_{ij} * \prod_{j=1}^{n}\prod_{i=1}^{z+2} e^{-(\lambda x_{ij})^2} * \prod_{i=1}^{n}\prod_{i=1}^{z+2}(1- e^{-(\lambda x_{ij})^2})^{\beta_i -1}$$

The log – likelihood function is given by

$$lnL(x_{ij}\,;\,\beta_i\,,\lambda\,)$$

$$= n\sum_{i=1}^{z+2} \ln \beta_i + 2n(z+2)\ln\lambda$$

$$+ \sum_{i=1}^{z+2}\sum_{i=1}^{n} ln\, x_{ij} - 2\sum_{i=1}^{z+2}\sum_{j=1}^{n}(\lambda\, x_{ij}) + (\beta_i - 1)\sum_{i=1}^{z+2}\sum_{j=1}^{n} ln(1- e^{-(\lambda x_{ij})^2})$$

Differentiating log – likelihood function partially with respect to β_i and equating it to zero will be

$$\frac{\partial}{\partial \beta_i} lnl = 0$$

$$\frac{n}{\beta_i} + \sum_{i=1}^{n} \ln(1- e^{-(\lambda x_{ij})^2}) = 0$$

$$(6)\hat{\beta}_{imle} = \frac{-n}{\sum_{j=1}^{n}\ln(1- e^{-(\lambda x_{ij})^2})}$$

For $i = 1\,,2\,,\ldots,z\,,z+1\,,z+2$ the estimates of Rs ; $i = 1\,,2\,,\ldots,z\,,z+1\,,z+2$ is given by

$$(7)\hat{R}_{smle} = 1 - \sum_{i=z+1}^{z+2} \hat{\beta}_{imle} * \sum_{m=1}^{z}(-1)^{m-1} \sum_{i1=i2\ldots iz=1}^{z} \int_0^1 y^{\sum_{j=1}^{m}\hat{\beta}_{ijmle} + \sum_{i=z+1}^{z+2}\hat{\beta}_{imle}} * dy$$

Note that $\hat{\beta}_{imle}$ is biased since $E(\hat{\beta}_{imle}) = \frac{n}{n-1}\beta_i \neq \beta_i$ for $i = 1\,,2\,,\ldots,z\,,z+1\,,z+2$

Hence $\hat{\beta}_{iub} = \frac{n-1}{n}\hat{\beta}_{imle} = \frac{n-1}{\sum_{j=1}^{n}\ln(1- e^{-(\lambda x_{ij})^2})}$

There fore

and var $(\hat{\beta}_{iub}) = \frac{\beta_i^{\,2}}{n-2}$ For $i = 1\,,2\,,\ldots,z\,,z+1\,,z+2 E(\hat{\beta}_{iub}) = \beta_i$

4- Shrinkage Estimation Method

In, 1968, Thompson proposed to shrink usual estimate $\hat{\beta}$ of the parameter β to prior in formation βo using weight factor $\varphi(\hat{\beta})$. Such that $0 \leq \varphi(\hat{\beta}) \leq 1$. we give the form of shrinkage estimator of β_i say $\hat{\beta}_{ish}$ will be

$$\hat{\beta}_{ish} = \varphi(\hat{\beta}_i)\hat{\beta}_{iub} + (1 - \varphi(\hat{\beta}_i))\beta io$$

Where

and $\beta_{io} = \frac{\ln(\frac{1}{2})}{\sum_{i=1}^{n}\ln(1- e^{-(\lambda x_{imed})^2})}\hat{\beta}_{iub} = \frac{n-1}{\sum_{j=1}^{n}\ln(1- e^{-(\lambda x_{ij})^2})}$

For $i = 1\,,2\,,\ldots,z\,,z+1\,,z+2$

Now, it is possible to use the “shrinkage estimation method (SH)” for estimate the parameter β_i of generalized Rayleigh distribution for three kinds of shrinkage estimation methods

4-1 – Constant Shrinkage Estimation Method(SH1)

The constant shrinkage weight factor will be $\varphi(\hat{\beta}_{iub}) = 0.01$. The constant shrinkage estimators of $\beta_i\ ; i = 1,2,\dots,z,z+1,z+2$ as follows

$\hat{\beta}_{ishl} = \varphi(\hat{\beta}_{iub})\hat{\beta}_{iub} + (1-\varphi(\hat{\beta}_{iub}))\beta_i o$

