Journal of Basrah Researches (Sciences) **51**(2), 133 (2025)
DOI: https://doi.org/10.56714/bjrs.51.2.10

Yang Transform Approach to Renewal Equations and Integral Equations

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A R T I C L E   I N F O

Received 18 August 2025
Revised 10 October 2025
Accepted 29 October 2025
Published 31 December 2025

K e y w o r d s :

Convolution Theorem ;Integra-Differential Equations; Integral Transforms ;Integral Equations.

Citation: Z. M. R. H. al fatlawy, J. Basrah Res. (Sci.) **50**(2), 133 (2025). DOI:https://doi.org/10.56714/bjrs.51.2.10

A B S T R A C T

Integral transforms provide powerful tools for engineers, physicists, and mathematicians. Many issues that arise in different areas of science and engineering, can be solved with ease and effectiveness using these methods. for solution using these integral Therefore, that exact solution has been obtained using very less computational work and spending very little time as well.

The main objective of this paper is to focus on studying and classifying integro-differential equations and integral equations, how to solve the integral equations using these yang transforms by taking some practical examples such as the renewal equation, to illustrate how to use these yang integral transforms accurately and effectively.Integral transformation methods provide simple and efficient techniques for solving many problems arising in various fields of science and engineering, particularly in solving integral equations. Therefore, exact solutions are achieved using minimal computational effort and a very short time.

It also explores practical examples, such as the renewal and integral equation, to demonstrate the effectiveness and accuracy of using these integral transformations.

1. Introduction

The integral equation is encountered in many scientific domains and applications oscillation theory, electrical engineering, economics, and medicine, filtration theory, game theory, control, queuing theory, elasticity, including fluid dynamics, plasticity, heat and mass transfer, among others. In mathematicsThese transforms have been used for solving different type of integral equations [1], [2], equations that have unknown functions appearing under the integral sign, are known as those integral equations:

$$\eta(\dot{x}) = \digamma(\dot{x}) + \lambda \int_{s(\dot{x})}^{r(\dot{x})} N(\dot{x},\dot{t})\, \eta(\dot{t})\, d\dot{t}$$

The kernel of an integral equation is denoted by $N(\dot{x},\dot{t})$, which is a known function of two variables. $s(\dot{x})$ and $r(\dot{x})$ are the limits of integration, and λ is that a constant parameter. It will be found that then unknown function is inside the integral sign [3]. Inside and outside the integral sign, the unknown function $\eta(\dot{x})$ can be found in numerous other situations. $N(\dot{x},\dot{t})$ and $\digamma(\dot{x})$ are predefined functions. Understanding the qualitative aspects of numerous processes and phenomena in a variety of natural science domains depends on the precise solution of these equations. Since

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ISSN: 1817-2695 (Print); 2411-524X (Online)
Online at: https://jou.jobrs.edu.iq

they are frequently derived from differential equations, integral equations are used in many different contexts [4]. In applied mathematics, engineering science and mathematical physics, integral transforms have been effectively utilized for nearly 200 years to solve a wide range of problems. The best integrals transform technique, in particular, can be used to derives the closed-form.

$$L\{f(\dot{x})\} = \int_0^\infty e^{-v\dot{x}} f(\dot{x})\, d\dot{x}, \quad \text{for } \dot{x} \geq 0$$

where v is real number and L is Laplace transform operator and in 2016 Xiao – Jung Yang introduced a new integral transform, It has been mainly applied to solve many problems such as a steady heat transfer problem but here we will use it to solution the integral equations [5].,The Yang transform of the function $f(\dot{x})$ is symbolizes by $\gamma\{f(\dot{x})\}$ and is defined as:

$$\gamma\{f(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}} f(\dot{x}) d\dot{x} \qquad , \ \dot{x} \geq 0$$

where are γ is Yang transform operator with $v \in (-\dot{x}_1, \dot{x}_2)$, the equations as well and a novel of that relationship between of the differential equation and integral equations is considered.

since the integral equations have many applications in real life, some important applications of these kinds of equations such as renewal equation, are solved using the previous integral transform. In this exploration, new apply the Yang transform to get a recipe of general arrangements of direct differential conditions. In section two, we checked on the properties of changes

And the significant hypothesis for certain capabilities through the connection among Laplace and Yang transform. In section the third, we accomplished recipes of the overall arrangement of integral

Conditions of first, second, and higher-request. At last, in the last segment, we applied general recipes got in [6].

