Zagreb Indices and Co-indices in 2-idempotent Divisor Graph

Authors Names	ABSTRACT
Abbas Hassan Mesto [a] *Husam Q. Mohammad*[b] Publication date: 20 /2 /2026 ***Keywords:*** Zagreb index, Zagreb co-index, direct product local ring, idempotent divisor graph.	This paper studies the Zagreb indices and co-indices of the graph $\Pi_2(R)$, where R is a direct product of two local ring of nilpotency 2. It focuses on cases where one component is involution, and on the general case where neither component is. Explicit formulas are derived based on the number of units in each component, and examples are given to support and illustrate the result.

1.Introduction

Let R be a finite commutative ring with identity $1 \neq 0$.In [4] 1988, Beck introduced the concept of zero divisor graph for commutative rings, establishing a foundational link between ring theory and graph theory. This graph, defined over a commutative ring with identity $1 \neq 0$, was further refined by Anderson and Livingston [2] in1999.Their work formalized the zero divisor graph $\Gamma(R)$, whose vertex set comprises the non-zero zero divisors $Z(R)^* = Z(R) - \{0\}$.Two distinct vertices z_1 and z_2 are adjacent in $\Gamma(R)$ if and only if $z_1 . z_2 = 0$. Subsequent studies by various authors (e.g., $[10], [14] - [16]$) expanded upon this framework.

In [13]2013, Mohammad and Shuker investigated zero divisor graphs for specific finite rings, characterized complete bipartite structures within such graphs. In [12] 2022, they proposed the idempotent divisor graph $\Pi(\mathrm{R})$ defined over the non-zero elements $R^* = R - \{0\}$. Here, distinct vertices a and b are adjacent if and only if $a.b = 0$ for some non-unit idempotent element $e \in R$(i.e., $e^2 = e$.

In [1]2022, Anderson and Badawi extended this discourse n-zero divisor graphs for commutative semi-group S with zero. For a positive integer n, the graph $\Gamma_n(S)$ has vertices $Z_n(S)^* = Z_n(S) - \{0\}$, where $Z_n(S)$ denotes the set of n-zero divisors. Adjacency in $\Gamma_n(S)$ occurs precisely when $ab = 0$ for distinct vertices a and b.

In [11]2025, mohammad and Mesto introduced a novel graph structure termed the n-idempotent divisor graph, which synthesize concepts from both n-zero divisor graph and idempotent divisor graph.

In [9] 1972, Gutman and Trinajstic' introduced the first Zagreb index, second Zagreb index.

In [7] 2008, the researcher Došlic' introduced two new indices associated with the first and second Zagreb indices, known as the Zagreb co-incidence, which are based on the degrees of non-adjacent vertices. In the third part of this paper, we were able to find the first and second Zagreb indices for a graph consisting of two local rings, one of which is involution, and other is not, in addition to solving an example using the graph method according to the result we obtained, and in another case when they are not involution. In the t part, we found first and second Zagreb co-index for the rings. In this paper we denote by u_S the number of unit elements of S and $u_{\acute{S}}$ the number of unit elements of $\acute{S}$ $R \times \acute{R}$ it means a direct product of two rings R and $\acute{R}$.

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2. Preliminaries

Definition 2.1 [6]: The degree of a vertex v in a graph G is the number of vertices adjacent to v, or equivalently, the number of edges incident to v. The degree of v is denoted by $deg_G v$ or simply deg(v).

Definition2.2[9]: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For an edge connecting vertices u and v denoted by uv, the **first Zagreb index** is defined as

$$M_1 = M_1(G) = \sum_{v\in V(G)}(deg(v))^2 \text{ or } M_1 = M_1(G) = \sum_{uv\in E(G)}[deg(u) + deg(v)]$$

And **second Zagreb index** $M_2 = M_2(G) = \sum_{uv\in E(G)}[deg(u).deg(v)]$

Definition2.3[7]: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ For an edge $uv \in E(G)$, the **first Zagreb co-index** is defined as

$$\overline{M_1} = \overline{M_1}(G) = \sum_{uv\notin E(G)}[deg(u) + deg(v)]$$

And **second Zagreb co-index** $\overline{M_2} = \overline{M_2}(G) = \sum_{uv\notin E(G)}[deg(u).deg(v)]$

Definition2.4[5]: A ring R is said to be **local ring** if it has a unique maximal ideal

Definition2.5:[8] An element u is called involution if $u^2 = 1$

Definition2.6 [3]: Nilpotency in ring theory refers to an element $a \in R$ such that $a^n = 0$ for some positive integer n. The smallest such n is called the index of nilpotency of a.

Definition2.7 [11] : Let $R_n = \{x^n : x \in R\}$. Then R_n is a semi-group of R with multiplicative 0. The **n-idempotent divisor graph** of R to be the simple graph $\Pi_n(R)$ with vertices in $R_n^* = R_n - \{0\}$ and distinct vertices a and b in R_n^* are adjacent if and only if $ab = e$.

Lemma 2.8 [11]:

Let S be a involution and $\acute{S}$ are a local rings with nilpotency 2 and not isomorphic with S, then

$$deg(v)_{v\in\Pi_2(R)} = \begin{Bmatrix} u_{\acute{S}} & if & v = (1,0) \\ 3 & if & v = (0,u_2), u_2 \neq 1 \\ 2 & & if\ v = (u_1,u_2)\ or\ v = (0,1) \end{Bmatrix}$$

, where $R \cong S \times \acute{S}$

Lemma 2.9 [11]:

Let $R \cong S \times \acute{S}$, where S and $\acute{S}$ are local rings of nilpotency 2 and not involution, then

$$deg(v)_{v\in\Pi_2(R)} = \begin{Bmatrix} u_{\acute{S}} & if & v=(1,0) \\ u_S & if & v=(0,1) \\ u_{\acute{S}}+1 & if & v=(u_1,0), u_1 \neq 1 \\ u_S+1 & if & v=(0,u_2), u_2 \neq 1 \\ 2 & if & v=(u_1,u_2) \end{Bmatrix}, \text{ where } R \cong S \times \acute{S}$$

Remark 2.11: If R is local ring, then R has even elements of $U(R)$.

3. Zagreb indices of 2-idempotent divisor graph

In this section, we successfully derived a result for calculating the first and second Zagreb indices of product two local rings : once when one of them is involution while the other is not, and another time when both are non-involution. We supported the findings by explain example.

Theorem 3.1: Let S to be involution and $\acute{S}$ are a local rings with nilpotency 2 and not isomorphic with S, then the first Zagreb index is $M_1(\Pi_2(R)) = \left(\frac{2u_{\acute{S}}^2+13u_{\acute{S}}-10}{2}\right)$, where $R \cong S \times \acute{S}$

Proof:

By lemma (2.8), we have $\left(\frac{u_{\acute{S}}}{2}-1\right)$ vertices of $(o,u_2), u_2 \in U(S_2)-\{1\}$, and $\frac{u_{\acute{S}}}{2}$ of vertices (u_1,u_2) so that

$$M_1(\Pi_2(R)) = \sum_{v\in V(\Pi_2 R)} (\deg(v))^2$$

$$= (\deg((1,0)))^2 + (\deg(0,1))^2 + \sum_{\substack{v=(0,u_2)\\ u_2\neq 1}} (\deg(v))^2 \sum_{v=(u_1,u_2)} (\deg(v))^2$$

$$= u_{\acute{S}}^2 + 2^2 + \left(\frac{u_{\acute{S}}}{2}-1\right)(3)^2 + \left(\frac{u_{\acute{S}}}{2}\right).2^2$$

$$= u_{\acute{S}}^2 + 4 + \frac{9}{2}u_{\acute{S}} - 9 + 2u_{\acute{S}} = u_{\acute{S}}^2 + \frac{13}{2}u_{\acute{S}} - 5 = \frac{2u_{\acute{S}}^2 + 13u_{\acute{S}} - 10}{2}$$

Theorem 3.2: Let $R \cong S \times \acute{S}$, where S and $\acute{S}$ are local rings of nilpotency 2 and not involution, then the first Zagreb index

$$M_1(\Pi_2(R)) = \frac{1}{2}[6(u_S.u_{\acute{S}}) + (u_S.u_{\acute{S}}^2) + (u_{\acute{S}}.u_S^2) - 3(u_S+u_{\acute{S}}) - 4].$$

Proof:

By lemma (2.9) We have $(\frac{u_S}{2}-1)$ vertices of $(u_1,0)$, $\left(\frac{u_{\acute{S}}}{2}-1\right)$ vertices of (o,u_2), where $u_i \in [\{U(S)$ or $U(\acute{S})\}]-\{1\}, i=1,2$ and $\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}$ of vertices of (u_1,u_2).

Therefore, $M_1\left((\Pi_2(R))\right) = \sum_{v \in V(\Pi_2 R)} (\deg(v))^2$

$$= (deg(1,0))^2 + (deg((0,1)))^2 + \sum_{u_1 \neq 1, v=(u_1,0)} (deg(v))^2 + \sum_{u_2 \neq 1, v=(0,u_2)} (deg(v))^2 + \sum_{(u_1,u_2)} (deg(v))^2$$

$$= {u_{\acute{S}}}^2 + {u_S}^2 + \left(\frac{u_S}{2}-1\right)(u_{\acute{S}}+1)^2 + \left(\frac{u_{\acute{S}}}{2}-1\right)(u_S+1)^2 + 2^2.\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right)$$

$$= {u_{\acute{S}}}^2 + {u_S}^2 + \left(\frac{u_S}{2}-1\right)({u_{\acute{S}}}^2+2u_{\acute{S}}+1) + \left(\frac{u_{\acute{S}}}{2}-1\right)({u_S}^2+2u_S+1) + u_S.u_{\acute{S}}$$

$$= {u_{\acute{S}}}^2 + {u_S}^2 + \frac{u_S.{u_S}^2}{2} + (u_S.u_{\acute{S}}) + \frac{u_S}{2} - {u_{\acute{S}}}^2 - 2u_{\acute{S}} - 1 + \frac{u_{\acute{S}}.{u_S}^2}{2} + u_S.u_{\acute{S}} + \frac{u_{\acute{S}}}{2} - {u_S}^2 - 2u_S - 1 + u_S.u_{\acute{S}}$$

$$= 3(u_S.u_{\acute{S}}) + \frac{1}{2}(u_S.{u_{\acute{S}}}^2 + u_{\acute{S}}.{u_S}^2) - \frac{3}{2}(u_{\acute{S}}) - \frac{3}{2}(u_S) - 2$$

$$= 3(u_S.u_{\acute{S}}) + \frac{1}{2}(u_S.{u_{\acute{S}}}^2 + u_{\acute{S}}.{u_S}^2) - \frac{3}{2}(u_S + u_{\acute{S}}) - 2$$

$$= \left[\frac{6(u_S.u_{\acute{S}}) + (u_S.{u_{\acute{S}}}^2 + u_{\acute{S}}.{u_S}^2) - 3(u_S+u_{\acute{S}}) - 4}{2}\right]$$

