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Using The Solution Of Abel's Integral Equation To Convert The Caputo Fractional Derivatives Equation To An Integral Equation And Solve It

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Abstract

To convert the arbitrary ordinary differential equations of Caputo fractional order to the integral equation we introduce in this paper the method by using the theorem of the Abel equation to get the analytical solution of the singular equation in [16]and [18] used it to convert the Caputo fractional differential equation to an integral equation and solve it to get an analytical solution to differential equations of fractional order. We solved many general fractional equations (The BagleyTorvik equation, the fractional logistic differential equation, the fractional Bratutype equation, and other examples) using this method and provided a general solution. The new formulas to solve the Riemann-Liouville fractional differential equation have been obtained and applied, illustrated by many examples.

Keyword : Fractional derivatives, Integral Fractional equation, Abel's integral equation.

استخدام حلول معادلات تكامل ابل لتحويل معادلات المشتقة الكسرية لكبوتو إلى معادلات تكاملية يمكن حلها

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ملخص البحث :

لتحويل اي معادلة تفاضلية اعتياديه ذات رتبه كسريه لمشتق كبوت إلى معادلة تكامليه استخدمنا في هذه الورقة البحثية طريقة باستخدام نظرية معادلة أبل للحصول على حل صحيح أو صريح وليس حل عددي ، حيث تم تحويل المعادلة التفاضلية إلى معادلة تكاملية يمكن إيجاد حلها بالطرق المعروفة ٠ تم حل معظم المعادلات المعروفة مثل معادلة باگلي تروفيك و معادلة المنطق التفاضلية و معادلة بريوت التفاضلية وغيرها من المعادلات الأخرى ٠ كذلك استخدمنا هذه الصيغ الجديدة معادلات ذات مشتقات ريمان ليفيول ذات الرتب الكسرية ٠ تم استخدام أمثلة تطبيقية متعددة لبيان صحة الورقة البحثية ودقتها.

الكلمات المفتاحية :المشتقات ذات الرتبة الكسرية، المعادلات التكاملية، معادلة ابل التكاملية

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1 Introduction and history

Generally speaking, mathematical modeling is the best tool to understand and analyze data. It can be used in many fields of science and technology, and many mathematical problems of these models can be solved with the help of ordinary/ partial differential equations theory or with integral equations theory [16]. Many initial and boundary value problems are associated with ordinary/partial differential equations that can be solved more effectively by integral equations methods. In general pure analysis mathematics, the Integral $g(x) = \int_0^x y(t)(x-t)^{-\alpha} dt$ equations are considered the most useful in the theories of stochastic processes and functional analysis. There are two general kinds of integral equations and Abel's integral equation is of a special kind of linear Volterra integral equation of the first kind and is usually solved via the Laplace transform method, which finally reduces it to a differentiation of fractional order [15]. from the first sight to the Abel equation, the reader can know this equation is a particular case of a linear Volterra equation of the first kind. In 1823, Abel, while generalizing the tautochrone problem, derived the equationwhere$g(x)$ is a given function and $y(t)$is an unknown function, This problem is nothing but to find a curve in two dimensions when a material point, having started its motion at a point of the curve with ordinate x without initial velocity and moving along the curve under the action of gravity without friction, will reach the x-axis in time is $t = g(x)/\sqrt{2}\_g$ such that g is the acceleration in free falling[14].

By and large, Abel's problem was the oldest to lead to the study of integral equations, and the distinctive feature of this equation can be derived directly from concrete problems of mechanics or physics (without passing through a differential equation)[14][16] [17].

The aim of this paper, to convert the arbitrary ordinary differential equations of Caputo fractional order to the integral equation by using the Abel equation theorem to get the analytical solution of the singular equation. The paper is organized as follows: In the second section, introduced Abel integral equation was introduced to convert the Riemann-Liouville and Caputo Fractional Derivative equations and examples In the last section, a brief conclusion and some remarks were given.