Hence, the estimates of Rs based on constant shrinkage estimation method of $\beta_i\ ; i = 1,2,\dots,z,z+1,z+2$ is given by

$$\hat{R}_{ssh1} = 1 - \sum_{i=z+1}^{z+2} \hat{\beta}_{ish1} * \sum_{m=1}^{z} (-1)^{m-1} \sum_{i1=i2\dots iz=1}^{z} \int_0^1 y^{\sum_{j=1}^{m} \hat{\beta}_{ijsh1} + \sum_{i=z+1}^{z+2} \hat{\beta}_{ish1}} * dy$$

(8)

4.2- Shrinkage Function Estimator(sh2)

We suggest the shrinkage function estimator (sh2) for estimate the parameters $\beta_i\ ; i = 1,2,\dots,z,z+1,z+2$ and the system reliability Rs based on the shrinkage weight function which is depends on sample size (n) will be

$\varphi(\hat{\beta}_{iub}) = e^{-n}$

The shrinkage function estimators of $\beta_i\ ; i = 1,2,\dots,z,z+1,z+2$ Will be

$\hat{\beta}_{ish2} = e^{-n}\ \hat{\beta}_{iub} + (1-e^{-n})\beta_i o$ where $\beta_i\ ; i = 1,2,\dots,z,z+1,z+2$

Hence the shrinkage function estimator of system reliability Rs based on shrinkage weight function is given by

$$\hat{R}_{ssh2} = 1 - \sum_{i=z+1}^{z+2} \hat{\beta}_{ish2} * \sum_{m=1}^{z} (-1)^{m-1} \sum_{i1=i2\dots iz=1}^{z} \int_0^1 y^{\sum_{i=1}^{m} \hat{\beta}_{ijsh2} + \sum_{i=z+1}^{z+2} \hat{\beta}_{ish2}} * dy$$

(9)

4.3 Squared Shrinkage Estimator (sh3)

Assume the squared shrinkage weight factor as

$$\gamma(\hat{\beta}_{iub}) = \left(\frac{\hat{\beta}_{iub} - E\left(\frac{\hat{\beta}_{iub}}{\beta_{io}}\right)}{\sqrt{var\left(\frac{\hat{\beta}_{iub}}{\beta_{io}}\right)}}\right)^2 * 0.001 \quad ; \ i = 1,2,\dots,z,z+1,z+2$$

Therefore, the squared shrinkage estimator $\hat{\beta}_{ish3}$ as follows

$\hat{\beta}_{ish3} = \gamma(\hat{\beta}_{iub})\hat{\beta}_{iub} + (1-\gamma(\hat{\beta}_{iub})) * \beta_{io}$

Hence, the estimates of Rs based on squared shrinkage method of $\beta_i\ ; i = 1,2,\dots,z,z+1,z+2$ is given by

$$\hat{R}_{ssh3} = 1 - \sum_{i=z+1}^{z+2} \hat{\beta}_{ish3} * \sum_{m=1}^{z} (-1)^{m-1} \sum_{i1=i2\dots iz=1}^{z} \int_0^1 y^{\sum_{i=1}^{m} \hat{\beta}_{ijsh3} + \sum_{i=z+1}^{z+2} \hat{\beta}_{ish3}} * dy$$

(10)

5- Least Squares Estimator (LSE)

The random samples strength X_{ij} have generalized Rayleigh distribution two parameters β_i and λ of size (n); $i = 1,2,\dots,z$ and $i = 1,2,\dots,n$ and stress random samples X_{z+1j}, X_{z+2j} follow generalized Rayleigh distribution with parameter $\beta_i; i = z+1, z+2$ and λ of size (n)

$$s = \sum_{j=1}^{n} [F(X_{ij}) - E(F(X_{ij})]^2 \text{ for } i = 1,2,\dots,z,z+1,z+2 \text{ and } j = 1,2,\dots,n$$

We have

$$F(X_{ij}) = (1-e^{-(\lambda x_{ij})^2})^{\beta_i} \text{ and } E(F(X_{ij}) = p_{ij}$$

Such that $p_{ij} = \frac{j}{n+1}$ for $i = 1,2,\dots,z,z+1,z+2$ and $j = 1,2,\dots,n$

Now, we consider $F(X_{ij}) - E\left(F(X_{ij})\right)$ *and* $\left(1-e^{-(\lambda x_{ij})^2}\right)^{\beta_i} = \frac{j}{n+1}$

$$s = \sum_{j=1}^{n} \left[\ln p_{ij} - \beta_i \ln(1 - e^{-(\lambda x_{ij})^2}) \right]^2$$

$$\frac{ds}{d\beta_i} = 2 \sum_{j=1}^{n} \left[\ln p_{ij} - \beta_i \ln(1 - e^{-(\lambda x_{ij})^2}) \right]^2 * (\ln \left(1 - e^{-(\lambda x_{ij})^2}\right) = 0$$

$$\hat{\beta}_{ilse} = \frac{\sum_{j=1}^{n} \ln p_{ij} \ln(1 - e^{-(\lambda x_{ij})^2})}{\sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^2} \text{ For } i = 1, 2, \dots, z, z+1, z+2 \text{ and } j = 1, 2, \dots, n$$