Theorem 3.3: Let S be involution and $\acute{S}$ are a local rings with nilpotency 2 and not isomorphic with S, then the second Zagreb index is $M_2(\Pi_2(R)) = \left[\frac{10{u_{\acute{S}}}^2+17u_{\acute{S}}-26}{4}\right]$

Proof:

The vertex (1,0) which has degree $u_{\acute{S}}$ adjacent with vertices $(0,u_2)$, $(0,1)$ and $(1,u_2)$ which have degrees 3 , 2 and 2 respectively. For the vertex (1,0), then

$$A = \sum_{(1,0)(0,u_2) \in E(\Pi_2(R))} deg\big((1,0)\big).deg\big((0,u_2)\big) + \sum_{(1,0)(0,1) \in E(\Pi_2(R))} deg\big((1,0)\big).deg\big((0,1)\big) + \sum_{(1,0)(1,u_2) \in E(\Pi_2(R))} deg\big((1,0)\big).deg\big((1,u_2)\big)$$

$$= \sum_{1 \neq u_2 (0,u_2) \in \Pi_2(R)} u_{\acute{S}}.3 + u_{\acute{S}}.2 + \sum_{1 \neq u_2 (0,u_2) \in \Pi_2(R)} u_{\acute{S}}.2$$

$$= u_{\acute{S}}.3.\left(\frac{u_{\acute{S}}}{2}-1\right) + u_{\acute{S}}.2 + u_{\acute{S}}.2.\frac{u_{\acute{S}}}{2} = \frac{3}{2}{u_{\acute{S}}}^2 - 3u_{\acute{S}} + 2u_{\acute{S}} + {u_{\acute{S}}}^2 = \frac{5}{2}{u_{\acute{S}}}^2 - u_{\acute{S}}$$

The vertex (0,1) has degree u_S adjacent only with two vertices (1,0) and (1,1) which have degrees $u_{\acute{S}}$ and 2 respectively.

For the vertex (0,1) ,then

$$B = \sum_{(0,1)(1,0)\in E(\Pi_2(R))} deg\big((0,1)\big).deg((1,0)) + \sum_{(0,1)(1,1)\in E(\Pi_2(R))} deg\big((0,1)\big).deg((1,1))$$

$$= 2.u_{\acute{s}} + 2.2 = 2.(u_{\acute{s}} + 2) = 2u_{\acute{s}} + 4$$

The vertex (1,1) has degree 2 adjacent only with two vertices (1,0) and (0,1) which have degrees $u_{\acute{s}}$ and 2 respectively.

For the vertex (1,1), then

$$C = \sum_{(1,1)(1,0)\in E(\Pi_2(R))} deg\big((1,1)\big).deg((1,0)) + \sum_{(1,1)(0,1)\in E(\Pi_2(R))} deg\big((1,1)\big).deg((0,1))$$

$$= 2.u_{\acute{s}} + 2.2 = 2.u_{\acute{s}} + 4.$$

The vertex $(0, u_2)$ has degree 3 adjacent with vertices $(1,0), (0, u_2^{-1})$ and $(1, u_2^{-1})$ which have degrees $u_{\acute{s}}$, 3 and 2 respectively.

For the vertex $(0, u_2)$, then

$$D = \sum_{(0,u_2)(1,0)\in E(\Pi_2(R))} deg((0,u_2)).deg((1,0)) + \sum_{(0,u_2)(0,u_2^{-1})\in E(\Pi_2(R))} deg((0,u_2)).deg((0,u_2^{-1}))$$

$$+ \sum_{(0,u_2)(1,u_2^{-1})\in E(\Pi_2(R))} deg\big((0,u_2)\big).deg((1,u_2^{-1})) = 3.u_{\acute{s}}.\left(\frac{u_{\acute{s}}}{2} - 1\right) + 3.3.\left(\frac{u_{\acute{s}}}{2} - 1\right) + 3.2.\left(\frac{u_{\acute{s}}}{2} - 1\right)$$

$$= \frac{3}{2}{u_{\acute{s}}}^2 - 3u_{\acute{s}} + \frac{9}{2}u_{\acute{s}} - 9 + 3u_{\acute{s}} - 6 = \frac{3}{2}{u_{\acute{s}}}^2 + \frac{9}{2}u_{\acute{s}} - 15.$$

The vertex $(1, u_2)$ has degree 2 adjacent with only two vertices (1,0) and $(0, u_2^{-1})$ which have degrees $u_{\acute{s}}$ and 3 respectively.

For the vertex $(1, u_2)$, then

$$E = \sum_{(1,u_2)(1,0)\in E(\Pi_2(R))} deg\big((1,u_2)\big).deg\big((1,0)\big) + \sum_{(1,u_2)(0,u_2^{-1})\in E(\Pi_2(R))} deg\big((1,u_2)\big).deg\big((0,u_2^{-1})\big)$$

$$= 2.u_{\acute{s}}.\left(\frac{u_{\acute{s}}}{2} - 1\right) + 2.3.\left(\frac{u_{\acute{s}}}{2} - 1\right) = {u_{\acute{s}}}^2 - 2u_{\acute{s}} + 3u_{\acute{s}} - 6 = {u_{\acute{s}}}^2 + u_{\acute{s}} - 6$$

Therefore,

$$A + B + C + D + E = \frac{5}{2}{u_{\acute{s}}}^2 - u_{\acute{s}} + 2u_{\acute{s}} + 4 + 2.u_{\acute{s}} + 4 + \frac{3}{2}{u_{\acute{s}}}^2 + \frac{9}{2}u_{\acute{s}} - 15 + {u_{\acute{s}}}^2 + u_{\acute{s}} - 6$$

$$= 5{u_{\acute{s}}}^2 + \frac{17}{2}u_{\acute{s}} - 13 = \frac{10{u_{\acute{s}}}^2 + 17u_{\acute{s}} - 26}{2}.$$

Whence the second Zagreb index

$$M_2(\Pi_2(R)) = \frac{1}{2}[A + B + C + D + E] = \left[\frac{10u_{\acute{S}}^2 + 17u_{\acute{S}} - 26}{4}\right]$$

Theorem 3.4: Let $R \cong S \times \acute{S}$, where S and $\acute{S}$ are local rings of nilpotency 2 and not involution, then the second Zagreb index of R

$$M_2\big(\Pi_2(R)\big) = \frac{1}{8}[8u_S u_{\acute{S}}^2 + 8u_S^2 u_{\acute{S}} - 4u_S^2 - 4u_{\acute{S}}^2 - 18u_S - 18u_{\acute{S}} + 10u_S u_{\acute{S}} + 2u_S^2 u_{\acute{S}}^2]$$

Proof:

Now by the same way of proof theorem (2.6), we get

$$A = \sum_{(1,0)(0,u_2)\in E(\Pi_2(R))} deg\big((1,0)\big).deg\big((0,u_2)\big) + \sum_{(1,0)(0,1)\in E(\Pi_2(R))} deg\big((1,0)\big).deg\big((0,1)\big) + \sum_{(1,0)(1,u_2)\in E(\Pi_2(R))} deg\big((1,0)\big).deg\big((1,u_2)\big)$$

$$= u_{\acute{S}}.(u_S + 1).\left(\frac{u_{\acute{S}}}{2} - 1\right) + u_{\acute{S}}u_S + u_{\acute{S}}2.\frac{u_{\acute{S}}}{2} = \left[\frac{3}{2}u_{\acute{S}}^2 + \frac{u_S u_{\acute{S}}^2}{2} - u_{\acute{S}}\right]$$

$$B = \sum_{(0,1)(u_1,0)\in E(\Pi_2(R))} deg\big((0,1)\big).deg\big((u_1,0)\big) + \sum_{(0,1)(u_1,1)\in E(\Pi_2(R))} deg\big((0,1)\big).deg\big((u_1,1)\big) + \sum_{(0,1)(1,0)\in E(\Pi_2(R))} deg\big((0,1)\big).deg\big((1,0)\big)$$

$$= u_S.(u_{\acute{S}} + 1)\left(\frac{u_S}{2} - 1\right) + u_S.2.\left(\frac{u_S}{2}\right) + u_S.u_{\acute{S}}.1 = \left[\frac{3}{2}u_S^2 + \frac{u_{\acute{S}}.u_S^2}{2} - u_S\right]$$

$$C = deg\big((1,1)\big).deg\big((1,0)\big) + deg\big((1,1)\big).deg\big((0,1)\big) = [2u_{\acute{S}} + 2u_S]$$

$$D = \sum_{(0,u_2)(1,0)\in E(\Pi_2(R))} deg\big((0,u_2)\big).deg\big((1,0)\big) + \sum_{(0,u_2)(u_1,0)\in E(\Pi_2(R))} deg\big((0,u_2)\big).deg\big((u_1,0)\big) + \sum_{(0,u_2)(u_1,u_2^{-1})\in E(\Pi_2(R))} deg\big((0,u_2)\big).deg((u_1,u_2^{-1})) + \sum_{(0,u_2)(0,u_2^{-1})\in E(\Pi_2(R))} deg\big((0,u_2)\big).deg\big((0,u_2^{-1})\big)$$

$$= (u_S + 1).u_{\acute{S}}.\left(\frac{u_{\acute{S}}}{2} - 1\right) + (u_S + 1).(u_{\acute{S}} + 1).\left(\frac{u_S}{2} - 1\right).\left(\frac{u_{\acute{S}}}{2} - 1\right) + (u_S + 1).2.\left(\frac{u_{\acute{S}}}{2} - 1\right).\left(\frac{u_S}{2}\right) + (u_S + 1).(u_S + 1).\left(\frac{u_{\acute{S}}}{2} - 1\right)$$

$$= \left[\frac{u_S.u_{\acute{S}}^2 + 3u_S.u_{\acute{S}} + u_S^2u_{\acute{S}}^2 + 3u_S^2u_{\acute{S}} - 10u_S^2 - 10u_S}{4}\right]$$