1.1 Abel's integral equation

While searching for the representation formula for the solution of a linear differential equation in such a manner to include boundary conditions or initial conditions explicitly, we arrive at an integral equation. The solution of the integral

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equation is much easier than the original boundary value or initial value problem [16]. In [16], the solution of Abel's integral equation. We can summarize 1- If

$$g(x) = \int_0^x \frac{y(t)}{(x-t)^\alpha} dt \qquad (1.1)$$

, where $x \in [a,b]$, and $0 < \alpha < 1$ such that g(x) is a known function and $y(t)$ is a unknown function then

$$y(t) = \frac{sin(\alpha\pi)}{\pi} \frac{d}{dt} \int_0^t \frac{g(x)}{(t-x)^{1-\alpha}} dx. \qquad (1.2)$$

and

$$y(t) = \frac{sin(\alpha\pi)}{\pi} [\frac{g(a)}{(t-a)^{1-\alpha}} + \int_0^t \frac{g'(x)}{(t-x)^{1-\alpha}} dx] \qquad (1.3)$$

2- More general form of the Abel singular integral equation it's given by

$$g(x) = \int_a^x \frac{y(t)}{(h(x)-h(t))^\alpha} dt \qquad (1.4)$$

, where $x \in [a,b]$, and $0 < \alpha < 1$ such that $g(x)$ is a known function and $y(t)$ is a unknown function and $h(t)$ is a strictly monotonically increasing and differentiable in $(a,b), h'(t) \neq 0$ then

$$y(t) = \frac{sin(\alpha\pi)}{\pi} \frac{d}{dt} \int_a^t \frac{h'(u)g(u)}{(h(t)-h(u))^{1-\alpha}} du. \qquad (1.5)$$

3- Another general form of the Abel singular integral equation it's given by

$$g(x) = \int_x^b \frac{y(t)}{(h(x)-h(t))^\alpha} dt \qquad (1.6)$$

, where $x \in [a,b]$, and $0 < \alpha < 1$ such that $g(x)$ is a known function and $y(t)$ is a unknown function and $h(t)$ is a strictly monotonically increasing and differentiable in $(a,b), h'(t) \neq 0$ then

$$y(t) = \frac{sin(\alpha\pi)}{\pi} \frac{d}{dt} \int_t^b \frac{h'(u)g(u)}{(h(u)-h(t))^{1-\alpha}} du. \qquad (1.7)$$

Theorem 1.1 *If g(x) is continuous in* $0 < x \leq X$ *and* $\lim_{t \to 0} x^\mu g(x) = C$, *where* $C \neq 0$ *and* $\mu < \alpha$ *then equation 1.1 has solution*

$$y(t) = \frac{sin(\alpha\pi)}{\pi} \frac{d}{dt} \int_0^t \frac{g(x)}{(t-x)^{1-\alpha}} dx.$$

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and this solution is continuous in $0 < x \leq X$ *and satisfies*

$$y(t) = (C + O(1))\frac{\Gamma(1-\mu)}{\Gamma(1-\alpha)\Gamma(\alpha-\mu)}t^{\alpha-\mu-1}$$

Furthermore, this solution is unique such that the function of the form $y(t) = t^{\beta}Y(t)$*,*

where $\beta > -1$ *and Y(t) is continuous.*

In [18] page (73), we can find proof of theorem1.1. This theorem it's useful in our work in this article

1.2 Riemann-Liouville and Caputo Fractional Derivative

Let $0 \leq \alpha, \beta \leq 1$ is a real number and $g : [a,b] \rightarrow R$ is continuose function then :

$$ {}_a^R I_u^{\alpha} g(u) = \frac{1}{\Gamma(\alpha)}\int_a^u (u-t)^{\alpha-1}g(t)dt \quad , \quad {}_u^R I_b^{\beta} g(u) = \frac{1}{\Gamma(\beta)}\int_u^b (t-u)^{\beta-1}g(t)dt \quad u \in [a,b] \tag{1.8}$$

is the Riemann-Liouville Fractional integrals of order α (RRLFI) and the left Riemann-Liouvill Fractional integrals of order β (LRLFI) respectively .