2. Basic Concepts of Integral Equations and Integro – Differential Equations

2.1 Integral Equations:

An integral equation is an equation containing an integral sign in which the unknown function $\boldsymbol{\eta}(\dot{t})$ appears.Ageneral integral equation is an equation be $\eta(\dot{x})$ expressed in the following form [7]:

$$\eta(\dot{x}) = f(\dot{x}) + \lambda \int_{s(x)}^{r(\dot{x})} N(\dot{x}, \dot{t})\, \eta(\dot{t})\, d\dot{t} \qquad \dots (1)$$

The kernel function of an integral equation is a known function of two variables, $\dot{x}$ and $\dot{t}$, denoted by $N(\dot{x}, \dot{t})$, since λ is a constant, $s(\dot{x})$ and $r(\dot{x})$ are the limits of integration. The driving term of the integral equation is often defined as that function $f(\dot{x})$,which is known beforehand. Furthermore, $\eta(\dot{x})$ does not only appear in the integral sign. Notably, $(\dot{x})$ and $r(\dot{x})$,the limits of integration, can be both variables, mixed variables, or constants. The left and right margins should both be 2.4 cm.

2. 2. Integro – Differential Equations:

Many scientific applications can be mathematically represented by an integral-differential equation, especially when initial value or boundary value problems are transformed into integral equations A general integro–differential equation involves both integrals and derivatives of the unknown function, and can be written in the following form [7]:

$$\eta^{(k)}(\dot{x}) = i\mathrm{F}(\dot{x}) + \lambda \int_a^b N(\dot{x}, \dot{t})\eta(\dot{t})\, d\dot{t}\, , \eta^{(k)} = \frac{d^k\eta}{d\dot{x}^k} \qquad \dots (2)$$

2.3. Connection Between Integral Equations and Differential Equations:

As previously discussed, many engineering and scientific applications, such as atmospheric radar, quantum mechanical scattering, and water waves, give rise to both integro–differential and integral equations. Importantly, integral equations and integro-differential equations can be obtained by converting initial value problems, even though they are derived from boundary value problems with specified boundary conditions. Although both conversions are reversible, the final conversion involved integral equations [7].

3.Some Useful Properties of Yang Transform:

3.1.YangTransform (γ – Transform)

In 2016, Xiao–Jung Yang introduced a new integral transform. It has been mainly applied to solve many problems, such as steady heat transfer problems. In this section, we apply it to solve integral equations [8], [9].

Definition: [10], [11] The γ - transform of a function $f(\dot{x})$ is denoted by $\gamma\,\{f(\dot{x})\}$ and is defined as:

$$\gamma\,\{f(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}} f(\dot{x})\, d\dot{x} \qquad , \quad \dot{x} \geq 0 \qquad \ldots(3)$$

where γ is Yang transform operator.

This definition is valid provided that the integral exists for some values of $v \in (-\dot{x}_1, \dot{x}_2)$

If we substitute this condition $\frac{\dot{x}}{v} = \dot{t}$,the equation (3) becomes:

$$\gamma\,\{F(x)\} = v \int_0^\infty e^{-t}\, F(vt)\, dt\,, \quad t > 0\,.$$

Since a few basic functions, the following tables display the previously well-known transforms.

Table 1: $(Y\ )$ –Transforms of some elementary functions.

$\boldsymbol{S.N}$	$f(\dot{x})$	$\boldsymbol{Y}\{f(\dot{x})\}$
1	1	v
2	$\dot{x}$	v^2
3	$\dot{x}^2$	$2!\,v^3$
4	$\dot{x}^n, n \in N$	$n! \cdot v^{n+1}$
5	$e^{a\dot{x}}$	$\dfrac{v}{1-av}$
6	$\sin a\dot{x}$	$\dfrac{a\,v^2}{1+a^2v^2}$
7	$\cos a\dot{x}$	$\dfrac{v}{1+a^2v^2}$
8	$\sinh a\dot{x}$	$\dfrac{av^2}{1-a^2v^2}$

So, using the Yang transform table for elementary functions is an important aid in differentiation, just as we rely on Laplace tables, but it is more flexible when applied to the renewal equation.

3.2.Convolution Theorem for γ - Transform

This is the main theorem to solve integral and differential equations, and it always plays an important role in a number of different physical applications [8].