$$E = \sum_{(u_1,0)(0,1)\in E(\Pi_2(R))} deg\big((u_1,0)\big).deg\big((0,1)\big) + \sum_{(u_1,0)(0,u_2)\in E(\Pi_2(R))} deg\big((u_1,0)\big).deg\big((0,u_2)\big) + \sum_{(u_1,0)(u_1^{-1},u_2)\in E(\Pi_2(R))} deg\big((u_1,0)\big).deg\big((u_1^{-1},u_2)\big) + \sum_{(u_1,0)(u_1^{-1},0)\in E(\Pi_2(R))} deg\big((u_1,0)\big).deg\big((u_1^{-1},0)\big)$$

$$= (u_{\acute{S}} + 1).u_S.\left(\frac{u_S}{2} - 1\right) + (u_{\acute{S}} + 1).(u_S + 1).\left(\frac{u_{\acute{S}}}{2} - 1\right).\left(\frac{u_S}{2} - 1\right) + (u_{\acute{S}} + 1).2.\left(\frac{u_S}{2} - 1\right).\left(\frac{u_{\acute{S}}}{2}\right) + (u_{\acute{S}} + 1).(u_{\acute{S}} + 1).\left(\frac{u_S}{2} - 1\right)$$

$$= \left[\frac{u_S^2u_{\acute{S}}. + 3u_S.u_{\acute{S}} + u_S^2u_{\acute{S}}^2 + 3u_S.u_{\acute{S}}^2 - 10u_{\acute{S}}^2 - 10u_{\acute{S}}}{4}\right]$$

$$F = \sum_{u_1,u_2\neq 1,(u_1,u_2)(0,u_2^{-1})\in E(\Pi_2(R))} deg\big((u_1,u_2)\big).deg\big((0,u_2^{-1})\big) + \sum_{u_1,u_2\neq 1,(u_1,u_2)(u_1^{-1},0)\in E(\Pi_2(R))} deg\big((u_1,u_2)\big).deg\big((u_1^{-1},0)\big)$$

$$= 2.(u_S + 1).\left(\frac{u_{\acute{S}}}{2} - 1\right).\left(\frac{u_S}{2} - 1\right) + 2.(u_{\acute{S}} + 1).\left(\frac{u_S}{2} - 1\right).\left(\frac{u_{\acute{S}}}{2} - 1\right)$$

$$= 2.(u_S + 1).\left(\frac{u_{\acute{S}} - 2}{2}\right).\left(\frac{u_S - 2}{2}\right) + 2.(u_{\acute{S}} + 1).\left(\frac{u_S - 2}{2}\right).\left(\frac{u_{\acute{S}} - 2}{2}\right)$$

$$= \frac{1}{2}[u_S^2u_{\acute{S}} - 2u_S^2 - u_Su_{\acute{S}} + 2u_S - 2u_{\acute{S}} + 4] + \frac{1}{2}[u_S.u_{\acute{S}}^2 - u_Su_{\acute{S}} - 2u_{\acute{S}}^2 + 2u_{\acute{S}} - 2u_S + 4]$$

$$G = \sum_{u_2\neq 1,(1,u_2)(1,0)\in E(\Pi_2(R))} deg\big((1,u_2)\big).deg\big((1,0)\big) + \sum_{u_2\neq 1,(1,u_2)(0,u_2^{-1})\in E(\Pi_2(R))} deg\big((1,u_2)\big).deg\big((0,u_2^{-1})\big) + \sum_{u_1\neq 1,(u_1,1)(0,1)\in E(\Pi_2(R))} deg\big((u_1,1)\big).deg\big((0,1)\big) + \sum_{u_1\neq 1,(u_1,1)(u_1^{-1},0)\in E(\Pi_2(R))} deg\big((u_1,1)\big).deg\big((u_1^{-1},0)\big)$$

$$= 2.u_{\acute{S}}.\left(\frac{u_{\acute{S}}}{2}-1\right)+2.(u_S+1).\left(\frac{u_{\acute{S}}}{2}-1\right)+2.u_S.\left(\frac{u_S}{2}-1\right)+2.(u_{\acute{S}}+1).\left(\frac{u_S}{2}-1\right)$$

$$= 2.u_{\acute{S}}.\left(\frac{u_{\acute{S}}-2}{2}\right)+2.(u_S+1).\left(\frac{u_{\acute{S}}-2}{2}\right)+2.u_S.\left(\frac{u_S-2}{2}\right)+2.(u_{\acute{S}}+1).\left(\frac{u_S-2}{2}\right)$$

$$= {u_{\acute{S}}}^2-2u_{\acute{S}}+u_Su_{\acute{S}}-2u_S+u_{\acute{S}}-2+{u_S}^2-2u_S+u_Su_{\acute{S}}-2u_{\acute{S}}+u_S-2$$

Therefore

$$A+B+C+D+E+F+G$$

$$\begin{aligned}
&= \frac{3}{2}{u_{\acute{S}}}^2+\frac{u_S{u_{\acute{S}}}^2}{2}-u_{\acute{S}}+\frac{3}{2}{u_S}^2+\frac{u_{\acute{S}}{u_S}^2}{2}-u_S+2u_{\acute{S}}+2u_S \\
&\quad +\frac{u_S{u_{\acute{S}}}^2+3u_Su_{\acute{S}}+{u_S}^2{u_{\acute{S}}}^2+3{u_S}^2u_{\acute{S}}-10{u_S}^2-10u_S}{4} \\
&\quad +\frac{{u_S}^2u_{\acute{S}}+3u_Su_{\acute{S}}+{u_S}^2{u_{\acute{S}}}^2+3u_S{u_{\acute{S}}}^2-10{u_{\acute{S}}}^2-10u_{\acute{S}}}{4} \\
&\quad +\frac{1}{2}[{u_S}^2u_{\acute{S}}-2{u_S}^2-u_Su_{\acute{S}}+2u_S-2u_{\acute{S}}+4] \\
&\quad +\frac{1}{2}[u_S{u_{\acute{S}}}^2-u_Su_{\acute{S}}-2{u_{\acute{S}}}^2+2u_{\acute{S}}-2u_S+4]+{u_{\acute{S}}}^2-2u_{\acute{S}}+u_Su_{\acute{S}}-2u_S+u_{\acute{S}}-2 \\
&\quad +{u_S}^2-2u_S+u_Su_{\acute{S}}-2u_{\acute{S}}+u_S-2
\end{aligned}$$

$$= \frac{1}{4}[8u_S{u_{\acute{S}}}^2+8{u_S}^2u_{\acute{S}}-4{u_S}^2-4{u_{\acute{S}}}^2-18u_S-18u_{\acute{S}}+10u_S.u_{\acute{S}}+2{u_S}^2{u_{\acute{S}}}^2]$$

Whence the second Zagreb index

$$M_2(\Pi_2(R)) = \tfrac{1}{2}[A+B+C+D+E+F+G]$$

$$= \frac{1}{8}[8u_S{u_{\acute{S}}}^2+8{u_S}^2u_{\acute{S}}-4{u_S}^2-4{u_{\acute{S}}}^2-18u_S-18u_{\acute{S}}+10u_Su_{\acute{S}}+2{u_S}^2{u_{\acute{S}}}^2]$$

Example 3.5: **(a)** To find first and second Zagreb indices of a graph $\boldsymbol{Z_4 \times Z_9}$

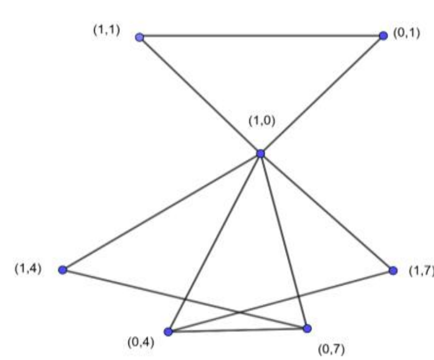

Figure 1. $\Pi_2(\boldsymbol{Z_4 \times Z_9})$

$$M_1(\Pi_2(R)) = \sum_{uv\in E(\Pi_2(R))} (deg(u) + deg(v)) = (2+2) + 4(2+6) + 2(3+6) + 2(2+3) + (3+3) = 70$$

$$M_2(\Pi_2(R)) = \sum_{uv\in E(\Pi_2(R))} (deg(u) . deg(v)) = (2.2) + 4(2.6) + 2(3.6) + 2(2.3) + (3.3) = 109$$

By theorem $M_1(\Pi_2(R)) = \frac{2u_{\acute{S}}{}^2+13u_{\acute{S}}-10}{2} = \frac{72+78-10}{2} = 70$

$$M_2(\Pi_2(R)) = \frac{10u_{\acute{S}}{}^2 + 17u_{\acute{S}} - 26}{4} = \frac{360+102-26}{4} = 109$$

(b) To find first and second Zagreb indices of a graph $\boldsymbol{Z_9} \times \boldsymbol{Z_9}$

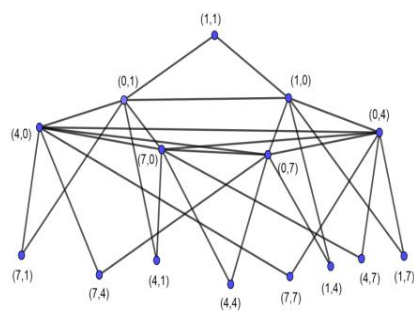

Figure 2. $\Pi_2(\boldsymbol{Z_9} \times \boldsymbol{Z_9})$

$$M_1(\Pi_2(R)) = \sum_{uv\in E(\Pi_2(R))} (deg(u) + deg(v))$$
$$= (6+6) + 6(2+6) + 4(6+7) + 6(7+7) + 12(7+2) = 304$$

$M_2(\Pi_2(R)) = \sum_{uv\in E(\Pi_2(R))}(deg(u) . deg(v)) = (6.6) + 6(2.6) + 4(6.7) + 6(7.7) + 12(7.2) = 738$

By theorem

$$M_1(\Pi_2(R)) = \frac{6(u_S.u_{\acute{S}}) + (u_S.u_{\acute{S}}{}^2 + u_{\acute{S}}.u_S{}^2) - 3(u_S + u_{\acute{S}}) - 4}{2} = \frac{608}{2} = 304$$

$M_2(\Pi_2(R)) = \frac{1}{8}[8u_Su_{\acute{S}}{}^2 + 8u_S{}^2u_{\acute{S}} - 4u_S{}^2 - 4u_{\acute{S}}{}^2 - 18u_S - 18u_{\acute{S}} + 10u_S.u_{\acute{S}} + 2u_S{}^2u_{\acute{S}}{}^2] = \frac{5904}{8} = 738$

4. Zagreb co-indices of 2-idempotent divisor graph

In this section we present computational results for the first and second Zagreb co-indices of pairs of local rings, we examining both of the cases where exactly one is involution while the other is not, and the case where neither ring admits such isomorphic, also we supported the findings by explain example.