$When \;\; u \in [a,b] and 0 \leq \alpha < 1$

$$ {}_a^R D_u^{\alpha} g(u) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{du}\int_a^u (u-t)^{-\alpha}g(t)dt \quad , \quad {}_u^R D_b^{\alpha} g(u) = \frac{1}{\Gamma(1-\alpha)}\frac{-d}{du}\int_u^b (t-u)^{-\alpha}g(t)dt \tag{1.9}$$

is the Riemann-Liouvill Fractional drivative of order α (RRLFD) and the left Riemann-Liouvill Fractional drivative of order α (LRLFD) respectively . If $0 \leq \alpha$ and $n + 1 \leq \alpha \leq n$ such that n is a positive integer number, the (RRLFD) and (LRLFD) are defined as

(1.10) $$ {}_a^R D_u^{\alpha} g(u) = \frac{d^n}{du^n} {}_a^R D_u^{-(n-\alpha)} g(u) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{du^n}\int_a^u (u-t)^{(n-\alpha-1)}g(t)dt.$$

(1.11) $$ {}_u^R D_b^{\alpha} g(u) = \frac{(-1)^n d^n}{du^n} {}_b^R D_u^{-(n-\alpha)} g(u) = \frac{1}{\Gamma(n \quad \alpha)}\frac{(-1)^n d^n}{du^n}\int_u^b (t-u)^{-\alpha}g(t)dt.$$

The Right and Left Caputo Fractional Drivative ,(RCFD) and (LCFD) of order α is :

(1.12) $$ {}_a^C D_u^{\alpha} g(u) = \frac{1}{\Gamma(1-\alpha)}\int_a^u (u-t)^{-\alpha}\dot{g}(t)dt.$$

(1.13) $$ {}_u^C D_b^{\alpha} g(u) = \frac{-1}{\Gamma(1 \quad \alpha)}\int_u^b (t-u)^{-\alpha}\dot{g}(t)dt.$$

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When $0 \leq$ α and $n + 1 \leq \alpha \leq n$ such that n is a positive integer

$${}^C_a D^\alpha_u g(u) = {}^C_a D_u^{-(n-\alpha)} g^{(n)}(u) = \frac{1}{\Gamma(n-\alpha)}\int_a^u (u-t)^{n-\alpha-1} g^{(n)}(t)dt.$$

number, the (RCFD) and (LCFD) are defined as:

(1.14)

$${}^C_u D^\alpha_b g(u) = (-1)^n {}^C_u D_b^{-(n-\alpha)} g^{(n)}(u) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_u^b (t-u)^{n-\alpha-1} g^{(n)}(t)dt. \quad (1.15)$$

Remark 1.1 ***The Fundamental Theorem of Fractional Calculus*** *or* ***Newton-***

$$C\,\alpha\,R\,\alpha \quad (1.16)$$

a $D_{u\cdot a} I_u\, g(u) = g(u)$

and

$$C\,\alpha\,R\,\alpha \quad (1.17)$$

u $D_{b\,\cdot u} I_b\, g(u) = g(u)$

Also where $0 < \alpha < 1$ *and* $g(u) \in AC[a,b]$ *or* $g(u) \in C[a,b]$ *then*

$${}^R_a I^\alpha_{u\cdot a}\, {}^C_a D^\alpha_u g(u) = g(u) - g(a) \quad (1.18)$$

Leibniz Theorem *means the derivative operation is inverse to the integral*

operation, and we can say that the left or right Caputo Fractional Derivative

operation and the left or right Riemann-Liouville Fractional Integral operation are

inverse to each other. Where $\alpha > 0$ *and* $g(u) \in L\infty(a,b)$ *or* $g(u) \in C[a,b]$ *then and*

$$Ru\; I_b \alpha g(u).C_u\, D_b \alpha = g(u) - g(b) \quad (1.19)$$

such that $L_\infty(a,b)$ = { *set of Lebesgue measurable functions on (a,b)*} , $AC[a,b]$ = { *set of space of functions which are absolutely continuous on [a,b]*} *and* $C[a,b]$ = { *set of space of functions which are continuous on [a,b]* } *see [19] and [8]*

