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A New Generated Lifetime Model: Theory, Simulation and Application to Stress-Strength Datasets

O. D. Adubisi

Department of Statistics, Federal University Wukari, Nigeria.

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ABSTRACT

In this article, a new continuous model called the exponentiated Gompertz generated skew- t (EGG_{ST}) distribution based on the exponentiated Gompertz generalized distribution is introduced. The new model is capable of fitting skewed, long and heavy tailed dataset and is more flexible than the skew-t distribution which is considered a special case. Related theoretical properties of the new model such as the quantile function, ordinary moment, probability weighted moment, order statistics, Rényi and Shannon entropies were investigated. The model parameters were estimated using the maximum likelihood estimation and simulation study carried out to examine the finite sample performance of these estimates. The applicability of the new model was illustrated by means of well-known breaking stress and strength datasets.

1. Introduction

Jones (2001) and Jones and Faddy (2003) established a tractable skewed extension of the symmetric student-t distribution known as the skew-t distribution by introducing a scaling factor $\sqrt{(a+b)/2}$ on the two degrees of freedom of the scaled Student-t distribution (Jones, 2002). Jones and Faddy (2003) implied that the skew-t(a,b) distributions are the distribution of the order statistics of the student-t distribution when (a) and (b) are integers. Jones (2004) provide detailed information on families of distributions arising from the distributions of order statistics because these distributions which are the skew-t distribution and the log F distribution appear to provide the most tractable instance of families with power and exponential tails. The skew-t distribution considered as an extension of the symmetric student-t distribution is derived by adding a skew parameter to the student-t distribution. Numerous authors such as Gupta *et al.* (2002), Johnson *et al.* (1995), Arellano-Valle and Genton (2005), Sahu *et al.* (2003), Azzalini and Capitanio (2003) and Hasan (2013) have introduced various forms of the skew-t distribution. Also, several authors have studied possible extensions and generalizations of the skew-t distribution such as the generalized skew-t distribution (Huang and Chen, 2006), generalized hyperbolic skew-t distribution (Aas and Haff, 2006), Balakrishnan skew-t distribution (Shafiei and Doostparast, 2014), Beta skew-t distribution (Shittu *et al.*, 2014), Exponentiated generalized skew-t distribution (Dikko and Agboola, 2017), Kumaraswamy skew-t distribution (Khamis *et al.*, 2017), Beta skew-t distribution (Basalamah *et al.*, 2018),

Corresponding author E-mail address: adubisiobinna@fuwukari.edu.ng
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Hybrid skew-t distribution (Adubisi *et al.*, 2022), flexible extended skew-t distribution (Adubisi *et al.*, 2023) and odd exponentiated skew-t distribution (Adubisi *et al.*, 2025). Cordeiro et al. (2016) proposed a new family of distributions which is called the exponentiated Gompertz generated family of distributions denoted as EGG-G. They extended this family of distributions to the gamma, normal and beta distributions by selecting EGG-G to be the corresponding distribution functions. In this article, a new skew-t distribution generalization based on the exponentiated Gompertz generated family of distributions is introduced. The new class of distribution called the exponentiated Gompertz generated skew-t (EGG_{ST}) distribution is quite capable of fitting and modeling skewed, long and heavy tailed datasets which is very crucial in this period of big data. It also contains the skew-t and other important distributions as sub-models.

The rest of this article is organized as follows. In Section 2, we define the new EGG_{ST} distribution and derive very useful mixture representations for the EGG_{ST} density and distribution functions. In section 4, related theoretical properties of the new distribution are presented. In section 4, the estimates of the model parameters are obtain using the maximum likelihood estimation (MLE) method. We perform a simulation study to assess the parameter estimates of the EGG_{ST} distribution in section 5. In section 6, we explore the flexibility and applicability of the new EGG_{ST} model by means of well-known real datasets while section 7 concludes the article.

2. Model Genesis

Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. Cordeiro *et al.* (2016) proposed the exponentiated Gompertz generated distribution denoted by $EGG-G(\alpha,\theta,\gamma)$ with cdf $F(x;\alpha,\theta,\gamma)$ and pdf $f(x;\alpha,\theta,\gamma)$ given as

$$F(x;\alpha,\theta,\gamma)=\left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-G(x)\right]^{-\gamma}\right)}\right\}^{\alpha} \tag{1}$$

$$f(x;\alpha,\theta,\gamma)=\frac{\alpha\theta g(x)e^{\frac{\theta}{\gamma}\left(1-\left[1-G(x)\right]^{-\gamma}\right)}}{\left[1-G(x)\right]^{1+\gamma}}\times\left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-G(x)\right]^{-\gamma}\right)}\right\}^{\alpha-1} \tag{2}$$

where $\alpha>0$, $\theta>0$ and $\gamma>0$ are three extra shape parameters. By taking $G(x)$ to be the cdf of the gamma, normal and beta distributions, Cordeiro *et al.* (2016) defined the EGG-Gamma, EGG-Normal and EGG-Beta distributions, respectively.

In this article, the $G(x)$ in (1) is the skew-t distribution function which leads to the introduction of a new distribution called the exponentiated Gompertz generated skew-t distribution denoted by $EGG_{ST}(\alpha,\theta,\gamma,\lambda)$ with cdf $F(x;\alpha,\theta,\gamma,\lambda)$ and pdf $f(x;\alpha,\theta,\gamma,\lambda)$ given as

$$F(x;\alpha,\theta,\gamma,\lambda)=\left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-G(x;\lambda)\right]^{-\gamma}\right)}\right\}^{\alpha} \tag{3}$$

$$f(x;\alpha,\theta,\gamma,\lambda)=\frac{\alpha\theta g(x;\lambda)e^{\frac{\theta}{\gamma}\left(1-\left[1-G(x;\lambda)\right]^{-\gamma}\right)}}{\left[1-G(x;\lambda)\right]^{1+\gamma}}\times\left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-G(x;\lambda)\right]^{-\gamma}\right)}\right\}^{\alpha-1} \tag{4}$$

where $x\in\Re$, $\alpha>0$, $\theta>0$, $\gamma>0$ and $G(x;\lambda)$, $g(x;\lambda)$ are the cdf and pdf of the skew-t distribution given by Jones (2001) and Jones and Faddy (2003) specified as

$$G(x;\lambda)=\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right),\qquad g(x;\lambda)=\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}$$

where $\lambda=a+b>0$, a and b are shape parameters. When $a=b$, the skew-t reduces to the usual symmetric student's t distribution on $2a$ degrees of freedom. Otherwise $a\neq b$ allows for skewness in the distribution.