Theorem 4.1 : Let S to be involution and $\acute{S}$ are a local rings with nilpotency 2 and not isomorphic with S, then the first Zagreb co-index is $\overline{M_1}(\Pi_2(R)) = \frac{1}{2}(25u_{\acute{S}} - 50)$, where $R \cong S \times \acute{S}$

Proof:

Clearly (1,0) adjacent with every other vertices. By lemma (2.8)

We have the vertex (0,1) which has degree 2 does not adjacent with two vertices $(0, u_2)$ and $(1, u_2), u_2 \neq 1$ which have degrees 3 and 2 respectively.

The vertex (0,1), then

$$A = \sum_{u_2\neq 1,(0,1)(0,u_2)\notin E(\Pi_2(R))} \big(deg\big((0,1)\big) + deg\big((0,u_2)\big)\big) + \sum_{(0,1)(1,u_2)\notin E(\Pi_2(R))} \big(deg\big((0,1)\big) + deg\big((1,u_2)\big)\big)$$

$$= (2+3)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) = 5\left(\frac{u_{\acute{S}} - 2}{2}\right) + 4\left(\frac{u_{\acute{S}} - 2}{2}\right) = \frac{1}{2}(5u_{\acute{S}} - 10 + 4u_{\acute{S}} - 8) = \frac{1}{2}(9u_{\acute{S}} - 18)$$

The vertex (1,1) which has degree 2 does not adjacent with two vertices $(0, u_2)$ and $(1, u_2),, u_2 \neq 1$ which have degrees 3 and 2 respectively

The vertex (1,1), then

$$B = \sum_{u_2\neq 1,(1,1)(0,u_2)\notin E(\Pi_2(R))} \big(de\ g\big((1,1)\big) + de\ g\big((0,u_2)\big)\big) + \sum_{u_2\neq 1,(1,1)(1,u_2)\notin E(\Pi_2(R))} \big(deg\big((1,1)\big) + deg\big((1,u_2)\big)\big)$$

$$= (2+3)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) = 5\left(\frac{u_{\acute{S}}-2}{2}\right) + 4\left(\frac{u_{\acute{S}}-2}{2}\right) = \frac{1}{2}(5u_{\acute{S}} - 10 + 4u_{\acute{S}} - 8) = \frac{1}{2}(9u_{\acute{S}} - 18)$$

The vertex $(0, u_2)$ which has degree 3 does not adjacent with vertices $(1, u_2), (0,1)$ and $(1,1)$, where $u_2 \neq 1$ which have degrees 2,2 and 2 respectively

The vertex $(0, u_2)$, then

$$C = \sum_{u_2 \neq 1,(0,u_2)(1,u_2)\notin E(\Pi_2(R))} \big(deg\big((0,u_2)\big) + deg\big((1,u_2)\big)\big) + \sum_{u_2 \neq 1,(0,u_2)(0,1)\notin E(\Pi_2(R))} \big(deg\big((0,u_2)\big) + deg((0,1))\ \big) + \sum_{u_2 \neq 1,(0,u_2)(1,1)\notin E(\Pi_2(R))} \big(deg\big((0,u_2)\big) + deg((1,1))\big)$$

$$= (3+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (3+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (3+2)\left(\frac{u_{\acute{S}}}{2} - 1\right)$$

$$= 5\left(\frac{u_{\acute{S}} - 2}{2}\right) + 5\left(\frac{u_{\acute{S}} - 2}{2}\right) + 5\left(\frac{u_{\acute{S}} - 2}{2}\right)$$

$$= 3\left(5\left(\frac{u_{\acute{S}} - 2}{2}\right)\right) = \frac{1}{2}(15u_{\acute{S}} - 30)$$

The vertex $(1,u_2), u_2 \neq 1$ which has degree 2 does not adjacent with vertices $(0,1)$, $(1,1)$, $(1,u_2^{-1})$ and $(0,u_2)$ which have degrees 2, 2, 2 and 3 respectively

The vertex $(1,u_2)$, then

$$D = \sum_{u_2 \neq 1,(1,u_2)(0,1)\notin E(\Pi_2(R))} (deg((1,u_2)) + deg\big((0,1)\big)) + \sum_{u_2 \neq 1,(1,u_2)(1,1)\notin E(\Pi_2(R))} (deg((1,u_2)) + deg((1,1))) + \sum_{u_2 \neq 1,(1,u_2)(1,u_2^{-1})\notin E(\Pi_2(R))} (deg\big((1,u_2)\big) + deg\big((1,u_2^{-1})\big)) + \sum_{u_2 \neq 1,(1,u_2)(0,u_2)\notin E(\Pi_2(R))} (deg\big((1,u_2)\big) + deg\big((0,u_2)\big))$$

$$= (2+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2+2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2+3)\left(\frac{u_{\acute{S}}}{2} - 1\right)$$

$$= 3\left(4\left(\frac{u_{\acute{S}} - 2}{2}\right)\right) + (2+3)\left(\frac{u_{\acute{S}} - 2}{2}\right) = \frac{1}{2}(17u_{\acute{S}} - 34)$$

Therefore

$$A + B + C + D = \frac{1}{2}(9u_{\acute{S}} - 18) + \frac{1}{2}(9u_{\acute{S}} - 18) + \frac{1}{2}(15u_{\acute{S}} - 30) + \frac{1}{2}(17u_{\acute{S}} - 34)$$
$$= \frac{1}{2}(50u_{\acute{S}} - 100)\ ) = 25u_{\acute{S}} - 50$$

Whence the first Zagreb co-index

$$\overline{M_1}(\Pi_2(R)) = \frac{1}{2}(A + B + C + D) = \frac{1}{2}(25u_{\acute{S}} - 50)$$

Theorem 4.2: Let $R \cong S \times \acute{S}$, where S and $\acute{S}$ are local rings of nilpotency 2 and not involution, then the first Zagreb co-index of R

$$\overline{M_1}(\Pi_2(R)) = \frac{1}{8}[u_S{}^2 . u_{\acute{S}}{}^2 + 44u_S u_{\acute{S}} - 44u_S - 44u_{\acute{S}} + 6u_S{}^2 u_{\acute{S}} + 6u_S u_{\acute{S}}{}^2 - 12u_S{}^2 - 12u_{\acute{S}}{}^2 - 16]$$

Proof:

The vertex (1,0) which has degree $u_{\acute{S}}$ does not adjacent with vertices $(u_1, 0)$ and $(u_1, u_2), u_1 \neq 1$ which have degrees $u_{\acute{S}} + 1$ and 2 respectively.

For the vertex (1,0), then

$$A = \sum_{u_1 \neq 1, (1,0)(u_1,0) \notin E(\Pi_2(R))} \big(deg((1,0)) + deg((u_1, 0))\big) + \sum_{u_1 \neq 1, (1,0)(u_1,u_2) \notin E(\Pi_2(R))} \big(deg((1,0)) + deg((u_1, u_2))\big)$$

$$= (u_{\acute{S}} + u_{\acute{S}} + 1).\left(\frac{u_S}{2} - 1\right) + (u_{\acute{S}} + 2).\left(\frac{u_S}{2} - 1\right)\left(\frac{u_{\acute{S}}}{2}\right)$$

$$= (2u_{\acute{S}} + 1)\left(\frac{u_S - 2}{2}\right) + (u_{\acute{S}} + 2)\left(\frac{u_S - 2}{2}\right)\left(\frac{u_{\acute{S}}}{2}\right)$$

$$= \frac{1}{2}(2u_S u_{\acute{S}} - 4u_{\acute{S}} + u_S - 2) + \frac{1}{4}(u_{\acute{S}} + 2)(u_S . u_{\acute{S}} - 2u_{\acute{S}})$$

$$= \frac{1}{4}(4u_S u_{\acute{S}} - 8u_{\acute{S}} + 2u_S - 4) + \frac{1}{4}(u_S u_{\acute{S}}{}^2 - 2u_{\acute{S}}{}^2 + 2u_S u_{\acute{S}} - 4u_{\acute{S}})$$

$$= \frac{1}{4}(6u_S u_{\acute{S}} - 12u_{\acute{S}} + u_S . u_{\acute{S}}{}^2 - 2u_{\acute{S}}{}^2 + 2u_S - 4)$$

The vertex (0,1) has degree u_S does not adjacent with vertices $(0, u_2)$ and $(u_1, u_2), u_2 \neq 1$ which have degrees $u_S + 1$, and 2 respectively.

For the vertex (0,1), then

$$B = \sum_{u_2 \neq 1, (0,1)(0,u_2) \notin E(\Pi_2(R))} \big(deg((0,1)) + deg((0, u_2))\big) + \sum_{u_2 \neq 1, (0,1)(u_1,u_2) \notin E(\Pi_2(R))} \big(deg((0,1)) + deg((u_1, u_2))\big)$$

$$= (u_S + u_S + 1)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (u_S + 2).\left(\frac{u_{\acute{S}}}{2} - 1\right).\left(\frac{u_S}{2}\right)$$

$$= (2u_S + 1)\left(\frac{u_{\acute{S}} - 2}{2}\right) + \frac{1}{4}(u_S + 2)(u_{\acute{S}} - 2)(u_S)$$

$$= \frac{1}{2}(2u_S u_{\acute{S}} - 4u_S + u_{\acute{S}} - 2) + \frac{1}{4}(u_S + 2)(u_S u_{\acute{S}} - 2u_S)$$

$$= \frac{1}{4}(6u_S u_{\acute{S}} - 12u_S + u_S{}^2 u_{\acute{S}} - 2u_S{}^2 + 2u_{\acute{S}} - 4)$$

The vertex (1,1) has degree 2 does not adjacent with vertices $(0,u_2), u_2 \neq 1$, $(u_1,0), u_1 \neq 1$ and (u_1,u_2) which have degrees $u_S + 1$, $u_{\acute{S}} + 1$ and 2 respectively.