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Remark 1.2 *There are many properties of The Riemann-Liouville and Caputo Fractional Derivative:*

1-Linearty :λand μ are constant then ${}^{C}{}_{a}D_u{}^{\alpha}(\mu g(u)+\lambda f(u))=\mu{}^{C}{}_{a}D_u{}^{\alpha}g(u)+\lambda{}^{R}{}_{a}D_u{}^{\alpha}f(u)$

2- ${}^{C}{}_{u}D_b{}^{\alpha}\zeta={}^{C}{}_{a}D_u{}^{\alpha}\zeta=0$ *such that ζ is constant . β .*

3- ${}^{R}_{a}D^{\alpha}_{u}(u-a)^{\beta}=\frac{\Gamma(\ \ +1)}{\Gamma(\beta+1-\alpha)}(u-a)^{\beta-\alpha}$

4- ${}^{R}_{a}D^{\alpha}_{u}\zeta=\frac{\zeta.(u-a)^{-\alpha}}{\Gamma(1-\alpha)}$ *and* ${}^{R}_{u}D^{\alpha}_{b}\zeta-\frac{\zeta.(b-u)^{-\alpha}}{\Gamma(1-\alpha)}$.

5- ${}^{C}_{u}D^{\alpha}_{b}g(u)={}^{R}_{u}D^{\alpha}_{b}g(u)-\sum_{k=0}^{r-1}\frac{g\quad(b)}{\Gamma(k-\alpha+1)}(b-u)^{k-\alpha}$ *and* $r-1<\alpha<r\in Z$

6- ${}^{C}_{a}D^{\alpha}_{u}g(u)={}^{R}_{a}D^{\alpha}_{u}g(u)\quad\sum^{r-1}\frac{g^{(k)}(a)}{}(u\quad a)^{k-\alpha}$ $-\quad{}_{k=0}\ \Gamma(k-\alpha-1)\quad-\quad and\quad-$

${}^{C}D^{\alpha}\ u^{m}\quad{}^{C}\ D^{\alpha}\ u^{m}$ *where and* $m<n$ *and* $n-1<\alpha<n$

$r1<\alpha<r\in Z$

7- ${}_{au}(\ \ )={}_{u}\ \ {}_{b}(\ \ )=0$

${}^{C}_{0}D^{\alpha}_{u}(u^{m})=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}u^{m-\alpha}$, *where and* $m>n$ *and* $n-1<\alpha<n$

${}^{C}_{a}D^{\alpha}_{u}(u-a)^{m}=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}(u-a)^{m-\alpha}$, *and*

${}^{C}_{0}D^{\alpha}_{u}(b-u)^{m}=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}(b-u)^{m-\alpha}$

${}^{C}_{a}D^{\alpha}_{u}g(u).{}^{C}_{a}D^{\alpha}_{u}f(u)=\frac{1}{\Gamma(n-\alpha)}{}^{C}_{a}D^{\alpha}_{u}(g(u).f(u))$

8-Where $g,f\in c^{n}[a,b],n-1<\alpha<n\in Z^{+}$ *we can write :. for more information see*

[2][10][9][3] [6][1][4][5][7]

9- respectively. One basic property needed for variational problems is an integration by parts formula. For the Caputo fractional derivative, we have (see e.g. Agrawal (2007b) and Kilbas et al. (2006)):

$$\int_a^b y_1(x)\cdot{}^{C}_{a}D^{\alpha}_{x}y_2(x)dx=\int_a^b y_2(x)\cdot{}_{x}D^{\alpha}_{b}y_1(x)dx+\left[y_2(x)\cdot{}_{x}I^{1-\alpha}_{b}y_1(x)\right]_a^b.$$

It can be observed that the left member of the left Caputo derivative, while on the right we have the right Riemann-Liouville derivative. Also, when α = 1, this formula becomes the usual integration by parts formula.