Hence, the estimates of Rs based on least squares estimator of $\beta_i ; i = 1, 2, \dots, z, z+1, z+2$ is given by

$$\hat{R}_{slse} = 1 - \sum_{i=z+1}^{z+2} \hat{\beta}_{ilse} * \sum_{m=1}^{z} (-1)^{m-1} \sum_{i1=i2\dots iz=1}^{z} \int_0^1 y^{\sum_{j=1}^{m} \hat{\beta}_{ijlse} + \sum_{i=z+1}^{z+2} \hat{\beta}_{ilse}} * dy$$

(11)

6-Bayes method (B)

Let x_{ij} follow generalized Rayleigh distribution with two parameters β_i and λ of size (n).Now, we have to find the Bayes estimate for β_i such that $i = 1, 2, \dots, z, z+1, z+2$ using non-information prior distribution g (β_i) based on modified extension of Jeffery prior and square loss function, as follow

The modified extension of Jeffery prior can be find by g $(\beta_i) \propto [I\ (\beta_i)]^c$

$$\text{Where } I\ (\beta_i) = -nE \left[\frac{\partial^2 \ln f(x_{ij}, \beta_i, \lambda)}{\partial \beta_i^2} \right]$$

There fore

$$g\ (\beta_i) = kn^c \beta_i^{-2c} \text{ for } i = 1, 2, \dots, z, z+1, z+2$$

The likelihood function $L(x_{ij}, \beta_i, \lambda)$ will be

$$L(x_{ij}, \beta_i, \lambda) = 2^n \beta_i^n \prod_{j=1}^{n} x_{ij}\, e^{-\lambda^2 \sum_{j=1}^{n} (x_{ij})^2} \prod_{j=1}^{n} (1 - e^{-(\lambda x_{ij})^2})^{\beta_i - 1}$$

The joint p.d.f. $H(x_{ij}, \beta_i, \lambda)$ is given by

$$H(x_{ij}, \beta_i) = 2^n \beta_i^n \lambda^{2n} \prod_{j=1}^{n} x_{ij}\, e^{-\lambda^2 \sum_{j=1}^{n} (x_{ij})^2} \prod_{j=1}^{n} (1 - e^{-(\lambda x_{ij})^2})^{\beta_i - 1} kn^c \beta_i^{-2c}$$

The marginal p.d.f. of x_{ij} will be

$$p(x_{ij}) = \int_0^\infty 2^n \beta_i^n \lambda^{2n} \prod_{j=1}^{n} x_{ij}\, e^{-\lambda^2 \sum_{j=1}^{n} (x_{ij})^2} \prod_{j=1}^{n} (1 - e^{-(\lambda x_{ij})^2})^{\beta_i - 1} kn^c \beta_i^{-2c}\, d\beta_i$$

Then the posterior distribution $\pi(x_{ij} ; \beta_i)$ for $i = 1, 2, \dots, z, z+1, z+2,\ \ j = 1, 2, \dots, n$ is given by

$$\pi(x_{ij} ; \beta_i) = \frac{H(x_{ij} ; \beta_i)}{p(x_{ij})}$$

$$= \frac{\beta_i^{n-2c}\ e^{-\beta_i \sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}}}{\int_0^\infty \beta_i^{n-2c}\ e^{-\beta_i \sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}}\, d\beta_i}$$

Now, we let

$$y = \beta_i \sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}$$

Therefore

$$\pi(x_{ij} ; \beta_i) = \frac{\beta_i^{n-2c} e^{-\beta_i \sum_{j=1}^{n} ln(1-e^{-(\lambda x_{ij})^2})^{-1}} \left[\sum_{j=1}^{n} ln(1-e^{-(\lambda x_{ij})^2})^{-1}\right]^{n-2c+1}}{\Gamma(n-2c+1)} \qquad (12)$$

