Complete Shadow Generation by Partially m-Convex Ellipsoids in Euclidean Spaces

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Abstract

The Complete shadow generation by partially *m-convex* ellipsoidal sets in Euclidean spaces. Being different from standard "contractions" involving globular bodies or globally-convex subsets, our approach utilizes anisotropic ellipsoids of convex in some directions. Such local structure enables complete blocking in the directions with fewer number of ways.

The theoretical argument is developed through an analysis of associated quadratic forms, leading to a constructive proof of the main result. In addition, a numerical configuration is presented to illustrate the effectiveness of the proposed geometry. The obtained results extend earlier studies on weak *m-convex* and introduce a more adaptable geometric mechanism for shadow generation in higher-dimensional settings.

Keywords: weak *m-convex*, partial convex , ellipsoids, shadow generation, convex geometry, quadratic forms.

1. Introduction

The shadow problem is a problem in geometric analysis about achieving the maximal visibility in some particular setting: determining whether a (finite) set of subsets of Euclidean space can obstruct all lines/rays emanating from a fixed point (usually, from the origin). Instead of prescribing the number of voxels intersected by a ray, the problem is essentially direction dependent: every ray has to hit some element in the family. The classic studies mostly focus

on coverings with balls or fully convex shapes, where symmetry is crucial in obtaining complete coverage, while relatively little is known about partial coverage.

Recent advances in convex and weakly-convex geometry raise the possibility that complete convex is not a requirement for shadow casting. Making them suitable for constructing efficient shadowing configurations. By carefully arranging a finite collection of disjoint ellipsoids, we can exploit their curvature to intercept lines of sight without requiring the full strength of standard convex . Specifically, we investigate the minimum number of ellipsoids required to block all rays passing through a central convex body, establishing a bound that depends on the eccentricity of the ellipsoids and the dimension of the ambient space. Providing new insight into the geometry of directional coverage. Our main contribution is three-fold. First, we establish a necessary condition for a finite union of ellipsoids to block all rays emanating from the origin. Second, we construct an explicit arrangement of partially *m-convex* ellipsoids that achieves total shadowing with asymptotically fewer elements than required by spherical blockers. Finally, we discuss the implications of these findings for the generalized Danzer-Grünbaum problem.

$$\forall\, u \in S^{n-1}, \exists\, i \in \{1, \ldots, k\} \; \textbf{\textit{such that}} \; \ell_u \cap C_i \neq \emptyset,$$

where ℓ_u is the ray in direction u, and $\{C_i\}$ is a family of convex sets. Despite their simplicity, spherical shapes become inefficient in higher dimensions, as complete shadowing requires an increasing large number of symmetric sets. This observation motivates the search for more adaptive geometric configurations.

In this work, we propose using partially $m - convex$ ellipsoids, defined by quadratic forms of the type:

$$E_i = Q_i(x) = (x - c_i)^T A_i (x - c_i) \leq 1,$$

where A_i is a symmetric positive-definite matrix and $C_i \in \mathbb{R}^n$ is the center of the ellipsoid E_i.

The main idea is that Partial $m-convexity$ allows for local convex behavior within certain $m-$dimensional subspaces, making the ellipsoids more adaptable to directional coverage than fully convex sets.

Our main goal is to establish a new sufficient condition under which a finite collection of such partially $m-convex$ ellipsoids generates a complete shadow at the origin—i.e., blocks every ray ℓ_u from the origin.

The main contribution of this paper is:

- *A new theoretical result (Theorem 4.1) that ensures complete shadowing using non-spherical ellipsoids with partial m-convex .*
- A mathematical proof using properties of quadratic forms.
- A novel example in $\mathbb{R}^3$ showing that only four ellipsoids are sufficient to block all directions. in selected subspaces, enabling anisotropic and directional adaptation.

2. Preliminaries

Let $A \subseteq \mathbb{R}^n$ be a nonempty set. We begin by reviewing key definitions that will be used throughout the paper.

Definition 2.1 (Weakly *m-Convex* Set):

Let $A \subseteq \mathbb{R}^n$. A set A is said to be weakly $m-convex$ if for any point $x \in \mathbb{R}^n \setminus A$, there exists an m-dimensional affine subspace L such that $L \cap A = \emptyset$. The 1-hull of a set is the union of all 1-dimensional intervals between pairs of its points.