2 Abel integral equation to convert The Riemann-Liouville and Caputo Fractional Derivative equations

From definition of Caputo fractianal derivative and equations 1.5 and 1.7 we can prove next two theorems

Theorem 2.1 *Where* $0<\alpha<1$ *and*

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$$ {}_{a}^{C}D_{x}^{\alpha}g(x)=\frac{1}{\Gamma(1-\alpha)}\int_{a}^{x}\frac{g'(t)}{(x-t)^{\alpha}}dt $$

, then $$ \frac{g'(t)}{\Gamma(1-\alpha)}=\frac{sin\alpha\pi}{\pi}\frac{d}{dt}\int_{a}^{t}\frac{{}_{a}^{C}D_{x}^{\alpha}g(x)}{(t-x)^{1-\alpha}}dx $$

Theorem 2.2 *Where* $0 < \alpha < 1$ *and*

$$ {}_{u}^{C}D_{b}^{\alpha}g(u)=\frac{-1}{\Gamma(1-\alpha)}\int_{u}^{b}\frac{g'(x)}{(u-x)^{\alpha}}dt $$

, then $$ \frac{g'(u)}{(-1)\Gamma(1-\alpha)}=\frac{sin\alpha\pi}{\pi}\frac{d}{dt}\int_{u}^{b}\frac{{}_{x}^{C}D_{b}^{\alpha}g(x)}{(x-u)^{1-\alpha}}dx $$

The converse of theorems 2.1 and 2.2 is true .We will illustrate two example to chake validity of last theorems to convert fractional defrantial equation of order α to integral equation and solve it to get analytical solution of it. Consider the flowing problems:

Example 2.1 *To solve fractional defrantial equation of order* 0.5 *with boundary condition*

$$ {}_{0}^{C}D_{x}^{0.5}g(x)-1=0, and \;\; g(0)=0. \qquad (2.1) $$

multiply both sides of equation 2.1 by $(\frac{\Gamma(1-\alpha)sin\alpha\pi}{\pi.(u-t)^{1-\alpha}})$ *and integrating both sides with recpect to x from 0 to t*

,than differentiating result with recpect to t we obtin :

$$ \int_{0}^{t}\frac{{}_{0}^{C}D_{x}^{0.5}g(x)}{(t-x)^{0.5}}dx-\int_{0}^{t}\frac{dx}{(t-x)^{0.5}}=0 $$

and $t - x > 0$ $$ \frac{sin(\pi/2)}{\pi}\frac{d}{dt}\int_{0}^{t}\frac{{}_{0}^{C}D_{x}^{0.5}g(x)}{(t-x)^{0.5}}dx-\frac{sin(\pi/2)}{\pi}\frac{d}{dt}\int_{0}^{t}\frac{dx}{(t-x)^{0.5}}=0 $$

by using theorem 2.1

$$ \frac{g'(t)}{\Gamma(0.5)}-\frac{1}{\pi}\frac{d}{dt}[2(t-x)^{0.5}]_{0}^{t}=0 $$

$$ g(t)=\frac{2\Gamma(0.5)}{\pi}.t^{0.5}+c $$

using boundary condition we get

$$ g(t)-\frac{2\Gamma(0.5)}{\pi}.t^{0.5} $$

Example 2.2 *To solve fractional defrantial equation of order* 1/3 *with boundary condition*

$$ {}_{0}^{C}D_{x}^{1/3}g(x)-x=0, and \;\; g(0)=0. \qquad (2.2) $$

multiply both side of equation 2.2 by $(\frac{\Gamma(1-\alpha)sin\alpha\pi}{\pi.(u-t)^{1-\alpha}})$ *and integrating both sides with recpect to x from 0 to t*