From now onward, we will denote the skew-t distribution as $ST(\lambda)$ and exponentiated Gompertz generated skew-t distribution as $EGG_{ST}(\xi)$, where $\xi=(\alpha,\theta,\gamma,\lambda)$ are the set of parameters.

2.1 Linear Representation

According to Cordeiro *et al.* (2016), using the binomial expansion for $\alpha \in \Re^{+}$, the pdf of the EGG_{ST} in (4) can be written as

$$f(x;\alpha,\theta,\gamma,\lambda) = g(x;\lambda)\sum_{i,j=0}^{\infty} \vartheta_{i,j}\left[1-G(x;\lambda)\right]^{-\gamma(j+1)-1} \quad (5)$$

where the binomial coefficient $\theta_{i,j}$, for all real numbers is expressed as $\theta_{i,j} = \alpha\theta\frac{(-1)^{i+j}}{j!}\binom{\alpha-1}{i}\left(\frac{(i+1)\theta}{\gamma}\right)^{j} e^{\frac{(i+1)\theta}{\gamma}}$. The index (i) in the sum of (5) stops at $\alpha-1$ given that α is an integer. If γ is an integer, then (5) is the density $ST(\lambda)$ multiplied by the infinite weighted power series of $\left[1-G(x;\lambda)\right]$. Also, if γ is not considered as an integer, the term $\left[1-G(x;\lambda)\right]^{-\gamma(j+1)-1}$ can be expanded as follows

$$\left[1-G(x;\lambda)\right]^{-\gamma(j+1)-1} = \sum_{k=0}^{\infty}(-1)^{k}\binom{-\gamma(j+1)-1}{k}G(x;\lambda)^{k}$$

Therefore, the density $f(x;\alpha,\theta,\gamma,\lambda)$ in (4) can be written as

$$f(x;\alpha,\theta,\gamma,\lambda) = g(x;\lambda)\sum_{i,j,k=0}^{\infty} \vartheta_{i,j,k} G(x;\lambda)^{k} \quad (6)$$

where

$$\vartheta_{i,j,k} = \alpha\theta\frac{(-1)^{i+j+k}}{j!}\binom{\alpha-1}{i}\left(\frac{\left[(i+1)\theta\right]}{\gamma}\right)^{j}\binom{-\gamma(j+1)-1}{k} e^{\frac{(i+1)\theta}{\gamma}}$$

Further, the density $f(x;\alpha,\theta,\gamma,\lambda)$, after applying the binomial expansion on $G(x;\lambda)^{k}$ in (6) can be expressed as

$$f(x;\alpha,\theta,\gamma,\lambda) = \vartheta_{i,j,k,l}\, x^{l}\left(\lambda + x^{2}\right)^{-\left(\frac{3+l}{2}\right)} \quad (7)$$

where

$$\vartheta_{i,j,k,l} = \frac{\alpha\theta\lambda}{2^{k+1}}\sum_{i,j,k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^{i+j+k}}{j!}\binom{\alpha-1}{i}\left(\frac{(i+1)\theta}{\gamma}\right)^{j} \times \binom{-\gamma(j+1)-1}{k}\binom{k}{l} e^{\frac{(i+1)\theta}{\gamma}}$$

The linear representation of cdf of EGG_{ST} in (3), using the binomial expansion for $\alpha \in \Re^{+}$, is given as

$$F(x;\alpha,\theta,\gamma,\lambda) = \vartheta_{i,j,k,l}\, x^{l}\left(\lambda + x^{2}\right)^{-\frac{l}{2}} \quad (8)$$

where

$$\vartheta_{i,j,k,l} = \frac{1}{2^{k}}\sum_{i,j,k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^{i+j+k}}{j!}\binom{\alpha}{i}\left(\frac{i\theta}{\gamma}\right)^{j}\binom{-j\gamma}{k}\binom{k}{l} e^{\frac{i\theta}{\gamma}}$$

Furthermore, an expansion for $\left[F(x)\right]^{s}$, where s is an integer, the binomial expansion is given as

$$F(x;\alpha,\theta,\gamma,\lambda)^{s} = \vartheta_{a,b,c,d,e}\, x^{e}\left(\lambda + x^{2}\right)^{-\frac{e}{2}} \quad (9)$$

where

$$\vartheta_{a,b,c,d,e} = \frac{1}{2^{d}}\sum_{a=0}^{s}\sum_{b,c,d=0}^{\infty}\sum_{e=0}^{d}\frac{(-1)^{b+c+d}}{c!}\binom{s}{a}\binom{\alpha}{b}\left(\frac{b\theta}{\gamma}\right)^{c} \times \binom{-c\gamma}{d}\binom{d}{e} e^{\frac{b\theta}{\gamma}}$$