Let f and g be functions with Yang transforms $T1_{(v)}$ and $T2_{(v)}$, respectively. Then, the Yang transform of their convolution is given by:

$$(f * \bar{g})(\dot{x}) = \int_0^x f(\dot{x})\, \bar{g}(\dot{x} - \dot{t})\, d\dot{t}$$

Then

$$\gamma\{(f * \bar{g})(\dot{x})\} = \gamma\{f(\dot{x})\}.\gamma\{\bar{g}(\dot{x})\}$$

3.3. Inverse of $\boldsymbol{\gamma}$ – Transform [8].

If $\gamma\,\{f(\dot{x})\}$ is the yang Transform of $f(\dot{x})$then the inverse of yang transform of $\gamma\,\{f(\dot{x})\}$ will be $f(\dot{x})$or mathematically:

$$\gamma^{-1}\{\gamma\,\{f(\dot{x})\}\} = f(\dot{x}).$$

where γ^{-1} is operator of the inverse γ - Transform and also then linearity property holds in the inverse γ – Transform. Thus:

$$\gamma^{-1}\left\{a\,\gamma\,\{f_1(\dot{x})\} + b\,\gamma\{f_2\,(\dot{x})\}\right\} = a\,\gamma^{-1}\{\gamma\{f_1(\dot{x})\}\} + b\,\gamma^{-1}\{\gamma\{f_2(\dot{x})\}\}$$
$$= a\,f_1(\dot{x}) + b f_2\,(\dot{x})\ .$$

4. Dualities Between Transforms

This section discusses the dualities between the Yang transform (γ-transform) and other well-known transforms, such as the Laplace transform, and highlights the relationships among them. the origin of the integral transforms is the Laplace transform can be traced back to celebrated work of P.S. Laplace) 1827-1749 (on probability theory in (1780s). In fact, Laplace's classic book includes some basic results of the Laplace transform which is one of the oldest and most commonly used integral transforms available in the mathematical literature .Laplace transform is defined for the function

Examples are provided using φ to represent the results of other transforms, emphasizing the significance of these relationships in the context of integral transforms The Yang transform is particularly useful for regeneration equations because it overcomes Laplace constraints with heavy-tailed distributions, simplifies the computation of recursive convolutions, and gives us direct and elegant solutions in the probabilistic domain [6].

4.1. Laplace -Yang Duality

The L - Transform of the function $f(\dot{x})$, for $\dot{x} \geq 0$, denoted by $L\{f(\dot{x})\}$ and is, defined as [12]:

$L\,[f(\dot{x})] = \int_0^\infty e^{-v\dot{x}}\,f(\dot{x})\,d\dot{x}\,,$

Similarly,the γ - Transform of a function $f(\dot{x})$, for $\dot{x} \geq 0$,is defined as:

$\gamma\{f(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}}\,f(\dot{x})\,d\dot{x}\,,$

Consequently,

$L\,\{f(\dot{x})\} = \int_0^\infty e^{-v\dot{x}}\,f(\dot{x})\,d\dot{x} = \int_0^\infty e^{-\frac{\dot{x}}{\frac{1}{v}}}\,f(\dot{x})\,d\dot{x}.$

From these definitions, the following relation holds:

$L\{f(\dot{x})\,\} = \varphi(\frac{1}{\tilde{v}})$

On the other hand, it can also be shown that:

$\gamma\,\{f(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}}\,f(\dot{x})\,d\dot{x} \;= \int_0^\infty e^{-\left(\frac{1}{v}\right)\dot{x}}\,f(\dot{x})\,d\dot{x} = \varphi\left(\frac{1}{\tilde{v}}\right).$

Therefore, we obtain

$\gamma\{f(\dot{x})\} = \varphi(\frac{1}{v})\ .$

4.2.lllustrative Examples

The following examples illustrate how Yang transforms can be applied to solve integral equations.

Example. Solve the following integral equation using integral transforms:

$$u(\dot{x}) = \frac{1}{6}\,\dot{x}^3 + \int_0^{\dot{x}}(\,\dot{t})\,u(\dot{t})\,d\dot{t} \qquad \ldots(4)$$

Solution:

Applying the Yang transform to both sides of the equation gives:

$\gamma\,\{u(\dot{x})\} = \gamma\,\left\{\frac{1}{6}\,\dot{x}^3\right\} + \gamma\,\{\int_0^{\dot{x}}(\dot{x}-\dot{t})\,u(\dot{t})\,d\dot{t}\}$

$\gamma\,\{u(\dot{x})\} = \frac{1}{6}\,.3!\;\tilde{v}^4 + \tilde{v}^2\,.\gamma\,\{u(\dot{x})\}$