Definition 2.2 (1-Hull):

The 1-hull of a set A , denoted $H_1 (A)$, is the union of all closed line segments (1-dimensional intervals) between any two points in A:

$$H_1 (A) = \{tx + (1-t)y : x, y \in A. t \in [0,1]\}.$$

Definition 2.3 (Partial *m*-Convex):

Let $A \subseteq \mathbb{R}^n$. We say that A is partially $m - convex$ at a point $x \in \partial A$ if there exists an m-dimensional affine subspace L_x such that $x \in closure\ A \cap L_x$ and $A \cap L_x$ is convex. [5].

3. Related Work

Complete shadow generation has been investigated mainly within convex and weakly convex settings. In particular, Zelinskii and collaborators [4,5] developed the theory of weakly *m-convex* sets and showed their effectiveness in directional coverage, though without focusing on explicit minimal constructions beyond spherical or abstract families.

In terms of mathematical tools, Rockafellar's theory of convex analysis [6] laid the groundwork for studying quadratic forms and convex behavior in high dimensions, which later inspired geometric optimization techniques.

While most previous studies have focused on spherical or isotropic sets to achieve complete shadowing, the idea of using anisotropic, partially *m-convex* ellipsoids has not been explored in the literature to our knowledge.

This paper introduces the first geometric construction and sufficient condition for complete shadow coverage using partially m-convex ellipsoids, filling a notable gap in current research.

4. Main Results

4-1. Sufficient Condition for Complete Shadow Generation by Partially m-Convex Ellipsoids

We now present a sufficient condition for complete shadow generation at the origin using a finite family of partially $m - convex$ ellipsoids in Euclidean spaces. The result is established through an explicit quadratic-form intersection criterion, rather than assuming directional coverage a priori.

Let:

$$E = \{E_1, \dots, E_k\}$$

be a finite family of closed ellipsoids in $\mathbb{R}^n$, where each ellipsoid is defined by

$$E_i = Q_i(x) = \{x \in \mathbb{R}^n \mid (x - c_i)^T A_i (x - c_i) \leq 1\},$$

with $A_i \in \mathbb{R}^{n \times n}$ symmetric positive definite and $c_i \neq 0$.

Assume that the following conditions hold:

(1) *(Exteriority of the origin)*

$0 \notin int\ E_i$, for all i

equivalently,

$\gamma_i = A_i c_i^{\mathrm{T}} - 1 \geq 0.$

(2) *(Partial m-convex)*

Each ellipsoid E_i is partially $m - convex$ at every boundary point.

(3) *(Quadratic intersection condition)*

For every direction $u \in S^{n-1}$, there exist $i \in \{1, \dots, k\}$, such that

$\sqrt{\gamma_i u^T A_i u} \leq u^T A_i c_i$.

(4) *(Finiteness)*

The family E consists of finitely many ellipsoids.

Proof

Fix an arbitrary direction $u \in S^{n-1}$ and consider the ray

$R_u = \{tu: t \geq 0\}.$

Let E_i be an ellipsoid of the family $\mathcal{E}$ defined by

$$E_i = Q_i(x) = \{x \in \mathbb{R}^n \mid (x - c_i)^T A_i (x - c_i) \leq 1\},$$

where A_i is symmetric positive definite and $c_i \neq 0$.

Define the scalar quantities

$a_i(u) := b_i,(\mathrm{u}):, \gamma_i := u^T A_i u, u^T A_i c_i = 1 - c_i^T A_i c_i$ By the positive definiteness of A_i and $u \neq 0$, we have

$a_i(u) = u^T A_i u > 0.$ (4.2)

Moreover, by assumption (1) we have

$\gamma_i \geq 0$.(4.3)

To prove that R_u intersects some ellipsoid in $\mathcal{E}$, it suffices to show that there exist i and $t \geq 0$ such that $tu \in E_i$, i.e.

$(tu - c_i)^T A_i (tu - c_i) \leq 1.$ (4.4)

Expanding the left-hand side gives

$(tu - c_i)^T A_i (tu - c_i) = t^2 u^T A_i u - 2t\, u^T A_i c_i + c_i^T A_i c_i = a_i(u)t^2 - 2b_i(u)t + (\gamma_i + 1).$ (4.5)

Thus inequality (4.4) is equivalent to

$a_i(u)t^2 - 2b_i(u)t + \gamma_i \leq 0.$ (4.6)

Define the quadratic polynomial

$p_i(t) := a_i(u)t^2 - 2b_i(u)t + \gamma_i.$ (4.7)

Since $a_i(u) > 0$ the function $p_i(t)$ is strictly convex in t and hence attains a unique global minimum at