To understand the effect of each parameter in determining the overall shape of the EGG_{ST} density, we present some graphs which depicts the shape of the EGG_{ST} distribution when three parameters are fixed while the fourth parameter is varied. In Figure 1, we fixed the parameters $(\theta = 1,\ \gamma = 0.4\ \ \lambda = 1)$ and plot the $EGG_{ST}(\alpha,1,0.4,1)$ density for selected values of α. It is observed from Figure 1, that the left tail of the EGG_{ST} density gets lighter as α increases. Likewise, we note that the parameter θ when varied and all other parameters are fixed $(\alpha = 1,\ \gamma = 1\ \ \lambda = 0.5)$, controls the right tail weight of the EGG_{ST} density as shown in Figure 2. Further, we observe from Figure 1 and 2 that as α and θ approach infinity the EGG_{ST} density tends to zero. Figure 3, depicts the effect of the γ parameter when varied on the shape of the EGG_{ST} density by fixing the other parameters $(\alpha = 1,\ \gamma = 1\ \ \lambda = 0.5)$. It is observed from Figure 3, that the γ parameter controls the leptokurtic behaviour of the EGG_{ST} density which tends to zero as γ increases. Figure 4, shows the effect of the parameter λ on the shape of the EGG_{ST} density by fixing the parameters $(\alpha = 1.5,\ \ \theta = 1\ \ \gamma = 0.5)$ and taking the parameter λ ranging from 0.5 to 30. Further, It is observed that as λ increases the EGG_{ST} density tend towards a flat curve.

3. Statistical Properties

In this section, some basic statistical properties of the EGG_{ST} distribution using the established expansions are investigated.

3.1 Quantile function

The EGG$_{ST}$ quantile function is by inverting (3). The quantile function $Q(u)$, $u \in (0,1)$ for the EGG$_{ST}$ is given as

$$Q(u)=\lambda^{\frac{1}{2}}\frac{\left[1-2\left(1-\left(\frac{\gamma}{\theta}\log\left[1-u^{\frac{1}{\alpha}}\right]\right)\right)^{-\frac{1}{\gamma}}\right]}{\left[1-\left(1-2\left(1-\left(\frac{\gamma}{\theta}\log\left[1-u^{\frac{1}{\alpha}}\right]\right)\right)^{-\frac{1}{\gamma}}\right)^{2}\right]^{\frac{1}{2}}} \quad (10)$$

where U is a uniform random variable on (0, 1).

The median, 25th and 75th percentiles are obtained by setting $u=0.5$, $u=0.25$ and $u=0.75$ in (10). Given that the uniform random variables are easily generated in most statistical softwares, the quantile function is considered very useful in generating the EGG$_{ST}$ random variates. Hence, Kenney and Keeping (1962) and Moors (1988) introduced a quantile-based approach for computing numerical values for the Bowley skewness and Moors kurtosis, respectively, given as

$$S_k=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}, \quad \text{and}$$

$$K_s=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$$

where $Q(.)$ is the EGG$_{ST}$ quantile function. Table 1, provides some numerical values for the median, 25th and 75th percentiles, skewness, kurtosis and interquartile range (IQR) for some carefully chosen parameter values. We observe that the median, 25th and 75th percentiles, and IQR decreases as λ increases across increasing values of α, θ and γ. Further, the skewness and kurtosis values decrease with the same numerical values as α, θ, γ and λ increases.

3.2 Moment

In this subsection, we derive the rth ordinary moment for the EGG$_{ST}$ distribution. Let $X \sim EGG_{ST}(\xi)$ be a random variable, the rth moment of X about the origin is expressed as

$$\mu_r=\mathrm{E}\left(X^r\right)=\int_{-\infty}^{+\infty}x^r f(x,\xi)\,dx \quad (11)$$

By inserting (7) in (11), we have

$$\mathrm{E}\left(X^r\right)=\int_{-\infty}^{+\infty}x^r\vartheta_{i,j,k,l}x^l\left(\lambda+x^2\right)^{-\left(\frac{3+l}{2}\right)}dx \quad (12)$$

The moment obtained after some algebra simplification is given as

$$\mu_r'=\mathrm{E}\left(X^r\right)=\vartheta_{i,j,k,l}\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+l+1}{2},\frac{2-r}{2}\right) \quad (13)$$

The mean and variance of EGG$_{ST}$ distribution using (13), are given as follows

$$\mathrm{E}(X)=\vartheta_{i,j,k,l}\lambda^{-\frac{1}{2}}\mathrm{B}\left(\frac{l+2}{2},\frac{1}{2}\right), \text{ and}$$

$$V(X)=\vartheta_{i,j,k,q}\lambda^{0}\mathrm{B}\left(\frac{l+3}{2},0\right)-\left[\vartheta_{i,j,k,q}\lambda^{-\frac{1}{2}}\mathrm{B}\left(\frac{l+2}{2},\frac{1}{2}\right)\right]^2$$

Furthermore, the moment generating function of EGG$_{ST}$ distribution is given as

$$M_X(t)=\vartheta_{i,j,k,l}\sum_{r=0}^{\infty}\frac{t^r}{r!}\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+l+1}{2},\frac{2-r}{2}\right) \quad (14)$$

and the characteristic function expressed as

$$\Phi_X(t)=\vartheta_{i,j,k,q}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+l+1}{2},\frac{2-r}{2}\right) \quad (15)$$

3.3 Probability Weighted Moments

The probability weighted moment (PWM) of a random variable X is a very useful mathematical quantity proposed by Greenwood *et al.* (1979). The PWM, $\tau_{r,s}$ is defined as

$$\tau_{r,s}=\mathrm{E}\left[X^rF(x)^s\right]=\int_{-\infty}^{+\infty}x^r f(x)\left(F(x)\right)^s dx \quad (16)$$

By inserting (7) and (9) in (16), the PWM of the EHL$_{ST}$ is given as

$$\tau_{r,s}=\omega^*\int_{-\infty}^{+\infty}x^{r+l+e}\left(\lambda+x^2\right)^{-\left(\frac{3+l+e}{2}\right)}dx \quad (17)$$