$\gamma\,\{u(\dot{x})\}(1-\tilde{v}^2) = \tilde{v}^4$

$\gamma\ \{\underset{\wedge}{u}(\dot{x})\} = \frac{\tilde{v}^4}{(1-\tilde{v}^2)}$

$\underset{\wedge}{u}(\dot{x}) = \gamma^{-1}\ \left\{\frac{v^4}{1-\tilde{v}^2}\right\}$

$\underset{\wedge}{u}(\dot{x}) = \gamma^{-1}\ \left\{\frac{v^2}{1-\tilde{v}^2}\ .\tilde{v}^2\right\} = \dot{x}.\sinh\dot{x}.$

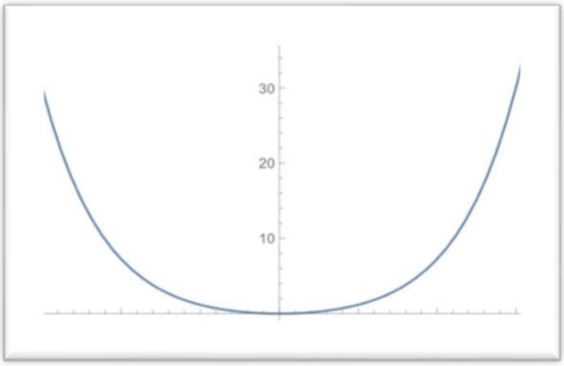

Fig.1: Graphical representation of the solution to the integral equation using the Yang transform: $u(\dot{x}) = \dot{x}\ .sinh\,\dot{x}$.

5. The Renewal Equation

Many important quantities in the study of regeneration processes can be expressed using a special type of integral equation known as the renewal equation.

, It plays a significant role in determining the expected number of renewals and in evaluating the average cost of management strategies .

$$\mathring{A}(\dot{t}) = \alpha(\dot{t}) + \int_0^t \mathring{A}(\dot{t}-\dot{t})\ d\mathit{f}(\underset{\wedge}{u})$$

The renewal equation can be expressed as a functional equation for Å, in terms of a known function F and α. Let's look at a specific

To illustrate, let us consider a specific example of a renewal equation [13], [14]:

$$b(t) = ax\, e^{-2x} + \int_0^t m(x)b(t-x)\, dx \qquad \text{….. (5)}$$

This equation can be solved using integral transforms, as demonstrated below.

$b(\dot{t}) = a\,\dot{x}\, e^{-2\dot{x}} + \int_0^{\dot{x}} m(\dot{x})\ .\ b(\dot{t}-\dot{x})\, d\dot{x}$

$\gamma\ \{b(\dot{t})\} = \gamma\ \{a\,\dot{x}\, e^{-2\dot{x}}\ \} +\ \gamma\ \{\int_0^t m(\dot{x})\ .\ b(\dot{t}-\dot{x})\, d\dot{x}\}$

$\gamma\{b(\dot{t})\} = a\ .v^2\ .\frac{v}{1+2v} + v^2\ .\gamma\ \{b(\dot{t})\}$

$\gamma\ \{b(\dot{t})\}(1-v^2) = \frac{a\, v^3}{1+\ 2v}$

$\gamma\{b(\dot{t})\} = a\ .\frac{v^3}{1+2v}\ .\frac{1}{1-v^2}$

$ib(\dot{t}) = a\ .\gamma^{-1}\ \{\frac{v^2}{1-v^2}\ .\frac{v}{1+2v}\}$

$\therefore\ ib(\dot{t}) = a\ .e^{-2\dot{t}}\ .\sinh\dot{t}$

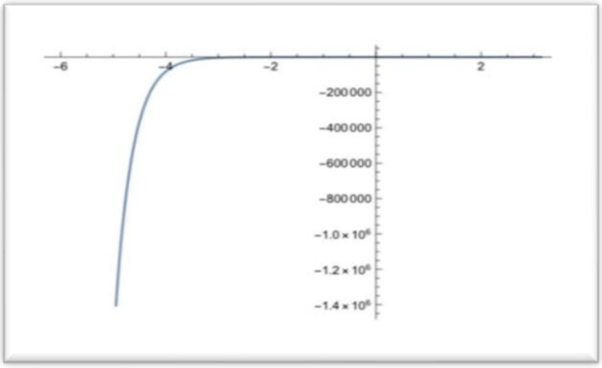

Fig.2. Graphical representation of the solution to the renewal equation using the Yang transform: $\mathrm{i}b(\mathrm{t}) = a\,.\,e^{-2\mathrm{t}}\,.\,sinh\,\mathrm{t}$

6. Conclusion and Future Work

In this paper, we focused on the study and classification of integral equations, particularly linear integral equations.