,than differentiating result with recpect to t we obtin :

$$\int_0^t \frac{{}_0^C D_x^{1/3} g(x)}{(t-x)^{2/3}} dx - \int_0^t \frac{x dx}{(t-x)^{2/3}} = 0$$

and $t - x > 0$ $$\frac{sin(\pi/3)}{\pi} \frac{d}{dt} \int_0^t \frac{{}_0^C D_x^{1/3} g(x)}{(t-x)^{2/3}} dx - \frac{sin(\pi/3)}{\pi} \frac{d}{dt} \int_0^t \frac{x dx}{(t-x)^{2/3}} = 0$$

by using theorem 2.1

$$\frac{g'(t)}{\Gamma(2/3)} - \frac{3^{0.5}}{2\pi} \frac{d}{dt} [\frac{9}{4} t^{4/3}] = 0$$

$$g(t) = \frac{9}{4} \frac{\Gamma(2/3) 3^{0.5}}{2\pi} . t^{4/3} + c$$

using boundary condition we get $c = 0$

$g(t) = \frac{9}{4} \frac{\Gamma(2/3) 3^{0.5}}{2\pi} . t^{4/3}$

Example 2.3 *To solve fractional defrantial equation of order α with boundary condition*

$$ {}_0^C D_x^{\alpha} g(x) - g(x) = 0, and \quad g(0) = 0. \qquad (2.3)$$

multiply both side of equation 2.3 by $(\frac{\Gamma(1-\alpha) sin \alpha \pi}{\pi (t-x)^{1-\alpha}})$ *and integrating both sides with recpect to x from 0 to t than differentiating result with recpect to t we obtin : and*

$$\frac{\Gamma(1-\alpha) sin\alpha\pi}{\pi} \frac{d}{dt} \int_0^t \frac{{}_0^C D_x^{\alpha} g(x)}{(t-x)^{1-\alpha}} dx - \frac{\Gamma(1-\alpha) sin\alpha\pi}{\pi} \frac{d}{dt} \int_0^t \frac{g(x) dx}{(t-x)^{1-\alpha}} = 0$$

$t - x > 0$

$g'(t) - \frac{\Gamma(1-\alpha) sin\alpha\pi}{\pi} \frac{d}{dt} \int_0^t \frac{g(x) dx}{(t-x)^{1-\alpha}} = 0$

$\frac{g'(t)}{\Gamma(1-\alpha)} = \frac{sin\alpha\pi}{\pi} \frac{d}{dt} \int_0^t \frac{g(x) dx}{(t-x)^{1-\alpha}}$

by using theorem 1.1 we get

$$\frac{g'(t)}{\Gamma(1-\alpha)} = (C + O(1)) \frac{\Gamma(1-\mu)}{\Gamma(1-\alpha)\Gamma(\alpha-\mu)} t^{\alpha-\mu-1}$$

$$g'(t) = (C + O(1)) \frac{\Gamma(1-\mu)}{\Gamma(\alpha \quad \mu)} t^{\alpha-\mu-1}$$

such that C is constant and μ any number μ < α. using boundary condition we get $c = 0$

Theorem 2.3 *Where* $0 < \alpha < 1$ *and* ${}_a^R D_u^{\alpha} g(u) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_a^u (u-x)^{-\alpha} g(x) dx$ *then*

$g(u) = \frac{sin(1-\alpha)\pi}{\pi} \frac{\Gamma(1-\alpha)}{(-1)^{-\alpha}} \int_a^u \frac{{}_a^R D_u^{\alpha} g(t)}{(u-t)^{1-\alpha}} dt$

To prove this theorem we can write The Left Riemann-Liouville Fractional Derivative as:

$${}^{R}_{a}D^{\alpha}_{u}g(u) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{du}\int_a^u \frac{g(x)}{(u-x)^{1-(-\alpha+1)}}dx$$

$${}^{R}_{a}D^{\alpha}_{u}g(u) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{du}\int_a^u \frac{g(x)}{(-1)^{\alpha}(x-u)^{1-(-\alpha+1)}}dx$$

multiply both sides of last equation by $\frac{sin(1-\alpha)\pi}{\pi}\frac{\Gamma(1-\alpha)}{(-1)^{-\alpha}}$

$$\frac{sin(1-\alpha)\pi}{\pi}\frac{\Gamma(1-\alpha)}{(-1)^{-\alpha}}.{}^{R}_{a}D^{\alpha}_{u}g(u) = \frac{sin(1-\alpha)\pi}{\pi}\frac{d}{du}\int_a^u \frac{g(x)}{(x-u)^{1-(-\alpha+1)}}dx$$

by using converse theorem 2.1 we get

$$g(u) = \frac{sin(1-\alpha)\pi}{\pi}\frac{\Gamma(1-\alpha)}{(-1)^{-\alpha}}\int_a^u \frac{{}^{R}_{a}D^{\alpha}_{u}g(t)}{(u-t)^{1-\alpha}}dt$$