So, after some algebraic simplifications, the PWM is given as

$$\tau_{r,s}=\omega^*\lambda^{\frac{r-2}{2}}\mathrm{B}\left(\frac{r+l+e+1}{2},\frac{2-r}{2}\right) \quad (18)$$

where

$$\omega^*=\frac{\alpha\theta\lambda}{2^{k+d+1}}\sum_{i,j,k,,b,c,d=0}^{\infty}\sum_{a=0}^{s}\sum_{l=0}^{k}\sum_{e=0}^{d}\frac{(-1)^{i+j+k+b+c+d}}{j!c!}\left(\frac{(i+1)\theta}{\gamma}\right)^j \times\left(\frac{c\theta}{\gamma}\right)^c\binom{\alpha-1}{i}\binom{-\gamma(j+1)-1}{k}\binom{k}{l} \times\binom{s}{a}\binom{\alpha}{b}\binom{-\gamma c}{d}\binom{d}{e}e^{\frac{((i+1)+b)\theta}{\gamma}}$$

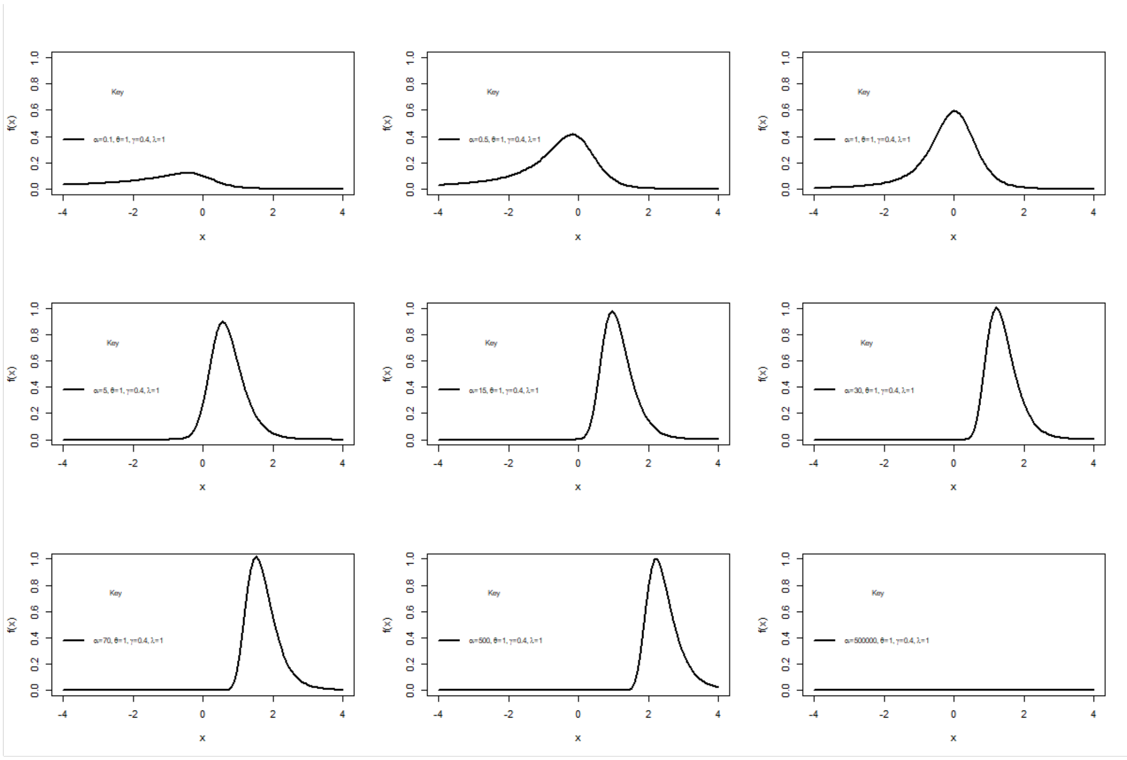

Figure 1. $EGG_{ST}\left(\alpha,1,0.4,1\right)$ density as the parameter α varies.