For the vertex (1,1), then

$$C = \sum_{u_2 \neq 1,(1,1)(0,u_2) \notin E(\Pi_2(R))} \left(deg\left((1,1)\right) + deg\left((0,u_2)\right)\right) + \sum_{u_1 \neq 1,(1,1)(u_1,0) \notin E(\Pi_2(R))} \left(deg\left((1,1)\right) + deg\left((u_1,0)\right)\right) + \sum_{u_1,u_2 \neq 1,(1,1)(u_1,u_2) \notin E(\Pi_2(R))} \left(deg\left((1,1)\right) + deg\left((u_1,u_2)\right)\right)$$

$$= (2 + u_S + 1)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2 + u_{\acute{S}} + 1)\left(\frac{u_S}{2} - 1\right) + (2 + 2).\left(\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right) - 1\right)$$

$$= (u_S + 3)\left(\frac{u_{\acute{S}} - 2}{2}\right) + (u_{\acute{S}} + 3)\left(\frac{u_S - 2}{2}\right) + 4\left(\frac{u_S.u_{\acute{S}}}{4} - 1\right)$$

$$= \frac{1}{2}(u_S.u_{\acute{S}} - 2u_S + 3u_{\acute{S}} - 6) + \frac{1}{2}(u_S.u_{\acute{S}} - 2u_{\acute{S}} + 3u_S - 6) + (u_S.u_{\acute{S}} - 4)$$

$$= \frac{1}{2}(4u_S.u_{\acute{S}} + u_S + u_{\acute{S}} - 20) = \frac{1}{4}(8u_S.u_{\acute{S}} + 2u_S + 2u_{\acute{S}} - 40)$$

The vertex $(0,u_2)$ has degree $u_S + 1$ does not adjacent with vertices $(0,1), (u_1,u_2)$ and $(u_1,1)$ which have degrees u_S , 2 and 2 respectively.

For the vertex $(0,u_2)$, then

$$D = \sum_{u_2 \neq 1,(0,u_2)(0,1) \notin E(\Pi_2(R))} \left(deg\left((0,u_2)\right) + deg\left((0,1)\right)\right) + \sum_{u_2 \neq 1,(0,u_2)(u_1,u_2) \notin E(\Pi_2(R))} \left(deg\left((0,u_2)\right) + deg\left((u_1,u_2)\right)\right) + \sum_{u_2 \neq 1,(0,u_2)(u_1,1) \notin E(\Pi_2(R))} \left(deg\left((0,u_2)\right) + deg\left((u_1,1)\right)\right)$$

$$= (u_S + 1 + u_S).\left(\frac{u_{\acute{S}}}{2} - 1\right) + (u_S + 1 + 2).\left(\frac{u_{\acute{S}}}{2} - 1\right).\left(\frac{u_S}{2}\right) + (u_S + 1 + 2).\left(\frac{u_{\acute{S}}}{2} - 1\right).\left(\frac{u_S}{2}\right)$$

$$= (2u_S + 1)\left(\frac{u_{\acute{S}} - 2}{2}\right) + (u_S + 3).\left(\frac{u_{\acute{S}} - 2}{2}\right).\left(\frac{u_S}{2}\right) + (u_S + 3).\left(\frac{u_{\acute{S}} - 2}{2}\right).\left(\frac{u_S}{2}\right)$$

$$= \frac{1}{2}(2u_S u_{\acute{S}} - 4u_S + u_{\acute{S}} - 2) + \frac{1}{4}({u_S}^2 u_{\acute{S}} - 2{u_S}^2 + 3u_S u_{\acute{S}} - 6u_S) + \frac{1}{4}({u_S}^2 u_{\acute{S}} - 2{u_S}^2 + 3u_S u_{\acute{S}} - 6u_S)$$

$$= \frac{1}{4}(10u_S u_{\acute{S}} - 20u_S + 2u_{\acute{S}} + 2{u_S}^2 u_{\acute{S}} - 4{u_S}^2 - 4)$$

The vertex $(u_1,0)$ has degree $u_{\acute{S}}+1$ does not adjacent with vertices $(1,0),(1,u_2)$ and (u_1,u_2) which have degrees $u_{\acute{S}}$, 2 and 2 respectively.

For the vertex $(u_1,0)$, then

$$E=\sum_{(u_1,0)(1,0)\notin E(\Pi_2(R))}\Big(deg\big((u_1,0)\big)+deg\big((1,0)\big)\Big)+\sum_{(u_1,0)(1,u_2)\notin E(\Pi_2(R))}\Big(deg\big((u_1,0)\big)+deg\big((1,u_2)\big)\Big)+\sum_{(u_1,0)(u_1,u_2)\notin E(\Pi_2(R))}\Big(deg\big((u_1,0)\big)+deg\big((u_1,u_2)\big)\Big)$$

$$=(u_{\acute{S}}+1+u_{\acute{S}}).\left(\frac{u_S}{2}-1\right)+(u_{\acute{S}}+1+2).\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}\right)+(u_{\acute{S}}+1+2).\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}\right)$$

$$=(2u_{\acute{S}}+1).\left(\frac{u_S-2}{2}\right)+\frac{1}{4}(\,u_{\acute{S}}+3)(u_Su_{\acute{S}}-2u_{\acute{S}})+\frac{1}{4}(\,u_{\acute{S}}+3)(u_Su_{\acute{S}}-2u_{\acute{S}})$$

$$=\frac{1}{2}(2u_Su_{\acute{S}}-4u_{\acute{S}}+u_S-2)+\frac{1}{4}(u_Su_{\acute{S}}{}^2-2u_{\acute{S}}{}^2+3u_Su_{\acute{S}}-6u_{\acute{S}}+\frac{1}{4}(u_Su_{\acute{S}}{}^2-2u_{\acute{S}}{}^2+3u_Su_{\acute{S}}-6u_{\acute{S}}$$

$$=\frac{1}{4}(10u_Su_{\acute{S}}-20u_{\acute{S}}+2u_S+2u_Su_{\acute{S}}{}^2-4u_{\acute{S}}{}^2-4).$$

The vertex (u_1,u_2) has degree 2 does not adjacent with vertices $(1,0),(0,1),(u_1,u_2)$, $(0,u_2)$ and $(u_1,0)$ which have degrees $u_{\acute{S}}$, u_S , 2 , u_S+1 and $u_{\acute{S}}+1, u_1,u_2\neq 1$ respectively.

For the vertex (u_1,u_2), then

$$F=\sum_{u_1,u_2\neq 1,(u_1,u_2)(1,0)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big)+deg\big((1,0)\big)\Big)+\sum_{u_1,u_2\neq 1,(u_1,u_2)(0,1)\notin E(\Pi_2(R))}\Big(deg((u_1,u_2))+deg\big((0,1)\big)\Big)+\sum_{u_1,u_2\neq 1,(u_1,u_2)(u_1,u_2)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big)+deg\big((u_1,u_2)\big)\Big)+\sum_{u_1,u_2\neq 1,(u_1,u_2)(0,u_2)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big)+deg\big((0,u_2)\big)\Big)+\sum_{u_1,u_2\neq 1,(u_1,u_2)(u_1,0)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big)+deg\big((u_1,0)\big)\Big)$$

$$=(2+u_{\acute{S}})\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}\right)+(2+u_S)\left(\frac{u_{\acute{S}}}{2}-1\right).\left(\frac{u_S}{2}\right)+(2+2).\left(\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right)-1\right).\left(\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right)-1\right)+(2+u_S+1)\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}-1\right)+(2+u_{\acute{S}}+1)\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}-1\right)$$

$$= \frac{1}{4}(u_{\acute{S}} + 2)(u_{S} - 2)(u_{\acute{S}}) + \frac{1}{4}(u_{S} + 2)(u_{\acute{S}} - 2)(u_{S}) + 4\left(\frac{u_{S}u_{\acute{S}} - 4}{4}\right).\left(\frac{u_{S}u_{\acute{S}} - 4}{4}\right) + (u_{S} + 3)\left(\frac{u_{S} - 2}{2}\right).\left(\frac{u_{\acute{S}} - 2}{2}\right) + (u_{\acute{S}} + 3)\left(\frac{u_{S} - 2}{2}\right).\left(\frac{u_{\acute{S}} - 2}{2}\right)$$

$$= \frac{1}{4}(u_{\acute{S}}^2 + 2u_{\acute{S}})(u_{S} - 2) + \frac{1}{4}(u_{S} + 2)(u_{S}u_{\acute{S}} - 2u_{S}) + \frac{1}{4}(u_{S}^2u_{\acute{S}}^2 - 4u_{S}u_{\acute{S}} - 4u_{S}u_{\acute{S}} + 16) + \frac{1}{4}(u_{S} + 3)(u_{S} - 2)(u_{\acute{S}} - 2) + \frac{1}{4}(u_{\acute{S}} + 3)(u_{S} - 2)(u_{\acute{S}} - 2)$$

$$= \frac{1}{4}[2u_{S}u_{\acute{S}}^2 - 4u_{\acute{S}}^2 + 2u_{S}^2u_{\acute{S}} - 12u_{S} - 2u_{S}u_{\acute{S}} - 4u_{S}^2 - 12u_{\acute{S}} + u_{S}^2u_{\acute{S}}^2 + 40]$$

The vertex $(1, u_2), u_2 \neq 1$ has degree 2 does not adjacent with vertices $(0, u_2)$ and $(u_1, 0)$ which have degrees $u_{S} + 1$ and $u_{\acute{S}} + 1$ and the vertex $(u_1, 1), u_1 \neq 1$ which has degree 2 does not adjacent with vertices $(u_1, 0)$ and $(0, u_2)$ which have $u_{\acute{S}} + 1$ and $u_{S} + 1$ respectively.

For the vertices $(1, u_2)$ and $(u_1, 1)$, then

$$G = \sum_{u_2 \neq 1, (1,u_2)(0,u_2) \notin E(\Pi_2(R))} \big(deg((1, u_2)) + deg((0, u_2))\big) + \sum_{u_2 \neq 1, (1,u_2)(u_1,0) \notin E(\Pi_2(R))} \big(deg((1, u_2)) + deg((u_1, 0))\big) + \sum_{u_1 \neq 1, (u_1,1)(u_1,0) \notin E(\Pi_2(R))} \big(deg((u_1, 1)) + deg((u_1, 0))\big) + \sum_{u_1 \neq 1, (u_1,1)(0,u_2) \notin E(\Pi_2(R))} \big(deg((u_1, 1)) + deg((0, u_2))\big)$$

$$= (2 + u_{S} + 1).\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2 + u_{\acute{S}} + 1).\left(\frac{u_{S}}{2} - 1\right)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2 + u_{\acute{S}} + 1).\left(\frac{u_{S}}{2} - 1\right) + (2 + u_{S} + 1).\left(\frac{u_{\acute{S}}}{2} - 1\right)\left(\frac{u_{S}}{2} - 1\right)$$

$$= \frac{1}{2}(u_{S} + 3).(u_{\acute{S}} - 2) + \frac{1}{4}(u_{\acute{S}} + 3).(u_{S} - 2).(u_{\acute{S}} - 2) + \frac{1}{2}(u_{\acute{S}} + 3).(u_{S} - 2) + \frac{1}{4}(u_{S} + 3).(u_{\acute{S}} - 2).(u_{S} - 2)$$

$$= \frac{1}{2}(u_{S}u_{\acute{S}} - 2u_{S} + 3u_{\acute{S}} - 6) + \frac{1}{4}(u_{\acute{S}} + 3)(u_{S}u_{\acute{S}} - 2u_{S} - 2u_{\acute{S}} + 4 + \frac{1}{2}(u_{S}u_{\acute{S}} - 2u_{\acute{S}} + 3u_{S} - 6) + \frac{1}{4}(u_{S} + 3)(u_{S}|U(S_2)| - 2u_{S} - 2|U(S_2)| + 4)$$

$$= \frac{1}{4}[2u_{S}u_{\acute{S}} - 4u_{S} + 6u_{\acute{S}} - 12 + u_{S}u_{\acute{S}}^2 - 2u_{S}u_{\acute{S}} - 2u_{\acute{S}}^2 + 4u_{\acute{S}} + 3u_{S}u_{\acute{S}} - 6u_{S} - 6u_{\acute{S}} + 12 + 2u_{S}u_{\acute{S}} - 4u_{S} + 6u_{\acute{S}} - 12 + u_{S}^2u_{\acute{S}} - 2u_{S}u_{\acute{S}} - 2u_{S}^2 + 4u_{S} + 3u_{S}u_{\acute{S}} - 6u_{S} - 6u_{\acute{S}} + 12]$$

$$= \frac{1}{4}[6u_{S}u_{\acute{S}} - 6u_{S} - 6u_{\acute{S}} - 2u_{S}^2 - 2u_{\acute{S}}^2 + u_{S}^2u_{\acute{S}} + u_{S}u_{\acute{S}}^2]$$