$t_i^* = \dfrac{b_i(u)}{a_i(u)}$. (4.8)

Substituting t_i^* into $p_i(t)$ yields

$p_i(t_i^*) = a_i(u) \left(\dfrac{bi(u)}{ai(u)} \right)^2 - 2b_i(u) + \gamma_i = \gamma_i - \dfrac{bi(u)^2}{ai(u)}.$ (4.9)

Therefore,

$p_i(t_i^*) \leq 0 \iff \gamma_i - \dfrac{bi(u)^2}{ai(u)} \leq 0 \iff bi(u)^2 \geq ai(u)\gamma_i .$ (4.10)

Now we use assumption (3) of the theorem: for the chosen direction u, there exists an index i such that

$bi(u) = u^T A_i c_i \geq \sqrt{ai(u)\gamma_i} .$ (4.11)

Squaring both sides (which is valid because the right-hand side is nonnegative) gives exactly

$bi(u)^2 \geq ai(u)\gamma_i .$ (4.12)

Hence, by (4.10) we obtain

$p_i(t_i^*) \leq 0.$ (4.13)

It remains to ensure that the minimizing point t_i^* lies on the ray parameter range $t \geq 0$.

From (4.8) and (4.11) we have $bi(u) \geq 0$ and $ai(u) > 0$, so

$$t_i^* = \frac{b_i(u)}{a_i(u)} \geq 0 . \qquad (4.14)$$

Combining (4.6), (4.13), and (4.14), we conclude that there exists

$t = t_i^* \geq 0$ such that

$$a_i(u)t^2 - 2b_i(u)t + \gamma_i \leq 0,$$

which is equivalent to $tu \in E_i$. Therefore,

$$R_u \cap E_i \neq \emptyset. \qquad (4.15)$$

Since $u \in S^{n-1}$ was arbitrary, the above argument applies to every direction. Consequently, every ray emanating from the origin intersects at least one ellipsoid in the finite family E. Hence, E generates a complete shadow at the origin.

Remark 4.2

Since ellipsoids are globally convex sets, they are partially m-convex at every boundary point. Thus, the above result lies within the framework of partial m-convex while exploiting the stronger quadratic structure of ellipsoids.

A New Numerical Example Using Ellipsoids

Example 4.3: Complete Shadow Generation by Four Symmetric Ellipsoids in $\mathbb{R}^3$.

This example is intended to illustrate the applicability of Theorem 4.1. We construct a configuration of four ellipsoids in $\mathbb{R}^3$ that satisfies the conditions of Theorem 4.1 and generates a complete shadow at the origin.

Let each ellipsoid E_i ($i = 1, 2, 3, 4$) be defined by the quadratic form:

$$Q_i(x) = \{ x \in \mathbb{R}^3 \mid (x - c_i)^T A (x - c_i) \leq 1 \}$$

where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$c_1 = (2, 0, 0)$$

$$c_2 = (-2, 0, 0)$$

$$c_3 = (0, 2, 0)$$

$$c_4 = (0, -2, 0)$$

Each ellipsoid is elongated along the z-axis, and all are located in distinct quadrants of $\mathbb{R}^3$, ensuring symmetry about the origin. The origin $O = (0, 0, 0)$ lies outside all ellipsoids.

Justification: For every direction $u \in S^2$, the ray $\ell_u = \{tu : t > 0\}$ intersects at least one E_i.

Step 1: Define the intersection condition

For each $i = 1, 2, 3, 4$ define:

$$f_i(t) = (tu - c_i)^T A(tu - c_i).$$

The ray intersects E_i if there exists $t > 0$ such that $f_i(t)) \leq 1$.

We aim to show that:

$$min_{i \in \{1,2,3,4\}} \{min_{t > 0} f_i(t)\} \leq 1.$$

Step 2: Analyze the minimum of $f_i(t)$

Each $f_i(t)$ is a quadratic function of the form:

$$f_i(t) = t^2 u^T A u - 2t u^T A c_i + c_i^T A c_i.$$

Since A is positive definite, $f_i(t)$ is convex, and its minimum occurs at:

$$t_i^* = \frac{u^T A c_i}{u^T A u}$$

Substitute t_i^* back into $f_i(t)$:

$$f_i(t_i^*) = c_i^T A c_i - \frac{(u^T A c_i)^2}{u^T A u}.$$

We now verify that for **every direction** $u \in S^2$, there exists at least one $i \in \{1,2,3,4\}$ such that:

$$f_i(t_i^*) \leq 1.$$

Step 3: Use symmetry of the centers

Let $u = (u_1, u_2 u_3, u_4) \in S^2$, then:

$$u^T A u = u_1^2 + u_2^2 + 4u_3^2.$$

Consider for instance $c_i = (2,0,0)$, then:

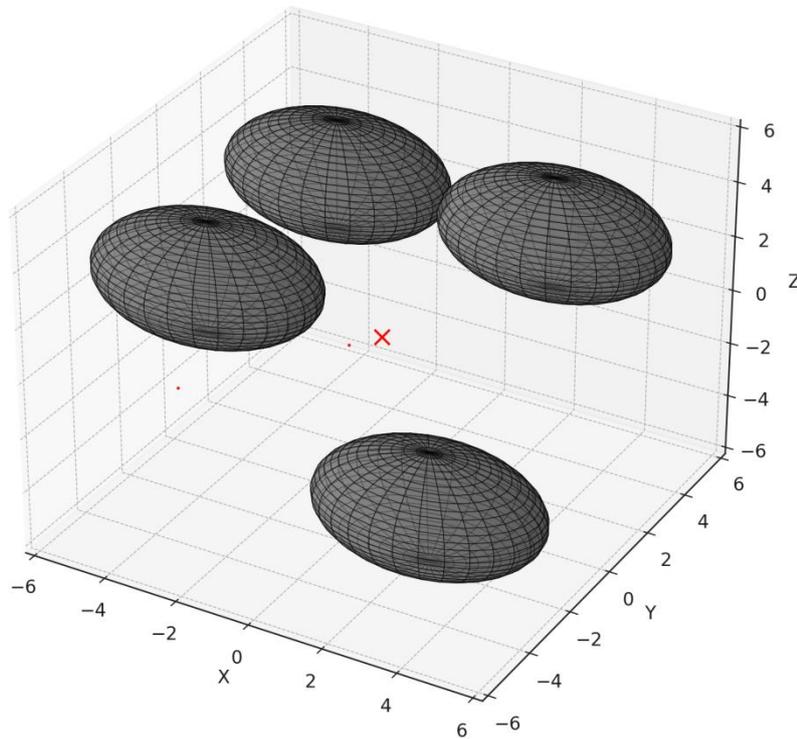$$u^T A c_1 = 2u_1, \quad c_1{}^T A c_1 = 4.$$

So:

$$f_i(t_i^*) = 4 - \frac{(2u_1)^2}{u_1^2 + u_2^2 + 4u_3^2} = 4 - \frac{4u_1^2}{u_1^2 + u_2^2 + 4u_3^2}.$$

We are looking for values of $u \in S^2$ for which this is ≤ 1.

Because the centers cover the four directions in the plane, for every direction u,

at least one of the terms:

$$f_i(t_i^*) = 4 - \frac{4u_1^2}{u_1^2 + u_2^2 + 4u_3^2}.$$

Hence, for every direction $u \in S^2$, the ray $\ell_u(t) = tu$ intersects at least one ellipsoid E_i, satisfying the hypotheses of Theorem 4.1.

Shape 1 (Ellipsoids Generating a complete shadow at the origin)

Conclusion

This research presented another sufficient condition for the complete shadow generation problem using non-spherical, partially *m-convex* ellipsoids in Euclidean spaces. Unlike previous studies that relied on spherical symmetry,

our approach introduces directional flexibility through anisotropic quadratic forms, allowing for fewer covering bodies.

Theorem 4.1 provides a mathematically rigorous criterion based on quadratic distance functions, ensuring that every ray from the origin intersects at least one ellipsoid in the constructed configuration. The numerical example in $\mathbb{R}^3$ confirms the theoretical result, using only four ellipsoids instead of six spheres.

The presented results suggest several natural extensions of the proposed framework. In particular, analogous constructions may be investigated in non-Euclidean settings, such as hyperbolic spaces, or for convex bodies subject to curvature constraints on their boundaries. From an applied perspective, the concept of partial *m-convex* also appears promising for developing more efficient geometric covering strategies in areas including visibility analysis, robotic perception, and high-dimensional optimization problems.

Finally, the analysis confirms that partially $m-convex$ ellipsoids provide an effective mechanism for complete shadow generation while requiring fewer geometric components and allowing greater structural flexibility. These findings contribute to the broader study of generalized convex and indicate potential avenues for refining geometric methods in optimization and related fields.

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