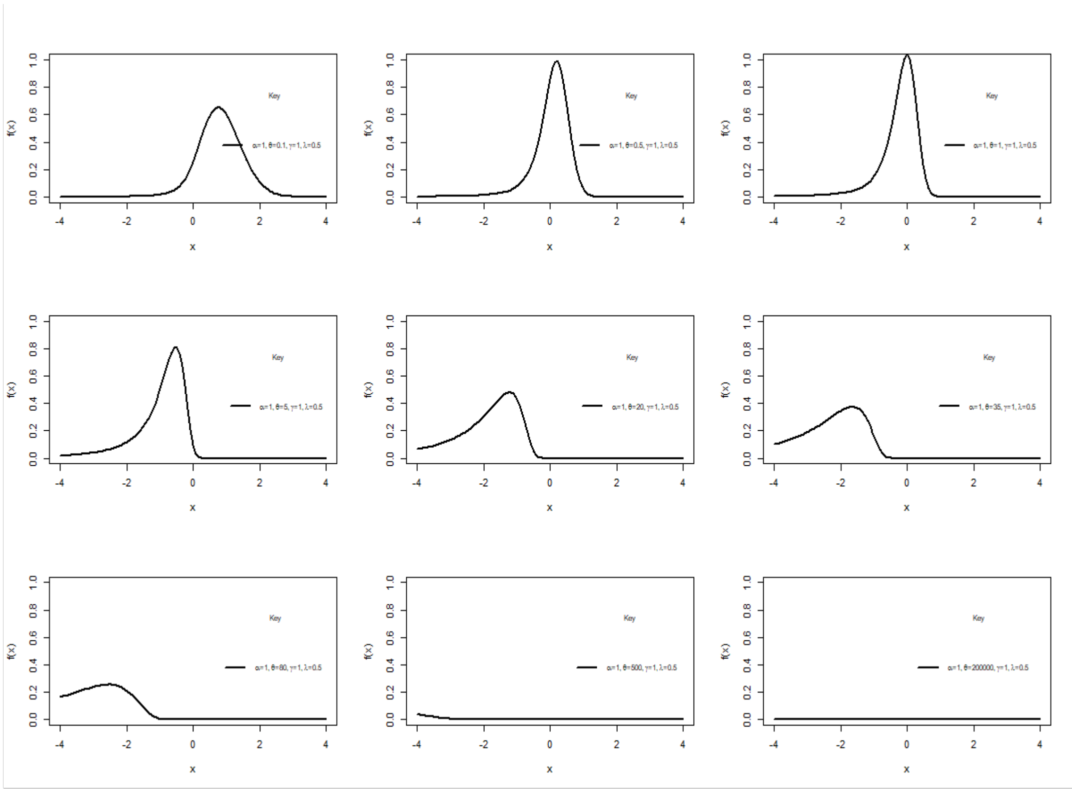

Figure 2: $EGG_{ST}\left(1,\theta,1,0.5\right)$ density as the parameter θ varies.

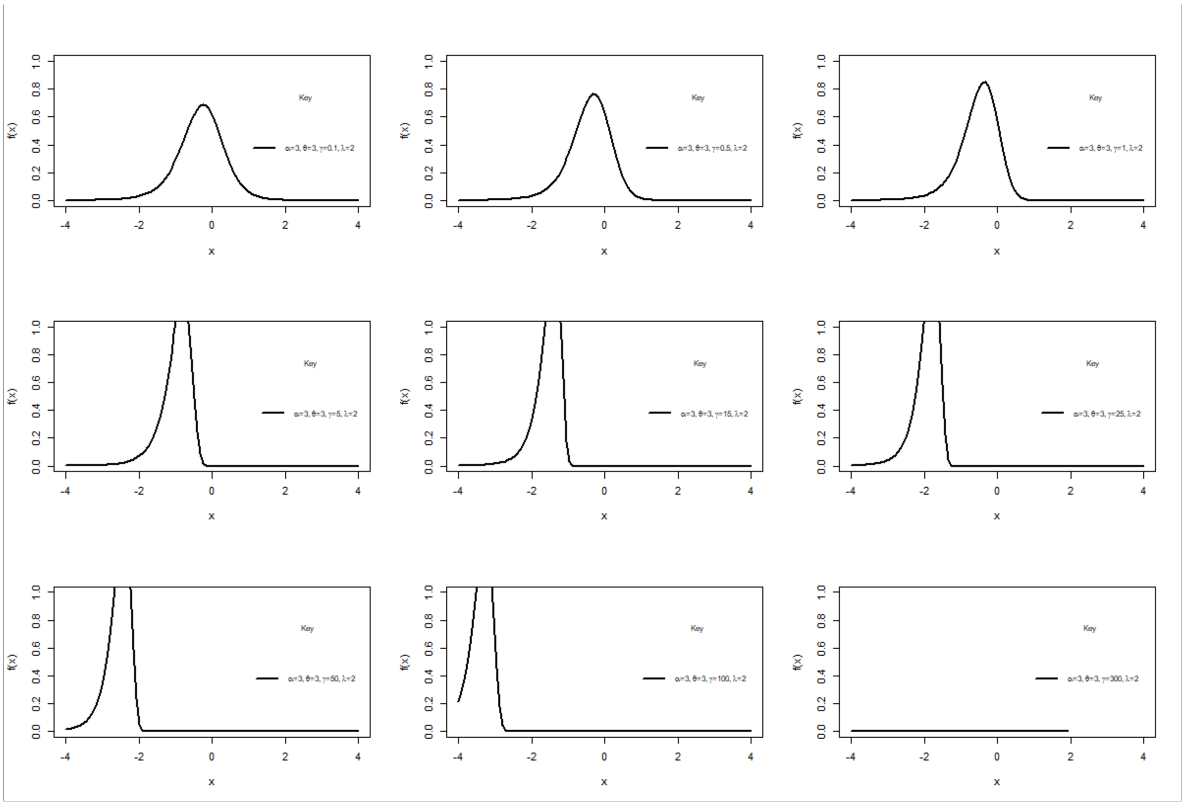

Figure 3: $EGG_{ST}\left(3,3,\gamma,2\right)$ density as the parameter γ varies.

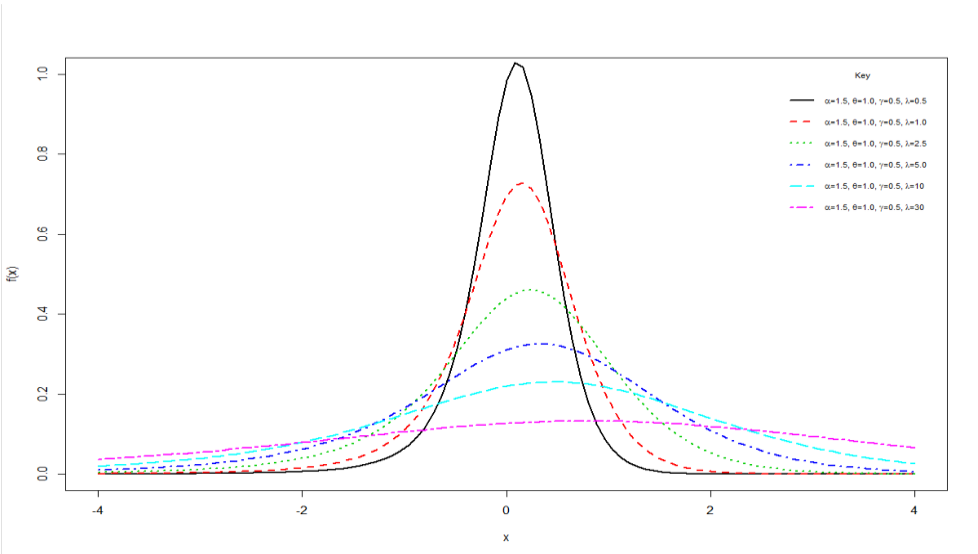

Figure 4: $EGG_{ST}\left(1.5,1,0.5,\lambda\right)$ density as the parameter λ varies.