Therefore,

$$
\begin{aligned}
A+B+C+D+E+F+G \\
&= \frac{1}{4}(6u_S u_{\acute{S}} - 12u_{\acute{S}} + u_S u_{\acute{S}}^2 - 2u_{\acute{S}}^2 + 2u_S - 4) \\
&+ \frac{1}{4}(6u_S u_{\acute{S}} - 12u_S + u_S^2 u_{\acute{S}} - 2u_S^2 + 2u_{\acute{S}} - 4) + \frac{1}{4}(8u_S . u_{\acute{S}} + 2u_S + 2u_{\acute{S}} - 40) \\
&+ \frac{1}{4}(10u_S u_{\acute{S}} - 20u_S + 2u_{\acute{S}} + 2u_S^2 - 4u_S^2 - 4 \\
&+ \frac{1}{4}(10u_S u_{\acute{S}} - 20u_{\acute{S}} + 2u_S + 2u_S u_S^2 - 4u_{\acute{S}}^2 - 4) \\
&+ \frac{1}{4}[2u_S u_{\acute{S}}^2 - 4u_{\acute{S}}^2 + 2u_S^2 u_{\acute{S}} - 12u_S - 2u_S u_{\acute{S}} - 4u_S^2 - 12u_{\acute{S}} + u_S^2 u_{\acute{S}}^2 + 40] \\
&+ \frac{1}{4}[6u_S u_{\acute{S}} - 6u_S - 6u_{\acute{S}} - 2u_S^2 - 2u_{\acute{S}}^2 + u_S^2 u_{\acute{S}} + u_S u_{\acute{S}}^2]
\end{aligned}
$$

$$
= \frac{1}{4}[u_S^2 u_{\acute{S}}^2 + 44u_S u_{\acute{S}} - 44u_S - 44u_{\acute{S}} + 6u_S^2 u_{\acute{S}} + 6u_S u_{\acute{S}}^2 - 12u_S^2 - 12u_{\acute{S}}^2 - 16]
$$

Whence the first Zagreb co-index

$$
\begin{aligned}
\overline{M_1}(\Pi_2(R)) &= \frac{1}{2}[A+B+C+D+E+F+G] \\
&= \frac{1}{8}[u_S^2 u_{\acute{S}}^2 + 44u_S u_{\acute{S}} - 44u_S - 44u_{\acute{S}} + 6u_S^2 u_{\acute{S}} + 6u_S u_{\acute{S}}^2 - 12u_S^2 - 12u_{\acute{S}}^2 - 16]
\end{aligned}
$$

Theorem 4.3 : Let S to be involution and $\acute{S}$ are a local rings with nilpotency 2 and not isomorphic with S, then the second Zagreb co-index is

$\overline{M_2}(\Pi_2(R)) = 14u_{\acute{S}} - 28$, where $R \cong S \times \acute{S}$

Proof:

By lemma (2.8)

We have the vertex (0,1) which has degree 2 does not adjacent with two vertices $(0, u_2)$ and $(1, u_2)$ which have degrees 3 and 2 respectively.

The vertex (0,1), then

$$
\begin{aligned}
A = \sum_{u_2 \neq 1, (0,1)(0,u_2) \notin E(\Pi_2(R))} \big(deg((0,1)).deg((0,u_2))\big) \\
+ \sum_{u_2 \neq 1, (0,1)(1,u_2) \notin E(\Pi_2(R))} \big(deg((0,1)).deg((1,u_2))\big)
\end{aligned}
$$

$$
= (2.3)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2.2)\left(\frac{u_{\acute{S}}}{2} - 1\right) = 6\left(\frac{u_{\acute{S}} - 2}{2}\right) + 4\left(\frac{u_{\acute{S}} - 2}{2}\right)
$$

$$
= 3(u_{\acute{S}} - 2) + 2(u_{\acute{S}} - 2) = 3u_{\acute{S}} - 6 + 2u_{\acute{S}} - 4 = 5u_{\acute{S}} - 10
$$

The vertex (1,1) which has degree 2 does not adjacent with two vertices $(0,u_2)$ and $(1,u_2)$ which have degrees 3 and 2 respectively

The vertex (1,1), then

$$B = \sum_{u_2 \neq 1 (1,1)(0,u_2) \notin E(\Pi_2(R))} \left(deg\big((1,1)\big) . deg\big((0,u_2)\big)\right) + \sum_{u_2 \neq 1,(1,1)(1,u_2) \notin E(\Pi_2(R))} \left(deg\big((1,1)\big) . deg\big((1,u_2)\big)\right)$$

$$= (2.3)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (2.2)\left(\frac{u_{\acute{S}}}{2} - 1\right) = 6\left(\frac{u_{\acute{S}}-2}{2}\right) + 4\left(\frac{u_{\acute{S}}-2}{2}\right)$$

$$= 3(u_{\acute{S}} - 2) + 2(u_{\acute{S}} - 2) = 3u_{\acute{S}} - 6 + 2u_{\acute{S}} - 4 = 5u_{\acute{S}} - 10$$

The vertex $(0,u_2)$ which has degree 3 does not adjacent with vertices $(1,u_2)$, (0,1) and (1,1) which have degrees 2 , 2 and 2 respectively

The vertex $(0,u_2)$, then

$$C = \sum_{u_2 \neq 1,(0,u_2)(1,u_2) \notin E(\Pi_2(R))} \left(deg\big((0,u_2)\big) . deg((1,u_2))\right) + \sum_{u_2 \neq 1,(0,u_2)(0,1) \notin E(\Pi_2(R))} \left(deg((0,u_2)). deg\big((0,1)\big)\right) + \sum_{u_2 \neq 1,(0,u_2)(1,1) \notin E(\Pi_2(R))} \left(deg\big((0,u_2)\big) . deg\big((1,1)\big)\right)$$

$$= (3.2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (3.2)\left(\frac{u_{\acute{S}}}{2} - 1\right) + (3.2)\left(\frac{u_{\acute{S}}}{2} - 1\right) = 6\left(\frac{u_{\acute{S}} - 2}{2}\right) + 6\left(\frac{u_{\acute{S}} - 2}{2}\right) + 6\left(\frac{u_{\acute{S}} - 2}{2}\right)$$

$$= 3\left(6\left(\frac{u_{\acute{S}} - 2}{2}\right)\right) = 9u_{\acute{S}} - 18$$

The vertex $(1,u_2)$ which has degree 2 does not adjacent with vertices $(0,1), (1,1), (1,u_2^{-1})$ and $(0,u_2)$ which have degrees 2 , 2, 2 and 3 respectively

The vertex $(1,u_2)$

$$D = \sum_{u_2 \neq 1,(1,u_2)(0,1) \notin E(\Pi_2(R))} \left(deg((1,u_2)) . deg\big((0,1)\big)\right) + \sum_{u_2 \neq 1,(1,u_2)(1,1) \notin E(\Pi_2(R))} \left(deg\big((1,u_2)\big) . deg\big((1,1)\big)\right) + \sum_{u_2 \neq 1,(1,u_2)(1,u_2) \notin E(\Pi_2(R))} \left(deg\big((1,u_2)\big) . deg\big((1,u_2)\big)\right)$$

$$+ \sum_{u_2 \neq 1,(1,u_2)(0,u_2^{-1}) \notin E(\Pi_2(R))} \left(deg\big((1,u_2)\big) . deg\big((0,u_2^{-1})\big)\right)$$

$$= (2.2)\left(\frac{u_{\acute{S}}}{2}-1\right)+(2.2)\left(\frac{u_{\acute{S}}}{2}-1\right)+(2.2)\left(\frac{u_{\acute{S}}}{2}-1\right)+(2.3)\left(\frac{u_{\acute{S}}}{2}-1\right)$$

$$= 3\left(4\left(\frac{u_{\acute{S}}-2}{2}\right)\right)+6\left(\frac{u_{\acute{S}}-2}{2}\right) = 9u_{\acute{S}}-18$$

Therefore,

$$A+B+C+D = 5u_{\acute{S}}-10+5u_{\acute{S}}-10+9u_{\acute{S}}-18+9u_{\acute{S}}-18 = 28u_{\acute{S}}-56$$

Whence the second Zagreb co-index

$$\overline{M_2}(\Pi_2(R)) = \frac{1}{2}[A+B+C+D] = \frac{1}{2}(28u_{\acute{S}}-56) = 14u_{\acute{S}}-28$$

Theorem 4.4: Let $R \cong S \times \acute{S}$, where S and $\acute{S}$ are local rings of nilpotency 2 and not involution, then the second Zagreb index of R

$$\overline{M_2}\big(\Pi_2(R)\big) = \frac{1}{8}[16u_S u_{\acute{S}}{}^2 - 32u_{\acute{S}}{}^2 - 24u_{\acute{S}} + 20u_S u_{\acute{S}} + 16u_S{}^2 u_{\acute{S}} - 32u_S{}^2 - 24u_S + u_S{}^2 u_{\acute{S}}{}^2]$$

Proof:

Now by the same way of proof theorem (4.2) we get

$$A = \sum_{u_1\neq 1,(1,0)(u_1,0)\notin E(\Pi_2(R))} \big(deg\big((1,0)\big).deg\big((u_1,0)\big)\big) + \sum_{u_1\neq 1,(1,0)(u_1,u_2)\notin E(\Pi_2(R))} \big(deg\big((1,0)\big).deg\big((u_1,u_2)\big)\big)$$

$$= (u_{\acute{S}}).(u_{\acute{S}}+1).\left(\frac{u_S}{2}-1\right)+(u_{\acute{S}}.2).\left(\frac{u_S}{2}-1\right)\left(\frac{u_{\acute{S}}}{2}\right) = \frac{1}{2}(u_{\acute{S}}{}^2+u_{\acute{S}})(u_S-2)+\frac{1}{2}(u_{\acute{S}}{}^2)(u_S-2)$$

$$= \frac{1}{2}(u_S u_{\acute{S}}{}^2 - 2u_{\acute{S}}{}^2 + u_S u_{\acute{S}} - 2u_{\acute{S}}) + \frac{1}{2}\ (u_S u_{\acute{S}}{}^2 - 2u_{\acute{S}}^2)$$

$$= \frac{1}{2}(2u_S u_{\acute{S}}{}^2 - 4u_{\acute{S}}{}^2 - 2u_{\acute{S}} + u_S u_{\acute{S}}) = \frac{1}{4}(4u_S u_{\acute{S}}{}^2 - 8u_{\acute{S}}{}^2 - 4u_{\acute{S}} + 2u_S u_{\acute{S}})$$

$$B = \sum_{u_2\neq 1,(0,1)(0,u_2)\notin E(\Pi_2(R))} \big(deg((0,1)).deg\big((0,u_2)\big)\big) + \sum_{u_2\neq 1,(0,1)(u_1,u_2)\notin E(\Pi_2(R))} \big(deg\big((0,1)\big).deg\big((u_1,u_2)\big)\big)$$

$$= (u_S).(u_S+1)\left(\frac{u_{\acute{S}}}{2}-1\right)+(u_S.2).\left(\frac{u_{\acute{S}}}{2}-1\right).\left(\frac{u_S}{2}\right) = \frac{1}{2}(u_S{}^2+u_S)(u_{\acute{S}}-2)+\frac{1}{2}(u_S{}^2)(u_{\acute{S}}-2)$$

$$= \frac{1}{2}(u_S{}^2 u_{\acute{S}} - 2u_S{}^2 + u_S u_{\acute{S}} - 2u_S + u_S{}^2 u_{\acute{S}} - 2u_S{}^2) = \frac{1}{2}(2u_S{}^2 u_{\acute{S}} - 4u_S{}^2 - 2u_S + u_S u_{\acute{S}})$$

$$= \frac{1}{4}(4u_S{}^2 u_{\acute{S}} - 8u_S{}^2 - 4u_S + 2u_S u_{\acute{S}})$$

$$C = \sum_{u_2\neq 1,(1,1)(0,u_2)\notin E(\Pi_2(R))} \left(deg((1,1)).deg\left((0,u_2)\right)\right) + \sum_{u_1\neq 1,(1,1)(u_1,0)\notin E(\Pi_2(R))} \left(deg\left((1,1)\right).deg\left((u_1,0)\right)\right) + \sum_{(1,1)(u_1,u_2)\notin E(\Pi_2(R))} \left(deg\left((1,1)\right).deg\left((u_1,u_2)\right)\right)$$

$$= (2.(u_S+1))\left(\frac{u_{\acute{S}}}{2}-1\right) + (2.(u_{\acute{S}}+1))\left(\frac{u_S}{2}-1\right) + (2.2).\left(\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right)-1\right)$$

$$= \frac{1}{2}(2u_S+2)(u_{\acute{S}}-2) + \frac{1}{2}(2u_{\acute{S}}+2)(u_S-2) + (u_S.u_{\acute{S}}-4)$$

$$= \frac{1}{2}(2u_S.u_{\acute{S}}-4u_S+2u_{\acute{S}}-4) + \frac{1}{2}(2u_S.u_{\acute{S}}-4u_{\acute{S}}+2u_S-4) + (u_Su_{\acute{S}}-4)$$

$$= \frac{1}{4}(12u_S.u_{\acute{S}}-4u_S-4u_{\acute{S}}-32)$$

$$D = \sum_{u_2\neq 1,(0,u_2)(0,1)\notin E(\Pi_2(R))} \left(deg\left((0,u_2)\right).deg\left((0,1)\right)\right) + \sum_{u_2\neq 1,(0,u_2)(u_1,u_2)\notin E(\Pi_2(R))} \left(deg\left((0,u_2)\right).deg\left((u_1,u_2)\right)\right) + \sum_{u_2\neq 1,(0,u_2)(u_1,1)\notin E(\Pi_2(R))} \left(deg\left((0,u_2)\right).deg\left((u_1,1)\right)\right)$$

$$= \left((u_S+1).(u_S)\right).\left(\frac{u_{\acute{S}}}{2}-1\right) + (u_S+1).(2).\left(\frac{u_{\acute{S}}}{2}-1\right).\left(\frac{u_S}{2}\right) + (u_S+1).(2).\left(\frac{u_{\acute{S}}}{2}-1\right).\left(\frac{u_S}{2}\right)$$

$$= \frac{1}{2}({u_S}^2+u_S)(u_{\acute{S}}-2) + \frac{1}{2}({u_S}^2+u_S)(u_{\acute{S}}-2) + \frac{1}{2}({u_S}^2+u_S)(u_{\acute{S}}-2)$$

$$= 3(\frac{1}{2}({u_S}^2u_{\acute{S}}-2u_S-2{u_S}^2+u_Su_{\acute{S}}) = \frac{1}{4}((6{u_S}^2u_{\acute{S}}-12u_S-12{u_S}^2+6u_Su_{\acute{S}})$$

$$E = \sum_{(u_1,0)(1,0)\notin E(\Pi_2(R))} \left(d(u_1,0).d(1,0)\right) + \sum_{(u_1,0)(1,u_2)\notin E(\Pi_2(R))} \left(d(u_1,0).d(1,u_2)\right) + \sum_{(u_1,0)(u_1,u_2)\notin E(\Pi_2(R))} \left(d(u_1,0).d(u_1,u_2)\right)$$

$$= (u_{\acute{S}}+1).(u_{\acute{S}}).\left(\frac{u_S}{2}-1\right) + (u_{\acute{S}}+1).(2).\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}\right) + (u_{\acute{S}}+1).(2).\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}\right)$$

$$= \frac{1}{2}({u_{\acute{S}}}^2+u_{\acute{S}}).(u_S-2) + \frac{1}{2}({u_{\acute{S}}}^2+u_{\acute{S}})(u_S-2) + \frac{1}{2}({u_{\acute{S}}}^2+u_{\acute{S}})(u_S-2)$$

$$=\frac{1}{2}(u_{S}u_{\acute{S}}{}^{2}-2u_{\acute{S}}{}^{2}+u_{S}u_{\acute{S}}-2u_{\acute{S}}+u_{S}u_{\acute{S}}{}^{2}-2u_{\acute{S}}{}^{2}+u_{S}u_{\acute{S}}-2u_{\acute{S}}+u_{S}u_{\acute{S}}{}^{2}-2u_{\acute{S}}{}^{2}+u_{S}u_{\acute{S}}-2u_{\acute{S}})$$

$$=\frac{1}{2}(3u_{S}u_{\acute{S}}{}^{2}+3u_{S}u_{\acute{S}}-6u_{\acute{S}}{}^{2}-6u_{\acute{S}})=\frac{1}{4}(6u_{S}u_{\acute{S}}{}^{2}+6u_{S}u_{\acute{S}}-12u_{\acute{S}}{}^{2}-12u_{\acute{S}})$$

$$\begin{aligned}F=&\sum_{u_1,u_2\neq 1,(u_1,u_2)(1,0)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big).deg\big((1,0)\big)\Big)\\&+\sum_{u_1,u_2\neq 1,(u_1,u_2)(0,1)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big).deg\big((0,1)\big)\Big)\\&+\sum_{u_1,u_2\neq 1,(u_1,u_2)(u_1,u_2)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big).deg\big((u_1,u_2)\big)\Big)\\&+\sum_{u_1,u_2\neq 1,(u_1,u_2)(0,u_2)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big).deg\big((0,u_2)\big)\Big)\\&+\sum_{u_1,u_2\neq 1,(u_1,u_2)(u_1,0)\notin E(\Pi_2(R))}\Big(deg\big((u_1,u_2)\big).deg\big((u_1,0)\big)\Big)\end{aligned}$$

$= (2).u_{\acute{S}})\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}\right)+(2).u_{S})\left(\frac{u_{\acute{S}}}{2}-1\right).\left(\frac{u_S}{2}\right)+$

$(2).(2).\left(\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right)-1\right).\left(\left(\frac{u_S}{2}.\frac{u_{\acute{S}}}{2}\right)-1\right)+(2).(u_S+1)\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}-1\right)+(2).(u_{\acute{S}}+1)\left(\frac{u_S}{2}-1\right).\left(\frac{u_{\acute{S}}}{2}-1\right)$

$$\begin{aligned}=&\frac{1}{4}(2u_{\acute{S}}{}^{2})(u_S-2)+\frac{1}{4}(2u_{S}{}^{2})(u_{\acute{S}}-2)+4\left(\frac{u_Su_{\acute{S}}-4}{4}\right).\left(\frac{u_Su_{\acute{S}}-4}{4}\right)\\&+\frac{1}{4}(2u_S+2)(u_S-2).(u_{\acute{S}}-2)+\frac{1}{4}(2u_{\acute{S}}+2)(u_S-2).(u_{\acute{S}}-2)\end{aligned}$$

$$\begin{aligned}=&\frac{1}{4}\left(2u_Su_{\acute{S}}{}^{2}-4u_{\acute{S}}{}^{2}\right)+\frac{1}{4}(2u_S{}^{2}u_{\acute{S}}-4u_S{}^{2})+\frac{1}{4}(u_S{}^{2}u_{\acute{S}}{}^{2}-8u_Su_{\acute{S}}+16)+\frac{1}{4}(2u_S{}^{2}-4u_S+2u_S\\&-4)(u_{\acute{S}}-2)+\frac{1}{4}(2u_{\acute{S}}{}^{2}-4u_{\acute{S}}+2u_{\acute{S}}-4)(u_S-2)\end{aligned}$$