Remark 2.1 *1- We can write theorem 2.1 and theorem 2.2 in the general form for all* $n-1<\alpha<n$ *such that* $n\in N$*. 2-We can't write theorem2.3 in the general form for all* $n-1<\alpha<n$ *such thatn* $\in N$*.* **Example 2.4** *To solve Bagley-Trovik equation (For more details about Bagley-Trovik equation see [7])*

$$g''(u) + {}^{C}_{0}D_u^{\alpha}g(u) - 2g^2(u) = 0, and\ g(0) = g'(0) = 0. \qquad (2.4)$$

$$2g^2(u) - g''(u) = {}^{C}_{0}D_u^{\alpha}g(u)$$

using equation of The Fundamental Theorem of Fractional Calculus 1.18 obtian

$$g(u) = -{}^{R}_{0}I^{\alpha}_{u}(2g^2(u) - g''(u)) = \frac{1}{\Gamma(\alpha)}\int_0^u \frac{-(2g^2(u)-g''(u))}{(u-t)^{\alpha-1}}$$

by using theorem 1.1 we get

$$-2g^2(t) + g''(t) = \frac{\Gamma(\alpha)sin\alpha\pi}{\pi}\frac{d}{dt}\int_0^t \frac{g(x)}{(t-x)^{2-\alpha}}dx$$

$$\frac{-2g^2(t)+g''(t)}{\Gamma(\alpha)} = (C+O(1))\frac{\Gamma(1-\mu)}{\Gamma(1-\alpha)\Gamma(\alpha-\mu)}u^{\alpha-\mu-1}$$

it's general solution of Bagley-Trovik equation. such that C is constant andμ*any number* $\mu<\alpha$*. we can initial condition we get C withe any number* $\mu<\alpha$*.*

Example 2.5 *To solve Fractional Logistic Diffreantail Equation (For more details about Fractional Logistic Diffreantail Equation see [12])*

$${}^{C}_{0}D^{\alpha}_{u}g(u) - \rho g(u)(1-g(u)) = 0, and\ \ g(0) = u_0\ \ t>0,\ \ \rho>0\ ,\ u_0>0\ ,\ 0<\alpha<1 \qquad (2.5)$$

using equation of The Fundamental Theorem of Fractional Calculus 1.18 obtian

$$g(u) - u_0 = {}^{R}_{0}I^{\alpha}_{u}\rho g(u)(1-g(u)) = \frac{1}{\Gamma(\alpha)}\int_0^u \frac{\rho g(u)(1-g(u))}{(u-t)^{\alpha-1}}$$

$$\frac{(g(u)-u_0)\Gamma(\alpha)}{\rho} = \int_0^u \frac{g(u)(1-g(u))}{(u-t)^{\alpha-1}}$$

by using theorem 1.1 we get

$$\frac{g(t)(1-g(t))\Gamma(\alpha)}{\rho} = \frac{\Gamma(\alpha)sin\alpha\pi}{\pi}\frac{d}{dt}\int_0^t \frac{(g(t)-u_0)}{(t-x)^{2-\alpha}}dx$$

using inatial condition we get C withe any number μ < α.

3 Conclusion

Integral equations are one of the most important mathematical tools in both pure and applied mathematics. Integral Equations play a very important role in modern science, such as in numerous problems in engineering and mechanics. Also The Riemann-Liouville and Caputo Fractional Differential equation provides a very useful framework to deal with physics and non local dynamics in Mechanics . In this work, we use a new method besides the Integral equations of Apel's formula to solve the Fractional Differential equation. We have proposed a new analytical technique to get the solution of solve some classes of Fractional Differential equations. We provide an example to illustrate our main results and to show the efficiency method.

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