Table 1: Median (M), 25th and 75th percentiles, skewness (Sk), kurtosis (Ks) and IQR

λ	α	θ	γ	**M**	**25th**	**75th**	**Sk**	**Ks**	**IQR**
0.5	0.8	0.2	0.1	1.0157	0.1515	3.6399	0.5045	1.9935	3.4885
	1.0	0.3	0.2	0.6944	0.1206	1.7109	0.2784	0.8422	1.5903
	1.2	0.4	0.3	0.5271	0.0928	1.1234	0.1571	0.4321	1.0305
1.0	0.8	0.2	0.1	1.4364	0.2142	5.1477	0.5045	1.9935	4.9334

	1.0	0.3	0.2	0.9820	0.1706	2.4196	0.2784	0.8422	2.2490
	1.2	0.4	0.3	0.7455	0.1312	1.5887	0.1571	0.4321	1.4574
1.5	0.8	0.2	0.1	1.7592	0.2624	6.3046	0.5045	1.9935	6.0422
	1.0	0.3	0.2	1.2027	0.2089	2.9634	0.2784	0.8422	2.7544
	1.2	0.4	0.3	0.9130	0.1607	1.9457	0.1571	0.4321	1.7850

3.4 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are order statistics obtained from the sample. According to David (1981), the pdf, $f_{p:n}(x)$ of the pth order statistics $X_{p:n}$ is defined as

$$f_{p:n}(x) = \frac{g(x)}{\mathrm{B}(p, n-p+1)} \left[G(x)\right]^{p-1} \left[1 - G(x)\right]^{n-p} \tag{19}$$

where, $G(x)$ and $g(x)$ are the cdf and pdf of the EGG$_{ST}$ distribution respectively, and $\mathrm{B}(.,.)$ is the beta function. Since $0 < G(x) < 1$ for $x > 0$ using the binomial theorem on $\left[1 - G(x)\right]^{n-p}$, we have

$$f_{p:n}(x) = \frac{1}{\mathrm{B}(p, n-p+1)} \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} \left[G(x)\right]^{p+l-1} g(x) \tag{20}$$

Expanding, after inserting (3) and (4) in (20), the pth order statistics for EGG$_{ST}$ distribution is given as

$$f_{p:n}(x) = \frac{1}{\mathrm{B}(p, n-p+1)} \vartheta_{i,j,l,k,m} x^m \left(\lambda + x^2\right)^{-\left(\frac{3+m}{2}\right)} \tag{21}$$

where,

$$\vartheta_{i,j,l,k,m} = \frac{\alpha\theta\lambda}{2^{k+1}} \sum_{l=0}^{n-p} \sum_{i=o}^{\alpha(p+l)-1} \sum_{j,k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^{l+i+j+k}}{j!} \binom{n-p}{l} \times \left(\frac{(i+1)\theta}{\gamma}\right)^j \binom{\alpha(p+l)-1}{i} \binom{-\gamma(j+1)-1}{k} \binom{k}{m} e^{\frac{(i+1)\theta}{\gamma}}$$

Setting $p = 1$ and $p = n$ in (21), the minimum and maximum order statistics can be derived. Furthermore, the rth moment of the pth order statistics for EGG$_{ST}$ distribution is expressed as

$$\mathrm{E}\left(X_{p:n}^r\right) = \int_{-\infty}^{+\infty} x^r f_{p:n}(x; \xi)\, dx \tag{22}$$

By inserting (21) in (22), we have

$$\mathrm{E}\left(X_{p:n}^r\right) = \frac{1}{\mathrm{B}(p, n-p+1)} \vartheta_{i,j,l,k,m} \int_{-\infty}^{+\infty} x^{r+m} \left(\lambda + x^2\right)^{-\left(\frac{3+m}{2}\right)} dx \tag{23}$$

So, after some algebraic simplifications, the rth moment of the pth order statistics is given as

$$\mathrm{E}\left(X_{p:n}^r\right) = \frac{1}{\mathrm{B}(p, n-p+1)} \vartheta_{i,j,l,k,m} \lambda^{\frac{r-2}{2}} \mathrm{B}\left(\frac{r+m+1}{2}, \frac{2-r}{2}\right) \tag{24}$$

3.5 Entropies

The entropy of a random variable X is a measure of the variation of uncertainty. According to Rényi (1961), the Rényi entropy of a random variable with pdf $f(x)$ is given as

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta} \log \int_{-\infty}^{+\infty} f(x)^\delta\, dx, \quad \delta > 0 \quad \text{and} \quad \delta \neq 1 \tag{25}$$

By inserting (4) in (25), applying the binomial theorem and some algebraic simplification, the Rényi Entropy of EGG$_{ST}$ distribution is given as

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta} \log\left(\vartheta_{i,j,k,l} \lambda^{\frac{1-3\delta}{2}} \mathrm{B}\left(\frac{l+1}{2}, \frac{3\delta-1}{2}\right)\right) \tag{26}$$

where,

$$\vartheta_{i,j,q} = \frac{(\alpha\theta\lambda)^\delta}{2^{l+\delta}} \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{i+j+k}}{j!} \binom{\delta(\alpha-1)}{i} \left(\frac{(i+\delta)\theta}{\gamma}\right)^j \times \binom{-\gamma(j+\delta)-\delta}{k} \binom{k}{l} e^{\frac{(i+\delta)\theta}{\gamma}}$$