$$\begin{aligned}=&\frac{1}{4}\left(2u_Su_{\acute{S}}{}^{2}-4u_{\acute{S}}{}^{2}\right)+\frac{1}{4}(2u_S{}^{2}u_{\acute{S}}-4u_S{}^{2})+\frac{1}{4}(u_S{}^{2}u_{\acute{S}}{}^{2}-8u_Su_{\acute{S}}+16)+\frac{1}{4}(2u_S{}^{2}u_{\acute{S}}-4u_Su_{\acute{S}}\\&+2u_Su_{\acute{S}}-4u_{\acute{S}}-4u_S^2+8u_S-4u_S+8)+\frac{1}{4}(2u_Su_{\acute{S}}{}^{2}-4u_Su_{\acute{S}}+2u_Su_{\acute{S}}-4u_S-4u_{\acute{S}}\\&+8u_{\acute{S}}-4u_{\acute{S}}+8\end{aligned}$$

$$=\frac{1}{4}[4u_Su_{\acute{S}}{}^{2}-8u_{\acute{S}}{}^{2}+4u_S{}^{2}u_{\acute{S}}-8u_S{}^{2}+u_S{}^{2}u_{\acute{S}}{}^{2}-12u_Su_{\acute{S}}+32]$$

$$G = \sum_{u_2\neq 1,(1,u_2)(0,u_2)\notin E(\Pi_2(R))} \left(deg\big((1,u_2)\big).deg\big((0,u_2)\big)\right) + \sum_{u_2\neq 1,(1,u_2)(u_1,0)\notin E(\Pi_2(R))} \left(deg\big((1,u_2)\big).deg\big((u_1,0)\big)\right) + \sum_{u_1\neq 1,(u_1,1)(u_1,0)\notin E(\Pi_2(R))} \left(deg\big((u_1,1)\big).deg\big((u_1,0)\big)\right) + \sum_{u_1\neq 1,(u_1,1)(0,u_2)\notin E(\Pi_2(R))} \left(deg\big((u_1,1)\big).deg\big((0,u_2)\big)\right)$$

$$= (2).(u_S+1).\left(\frac{u_{\acute{S}}}{2}-1\right) + (2).(u_{\acute{S}}+1).\left(\frac{u_S}{2}-1\right)\left(\frac{u_{\acute{S}}}{2}-1\right) + (2).(u_{\acute{S}}+1).\left(\frac{u_S}{2}-1\right) + (2(u_S+1).\left(\frac{u_{\acute{S}}}{2}-1\right)\left(\frac{u_S}{2}-1\right)$$

$$= \frac{1}{2}(2u_S+2).(u_{\acute{S}}-2) + \frac{1}{2}(u_{\acute{S}}+1).(u_S-2).(u_{\acute{S}}-2) + \frac{1}{2}(2u_{\acute{S}}+2).(u_S-2) + \frac{1}{2}(u_S+1).(u_{\acute{S}}-2).(u_S-2)$$

$$= \frac{1}{2}(2u_Su_{\acute{S}} - 4u_S + 2u_{\acute{S}} - 4) + \frac{1}{2}(u_Su_{\acute{S}} - 2u_{\acute{S}} + u_S - 2)(u_{\acute{S}}-2) + \frac{1}{2}(2u_Su_{\acute{S}} - 4u_{\acute{S}} + 2u_S - 4) + \frac{1}{2}(u_Su_{\acute{S}} - 2u_S + u_{\acute{S}} - 2)(u_S-2)$$

$$= \frac{1}{2}[2u_Su_{\acute{S}} - 4u_S + 2u_{\acute{S}} - 4 + u_Su_{\acute{S}}^2 + u_Su_{\acute{S}} - 2u_{\acute{S}}^2 - 2u_{\acute{S}} - 2u_Su_{\acute{S}} + 4u_{\acute{S}} - 2u_S + 4 + 2u_Su_{\acute{S}} - 4u_{\acute{S}} + 2u_S - 4 + u_S^2u_{\acute{S}} - 2u_S^2 - 2u_S + u_Su_{\acute{S}} - 2u_Su_{\acute{S}} + 4u_S - 2u_{\acute{S}} + 4]$$

$$= \frac{1}{2}[2u_Su_{\acute{S}} - 2u_S - 2u_{\acute{S}} - 2u_S^2 - 2u_{\acute{S}}^2 + u_S^2u_{\acute{S}} + u_Su_{\acute{S}}^2]$$

$$= \frac{1}{4}[4u_Su_{\acute{S}} - 4u_S - 4u_{\acute{S}} - 4u_S^2 - 4u_{\acute{S}}^2 + 2u_S^2u_{\acute{S}} + 2u_Su_{\acute{S}}^2]$$

Therefore ,

$$A + B + C + D + E + F + G$$

$$= \frac{1}{4}(4u_Su_{\acute{S}}^2 - 8u_{\acute{S}}^2 - 4u_{\acute{S}} + 2u_Su_{\acute{S}}) + \frac{1}{4}(4u_S^2u_{\acute{S}} - 8u_S^2 - 4u_S + 2u_Su_{\acute{S}}) + \frac{1}{4}(12u_S.u_{\acute{S}} - 4u_S - 4u_{\acute{S}} - 32) + \frac{1}{4}((6u_S^2u_{\acute{S}} - 12u_S - 12u_S^2 + 6u_Su_{\acute{S}}) + \frac{1}{4}(6u_Su_{\acute{S}}^2 + 6u_Su_{\acute{S}} - 12u_{\acute{S}}^2 - 12u_{\acute{S}}) + \frac{1}{4}[4u_Su_{\acute{S}}^2 - 8u_{\acute{S}}^2 + 4u_S^2u_{\acute{S}} - 8u_S^2 + u_S^2u_{\acute{S}}^2 - 12u_Su_{\acute{S}} + 32] + \frac{1}{4}[4u_Su_{\acute{S}} - 4u_S - 4u_{\acute{S}} - 4u_S^2 - 4u_{\acute{S}}^2 + 2u_S^2u_{\acute{S}} + 2u_Su_{\acute{S}}^2]$$

$$= \frac{1}{4}[16u_{S}u_{\acute{S}}^{2} - 32u_{\acute{S}}^{2} - 24u_{\acute{S}} + 20u_{S}u_{\acute{S}} + 16u_{S}^{2}u_{\acute{S}} - 32u_{S}^{2} - 24u_{S} + u_{S}^{2}u_{\acute{S}}^{2}]$$

Whence the second Zagreb co-index

$$\overline{M_2}(\Pi_2(R)) = \frac{1}{2}[A + B + C + D + E + F + G]$$
$$= \frac{1}{8}[16u_{S}u_{\acute{S}}^{2} - 32u_{\acute{S}}^{2} - 24u_{\acute{S}} + 20u_{S}u_{\acute{S}} + 16u_{S}^{2}u_{\acute{S}} - 32u_{S}^{2} - 24u_{S} + u_{S}^{2}u_{\acute{S}}^{2}]$$

Example 4.5: (a) To find first and second Zagreb co-indices of a graph $\boldsymbol{Z_4 \times Z_9}$

From Fig.1 We get

$$\overline{M_1}(\Pi_2(R)) = \sum_{uv \notin E(\Pi_2(R))} (deg(u) + deg(v)) = 5(2+2) + 6(2+3) = 50$$

$$\overline{M_2}(\Pi_2(R)) = \sum_{uv \notin E(\Pi_2(R))} (deg(u) \,.\, deg(v)) = 5(2.2) + 6(2.3) = 56$$

By theorem

$$\overline{M_1}(\Pi_2(R)) = \frac{1}{2}(25u_{\acute{S}} - 50) = \frac{1}{2}(25.6 - 50) = \frac{1}{2}(100) = 50$$

$$\overline{M_2}(\Pi_2(R)) = 14u_{\acute{S}} - 28 = 14(6) - 28 = 84 - 28 = 56$$

(b) To find first and second Zagreb co-indices of a graph $\boldsymbol{Z_9 \times Z_9}$ From Fig.2 we get

$$\overline{M_1}(\Pi_2(R)) = \sum_{uv \notin E(\Pi_2(R))} (deg(u) + deg(v))$$

$$= 24(2+7) + 36(2+2) + 12(6+2) + 4(6+7) = 508$$

$$\overline{M_2}(\Pi_2(R)) = \sum_{uv \notin E(\Pi_2(R))} (deg(u) \,.\, deg(v)) = 24(2.7) + 36(2.2) + 12(6.2) + 4(6.7) = 792$$

By theorem

$$\overline{M_1}(\Pi_2(R)) = \frac{1}{8}[u_{S}^{2}u_{\acute{S}}^{2} + 44u_{S}u_{\acute{S}} - 44u_{S} - 44u_{\acute{S}} + 6u_{S}^{2}u_{\acute{S}} + 6u_{S}u_{\acute{S}}^{2} - 12u_{S}^{2} - 12u_{\acute{S}}^{2} - 16]$$

$$= \frac{1}{8}[6^2(6^2) + 44(6)(6) - 44(6) - 44(6) + 6(6)^2(6) + 6(6)(6)^2 - 12(6)^2 - 12(6)^2 - 16]$$

$$= \frac{1}{8}[1296 + 1584 - 264 - 264 + 1296 + 1296 - 432 - 432 - 16] = \frac{1}{8}[4064] = 508$$

$$\overline{M_2}(\Pi_2(R)) = \frac{1}{8}[16u_{S}u_{\acute{S}}^{2} - 32u_{\acute{S}}^{2} - 24u_{\acute{S}} + 20u_{S}u_{\acute{S}} + 16u_{S}^{2}u_{\acute{S}} - 32u_{S}^{2} - 24u_{S} + u_{S}^{2}u_{\acute{S}}^{2}]$$

$$= \frac{1}{8}[16u_S u_{\acute{S}}{}^2 - 32u_{\acute{S}}{}^2 - 24u_{\acute{S}} + 20u_S u_{\acute{S}} + 16u_S{}^2 u_{\acute{S}} - 32u_S{}^2 - 24u_S + u_S{}^2 u_{\acute{S}}{}^2]$$

$$\overline{M_2}(\Pi_2(R)) = \frac{1}{8}[16(6)(6)^2 - 32(6)^2 - 24(6) + 20(6)(6) + 16(6)^2(6) - 32(6)^2 - 24(6) + (6)^2(6)^2]$$

$$= \frac{1}{8}[3456 - 1152 - 144 + 720 + 3456 - 1152 - 144 + 1296] = \frac{1}{8}[6336] = 792$$

4. Conclusions

This study presents explicit formulas for the first and second Zagreb indices and their co-indices for the graph $\Pi_2(R)$, where $R = S \times \acute{S}$ and both components are local rings of nilpotency 2.

The findings highlight how the unit structures of the components affect the topology of graph, contributing to the connection between ring theory, and graph theory and offering direction for future studies on topological indices in algebraic graphs.

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