Now, by using square error loss function which id defined below

$$L(\hat{\beta}_i ,\beta_i) = k(\hat{\beta}_i - \beta_i)^2$$
$$R(\hat{\beta}_i ,\beta_i) = E[L(\hat{\beta}_i ,\beta_i)]$$
$$R(\hat{\beta}_i ,\beta_i) = \int_0^{\infty} k(\hat{\beta}_i - \beta_i)^2 * \pi(x_{ij} ; \beta_i) d\beta_i$$

$$= \int_0^{\infty} \frac{k(\hat{\beta}_i - \beta_i)^2 \beta_i^{n-2c} e^{-\beta_i \sum_{j=1}^{n} ln(1-e^{-(\lambda x_{ij})^2})^{-1}} \left[\sum_{j=1}^{n} ln(1-e^{-(\lambda x_{ij})^2})^{-1}\right]^{n-2c+1}}{\Gamma(n-2c+1)} d\beta_i$$

$$\frac{dR}{d\hat{\beta}_i} = \int_0^{\infty} 2k(\hat{\beta}_i - \beta_i) * \frac{\beta_i^{n-2c} e^{-\beta_i \sum_{j=1}^{n} ln(1-e^{-(\lambda x_{ij})^2})^{-1}} \left[\sum_{j=1}^{n} ln(1-e^{-(\lambda x_{ij})^2})^{-1}\right]^{n-2c+1}}{\Gamma(n-2c+1)} d\beta_i = 0$$

We let

$$y = \beta_i \sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}$$
$$\beta_i = \frac{y}{\sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}}$$
$$d\beta_i = \frac{dy}{\sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}}$$

There fore

$$\hat{\beta}_i = \frac{\Gamma(n - 2c + 1)}{\Gamma(n - 2c + 1) \sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}}$$

Then, we have

$$\hat{\beta}_{iB} = \frac{(n - 2c + 1)}{\sum_{j=1}^{n} \ln(1 - e^{-(\lambda x_{ij})^2})^{-1}}$$

We assume c=2 therefore the Bayes estimator for $\beta_i(i = 1 ,2 ,\dots, z ,z + 1 ,z + 2 , i = 1 ,2 ,\dots, n)$ is given by

$$\hat{\beta}_{iB} = \frac{n-3}{\sum_{j=1}^{n} \ln(1-e^{-(\lambda x_{ij})^2})^{-1}} \qquad (13)$$

Hence the estimates of Rs based on Bayes estimator for $\beta_i(i = 1 ,2 ,\dots, z ,z + 1 ,z + 2 , i = 1 ,2 ,\dots, n)$ is given by

$$\hat{R}_{SB} = 1 - \sum_{i=z+1}^{z+2} \hat{\beta}_{iB} * \sum_{m=1}^{z} (-1)^{m-1} \sum_{i1=i2\dots iz=1}^{z} \int_0^1 y^{\sum_{j=1}^{m} \hat{\beta}_{iB} + \sum_{i=z+1}^{z+2} \hat{\beta}_{iB}} * dy \qquad (14)$$

7- Simulation Study

A simulation study will be conducted to show the estimator behavior of series system R_s for generalized Rayleigh distribution by generating 1000 samples of different size for different value of z and the parameters (λ , β_1, β_2 , … , β_{k+1}) as in tables. by the simulation study for the parameters considered, the value of maximum likelihood, shrinkage estimation (three type), least square and Bayes method for R_s with MSE are present in tables (1-6). "Monte Carlo simulation" is performed to compare the performances of the different methods of estimation for R_s . Math lab program was used in this research to estimate the distribution parameters

.: estimates of R_s 1Table

$k = 4, m = 10, n = 15, \lambda = 3, a_1 = 3, a_2 = 3.1, a_3 = 3.2, a_4 = 3.3$

α_5	α_6	R_s	$\hat{R}_{smle}$	$\hat{R}_{ssh1}$	$\hat{R}_{ssh2}$	$\hat{R}_{ssh3}$	$\hat{R}_{slse}$	$\hat{R}_{sB}$
3.5	4	0.047088	0.049715	0.047089	0.047136	0.047134	0.052358	0.0380761
4.5	5	0.028165	0.030291	0.028166	0.028187	0.028171	0.031228	0.022559
5.5	6	0.017905	0.019864	0.017905	0.017927	0.017922	0.021168	0.014476
6.5	7	0.011937	0.012856	0.011938	0.011942	0.011939	0.013861	0.009189