Furthermore, the q-entropy is defined as

$$\mathrm{H}_{(\delta)} = \frac{1}{\delta-1} \log\left(1 - \int_{-\infty}^{+\infty} f(x)^\delta\, dx\right), \quad \delta > 0 \quad \text{and} \quad \delta \neq 0 \tag{27}$$

Therefore, the q-entropy of EHL$_{ST}$ distribution is given as

$$\mathrm{H}_{(\delta)} = \frac{1}{\delta-1} \log\left(1 - \left(\vartheta_{i,j,k,l} \lambda^{\frac{1-3\delta}{2}} \mathrm{B}\left(\frac{l+1}{2}, \frac{3\delta-1}{2}\right)\right)\right) \tag{28}$$

4. Model Estimation

In this section, the maximum likelihood estimates (MLE's) of the EGG$_{ST}$ parameters are determined based on complete samples. Consider a sample $x_1, x_2, \ldots, x_n$ observed from the EGG$_{ST}$ distribution with unknown parameter vector $\xi = (\alpha, \theta, \gamma, \lambda)^T$. The log-likelihood function, say $l(\xi)$, is given as

$$l(\xi)=n\log\alpha+n\log\theta+n\log\lambda-n\log 2-3/2\sum_{i=1}^{n}\log\left(\lambda+x_i^2\right)-(\gamma+1)\sum_{i=1}^{n}\log\left(z_i\right)+\sum_{i=1}^{\infty}\log\left(1-l_i\right)+(\alpha-1)\sum_{i=1}^{n}\log\left(l_i\right)$$
(29)

where $z_i=\left(1-\frac{1}{2}\left(1+\frac{x_i}{\sqrt{\lambda+x_i^2}}\right)\right)$ and $l_i=1-e^{\frac{\theta}{\gamma}\left(1-\left(1-\left(\frac{1}{2}\left(1+\frac{x_i}{\sqrt{\lambda+x_i^2}}\right)\right)\right)^{-\gamma}\right)}$

The associated score function $U(\xi)=\left[\frac{\partial l(\xi)}{\partial\alpha},\frac{\partial l(\xi)}{\partial\theta},\frac{\partial l(\xi)}{\partial\gamma},\frac{\partial l(\xi)}{\partial\lambda}\right]^T$ components are given as

$$U_\alpha(\xi)=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(l_i\right),$$

$$U_\theta(\xi)=\frac{n}{\theta}-\sum_{i=1}^{n}\frac{\left(l_i\right)^{\theta}}{1-\left(l_i\right)}+(\alpha-1)\sum_{i=1}^{n}\frac{\left(l_i\right)^{\theta}}{\left(l_i\right)},$$

$$U_\gamma(\xi)=-\sum_{i=1}^{n}\log\left(z_i\right)-\sum_{i=1}^{n}\frac{\left(l_i\right)^{\gamma}}{1-\left(l_i\right)}+(\alpha-1)\sum_{i=1}^{n}\frac{\left(l_i\right)^{\gamma}}{\left(l_i\right)},$$

and

$$U_\lambda(\xi)=\frac{n}{\lambda}-\frac{3}{2}\sum_{i=1}^{n}\frac{1}{\left(\lambda+x_i^2\right)}-(\gamma+1)\sum_{i=1}^{n}\frac{x_i}{4\left(\lambda+x_i^2\right)^{3/2}\left(z_i\right)}+\theta\sum_{i=1}^{n}\frac{x_i\left(z_i\right)^{-\gamma}}{4\left(\lambda+x_i^2\right)^{3/2}\left(z_i\right)}-\theta(\alpha-1)\sum_{i=1}^{n}\frac{x_i\left(z_i\right)^{-\gamma}\left(1-l_i\right)}{4\left(\lambda+x_i^2\right)^{3/2}\left(z_i\right)\left(l_i\right)}$$

The maximum likelihood estimates (MLEs) of the parameters can be obtained by solving the components of the score vector simultaneously. Estimation of each parameter can be obtained using one of the various numerical procedures available on different computational softwares. We used the *optim* and *AdequacyMode*l functions which are available in R software to do so.

5. Simulation Study

In this section, we appraise the efficiency and flexibility of the EGG_{ST} distribution using simulation study. The simulation is carried out as follows:

- Data are generated using the quantile function of the EGG_{ST} distribution.

$$x=\lambda^{\frac{1}{2}}\frac{\left[1-2\left(1-\left(\frac{\gamma}{\theta}\log\left[1-u^{\frac{1}{\alpha}}\right]\right)\right)^{-\frac{1}{\gamma}}\right]}{\left[1-\left(1-2\left(1-\left(\frac{\gamma}{\theta}\log\left[1-u^{\frac{1}{\alpha}}\right]\right)\right)^{-\frac{1}{\gamma}}\right)^2\right]^{\frac{1}{2}}},\quad 0<u<1$$

- The selected parameter values are set as follows: $\alpha=1.5$, $\theta=1.7$, $\gamma=2.0$ and $\lambda=0.5$.
- The selected sample sizes are $n=30,50,150,250,300$ and 1000 .
- Generated 10,000 samples for each sample size.

The performance of the estimates is evaluated through the average estimates (MEs), absolute bias, variance, mean square errors (MSE) and root mean square errors (RMSE) for the different sample sizes. The absolute bias, MSE and RMSE are computed for $S=\hat{\alpha},\hat{\theta},\hat{\gamma},\hat{\lambda}$, using

$$Abs\hat{B}ias_s=\left|\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{S}_i-S\right)\right|$$

$$\hat{M}SE_s=\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{S}_i-S\right)^2$$

$$\hat{R}MSE_s=\sqrt{\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{S}_i-S\right)^2}$$

The simulation numerical results tabulated in Table 2, show that the bias decreases as the sample size increases. The MSE and RMSE for each parameter decreases as the sample size increases. This shows the consistency property of the MLEs.