In this study, the Yang transform was applied to solve the regeneration equation and some types of integral equations. The results showed that this method provides accurate solutions in a simplified manner compared to traditional methods, reducing computational complexity and providing straightforward and clear solutions. Furthermore, analytical comparisons confirmed the effectiveness of the Yang transform as a superior tool for solving these types of equations, especially when compared to classical transformations such as the Laplace transform.We demonstrated how well-known integral transforms, such as the Laplace and Yang transforms, can be used to solve such equations with improved accuracy and reduced error.

As an application, the renewal equation was discussed and solved using the Yang transform.

This study highlights the efficiency of the Yang transform as an alternative to traditional methods, and future research can further expand its applications to broader classes of integral and integro–differential equations. So, for future work, we recommend the following:

Apply these integral transforms to solve non-linear integral equations and explore their interrelationships.

1.Yang transforms: various sets of regeneration equations, especially those characterized by being nonlinear or high-order, with the aim of exploring its effectiveness in dealing with more complex and challenging cases.

2. Applications to Advanced Integral Equations: Investigating the possibility of using the Yang transform to solve problems related to the Fredholm and Volterra equations, in addition to integro-differential equations, thus contributing to opening new horizons for its employment in various mathematical and scientific applications.

3. Numerical Verification: Designing and developing numerical algorithms based on the Yang transform, capable of handling cases that lack closed-form analytical solutions, while testing their accuracy and effectiveness on software platforms such as MATLAB or Python to ensure the practical feasibility of the results.

4. Real-World Applications: Leveraging the solutions resulting from the Yang transform for application in vital fields including reliability theory, probabilistic models, and stochastic process analysis, with the aim of developing practical models with broad practical applications.

5. Innovation and integration with other methods: Explore the possibility of integrating the Yang transform with additional analytical techniques, such as the analytical homotopy method or the wavelet transform, to develop hybrid tools that contribute to enhancing the effectiveness of solutions and enriching the mathematical methods used.and Extend the method to integro–differential equations.
6.Investigate further applications of these transforms to non-linear problems and integro–differential equations.

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Journal of Basrah Researches (Sciences) **51**(2), 133 (2025)
DOI: https://doi.org/10.56714/bjrs.51.2.10

نـهج تحويل يانـغ للمعادلات التجديدية والمعادلات التكاملية

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معلومات البحث

الاستلام	18 اب 2025
المراجعة	10 تشرين أول 2025
القبول	29 تشرين أول 2025
النشر	31 كانون أول 2025

الكلمات المفتاحية

نظرية الالتفاف؛ المعادلات التكاملية التفاضلية؛ التحويلات التكاملية؛ المعادلات التكاملية.

Citation: Z. M. R. H. al fatlawy, J. Basrah Res. (Sci.) **50**(2), 133 (2025). DOI:https://doi.org/10.56714/bjrs.51.2.10

الملخص

تُوفر التحويلات التكاملية أدوات فعّالة للمهندسين والفيزيائيين والرياضيين. يمكن حل العديد من المشكلات التي تنشأ في مختلف مجالات العلوم والهندسة بسهولة وفعالية باستخدام طرق تحويلات يانغ التكاملية. لذلك، تم الحصول على هذا الحل الدقيق باستخدام جهد حسابي أقل بكثير وقضاء وقت قصير جدًا أيضًا.

الهدف الرئيسي من هذه الورقة هو التركيز على دراسة وتصنيف المعادلات التكاملية التفاضلية والمعادلات التكاملية، وكيفية حل المعادلات التكاملية باستخدام تحويلات يانغ هذه من خلال أخذ بعض الأمثلة العملية مثل معادلة التجديد، لتوضيح كيفية استخدام تحويلات يانغ التكاملية بدقة وفعالية. توفر طرق التحويل التكاملي تقنيات بسيطة وفعالة لحل العديد من المشكلات التي تنشأ في مختلف مجالات العلوم والهندسة، وخاصة في حل المعادلات التكاملية. لذلك، يتم تحقيق حلول دقيقة باستخدام الحد الأدنى من الجهد الحسابي ووقت قصير جدًا.

كما يستكشف أمثلة عملية، مثل معادلة التجديد والتكامل، لإثبات فعالية ودقة استخدام هذه التحويلات التكاملية.

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ISSN: 1817-2695 (Print); 2411-524X (Online)
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