.: MSE for estimates of R_s 2Table

$k = 4, m = 10, n = 15, \lambda = 3, a_1 = 3, a_2 = 3.1, a_3 = 3.2, a_4 = 3.3$

α_5	α_6	$\hat{R}_{smle}$	$\hat{R}_{ssh1}$	$\hat{R}_{ssh2}$	$\hat{R}_{ssh3}$	$\hat{R}_{slse}$	$\hat{R}_{sB}$	Best
3.5	4	0.000618245889	0.000000000001	0.000000079703	0.000000090396	0.001493115480	0.000488832327	$sh1$
4.5	5	0.000281798105	0.0000000000004	0.000000035811	0.000000088695	0.000684506663	0.000203152851	$sh1$
5.5	6	0.000137134102	0.0000000000002	0.000000015222	0.000000024920	0.000383009834	0.000090852483	$sh1$
6.5	7	0.000075327246	0.00000000000009	0.000000008083	0.000000025493	0.000172961578	0.000050270334	$sh1$

.: estimates of R_s 3Table

$k = 3, m = 10, n = 15, \lambda = 3, a_1 = 3, a_2 = 3.1, a_3 = 3.2$

α_5	α_6	R_s	$\hat{R}_{smle}$	$\hat{R}_{ssh1}$	$\hat{R}_{ssh2}$	$\hat{R}_{ssh3}$	$\hat{R}_{slse}$	$\hat{R}_{sB}$
3.5	4	0.073198	0.076729	0.073198	0.073255	0.073240	0.076825	0.061528
4.5	5	0.048016	0.051122	0.048017	0.048055	0.048034	0.053234	0.040141
5.5	6	0.033221	0.035529	0.033222	0.033245	0.033256	0.037181	0.027452
6.5	7	0.023949	0.025876	0.023949	0.023962	0.023958	0.028428	0.019734

.: MSE for estimates of R_s 4Table

$k = 3, m = 10, n = 15, \lambda = 3, a_1 = 3, a_2 = 3.1, a_3 = 3.2$

α_5	α_6	$\hat{R}_{smle}$	$\hat{R}_{ssh1}$	$\hat{R}_{ssh2}$	$\hat{R}_{ssh3}$	$\hat{R}_{slse}$	$\hat{R}_{sB}$	Best
3.5	4	0.001123856115	0.000000000002	0.000000152129	0.000000224798	0.002408083902	0.000938991779	$sh1$
4.5	5	0.000587463195	0.0000000000009	0.000000072316	0.000000095637	0.001472001681	0.000456818451	$sh1$
5.5	6	0.000357077249	0.0000000000006	0.000000043047	0.000000256108	0.000850691467	0.000265332312	$sh1$
6.5	7	0.000212461579	0.0000000000003	0.000000024581	0.000000034670	0.000589322030	0.000150577758	$sh1$

.: MLE for estimates of R_s 5Table

$k = 2, m = 10, n = 15, \lambda = 3, a_1 = 3, a_2 = 3.1$

α_5	α_6	R_s	$\hat{R}_{smle}$	$\hat{R}_{ssh1}$	$\hat{R}_{ssh2}$	$\hat{R}_{ssh3}$	$\hat{R}_{slse}$	$\hat{R}_{sB}$
3.5	4	0.129637	0.132906	0.129638	0.129686	0.129650	0.137417	0.112705
4.5	5	0.095006	0.099592	0.095006	0.095063	0.095046	0.099979	0.083357

5.5	6	0.0726344	0.076815	0.0726349	0.072674	0.072661	0.078550	0.063632
6.5	7	0.057340	0.060657	0.057341	0.05736	0.057347	0.061320	0.0498560

.: MSE for estimates of R_s 6Table

$k = 2, m = 10, n = 15, \lambda = 3, a_1 = 3, a_2 = 3.1$

α_5	α_6	$\hat{R}_{smle}$	$\hat{R}_{ssh1}$	$\hat{R}_{ssh2}$	$\hat{R}_{ssh3}$	$\hat{R}_{slse}$	$\hat{R}_{sB}$	Best
3.5	4	0.002488694600	0.000000000004	0.000000326492	0.000000464847	0.005197249764	0.002258927140	$sh1$
4.5	5	0.001533216780	0.000000000002	0.000000197319	0.000000373615	0.003256183614	0.001295291295	$sh1$
5.5	6	0.001003994776	0.000000000002	0.000000132490	0.000000096421	0.002053004684	0.000811496703	$sh1$
6.5	7	0.000772499720	0.000000000001	0.000000093630	0.000000185630	0.001401497280	0.000609853792	$sh1$

Conclusions

From estimations reliability of (z+2) components series system of the “stress - strength model” that are subject to one of the stresses, while the “stress and the strength” follow generalized Rayleigh distribution of the tables 1-6 which contain component for strength and to one of the stresses, one can find the proposal shrinkage estimation method using constant shrinkage estimation method (sh1), performance good behavior and it is the best estimator than the others in the sense of MSE.

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