Table 2: Simulation results from EGG_{ST} distribution

Sample size	Parameter	Average estimate	AbsBias	Variance	MSE	RMSE

30	α	1.7418	0.2418	0.6769	0.7353	0.8575
	θ	1.8355	0.1355	0.5864	0.6048	0.7777
	γ	2.1425	0.1425	0.4309	0.4512	0.6717
	λ	0.5476	0.0476	0.0864	0.0886	0.2977
50	α	1.6660	0.1660	0.3953	0.4229	0.6503
	θ	1.7750	0.0750	0.3831	0.3887	0.6234
	γ	2.1101	0.1101	0.3086	0.3208	0.5664
	λ	0.5419	0.0419	0.0538	0.0555	0.2357
150	α	1.5639	0.0639	0.1011	0.1051	0.3242
	θ	1.7028	0.0028	0.1238	0.1238	0.3518
	γ	2.0724	0.0724	0.1270	0.1322	0.3636
	λ	0.5286	0.0286	0.0179	0.0187	0.1368
250	α	1.5389	0.0389	0.0544	0.0559	0.2365
	θ	1.6865	0.0135	0.0694	0.0696	0.2638
	γ	2.0647	0.0647	0.0789	0.0831	0.2882
	λ	0.5237	0.0237	0.0109	0.0115	0.1072
300	α	1.5351	0.0351	0.0458	0.0471	0.2169
	θ	1.6909	0.0091	0.0583	0.0583	0.2415
	γ	2.0528	0.0528	0.0688	0.0716	0.2675
	λ	0.5204	0.0204	0.0089	0.0093	0.0965
1000	α	1.5164	0.0164	0.0130	0.0133	0.1152
	θ	1.6831	0.0169	0.0131	0.0134	0.1157
	γ	2.0374	0.0373	0.0198	0.0212	0.1456
	λ	0.5149	0.0149	0.0036	0.0038	0.0619

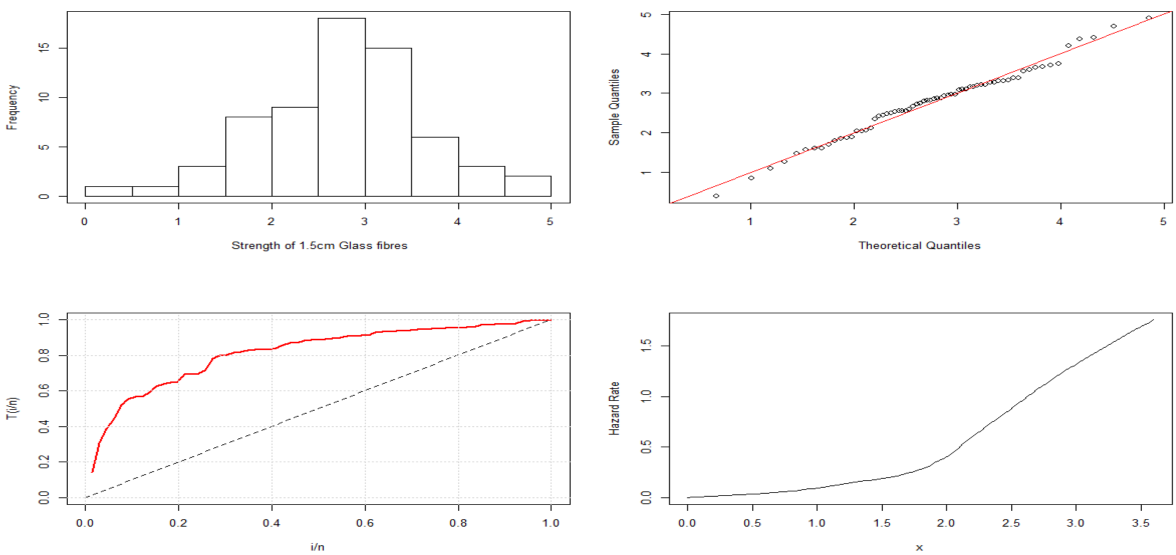

Figure 5: Descriptive plots for the first dataset.

6. Applications

In this section, we illustrate the flexibility of the EGG_{ST} model using well-known real datasets. The following criteria which include the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are used for model's comparison. We also consider the negative log-likelihood, Cramer-von Mises (W*), Anderson Darling (A*), Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value.

The first dataset used here is the breaking stress of the 50 mm length (GPa) carbon fibres, reported by Nicholas and Padgett (2006). The observations are: 0.39, 1.08, 4.70, 1.25, 1.47, 1.57, 1.61, 0.85, 1.61, 1.69, 1.80, 1.84,1.87, 1.89, 2.03, 2.03, 2.05,.12, 2.35, 2.41, 2.43, 2.48, 2.50,2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82,.90,2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15,3.15, 3.19, 3.22, 3.22, 3.27,.28, 3.31, 3.31, 3.33, 3.39, 3.39,3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42. This data has been previously analysed by Reyad et al. (2018), Yousof *et al.* (2017), Al-Aqtash *et al.* (2014) and Cordeiro and Lamonte (2011). The second dataset on the strengths of 1.5 cm glass fibres were obtained by workers at the UK National Physical Laboratory. The observations are: 1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288,1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.460, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.670, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.910, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197,4.121. Authors such as Eghwerido *et al.* (2020), Reyad and Othman (2017), Merovci *et al.* (2016), Bourguignon *et al.* (2014) and Smith and Naylor (1987) have previously used this data.

Figure 5 and 6 present the histogram, normal Q-Q plot, TTT and hazard rate plots of the first and second datasets. The histograms show that the datasets are skewed to the left while the TTT plots show that we have an increasing hazard rate which the EGG_{ST} model is capable of handling.

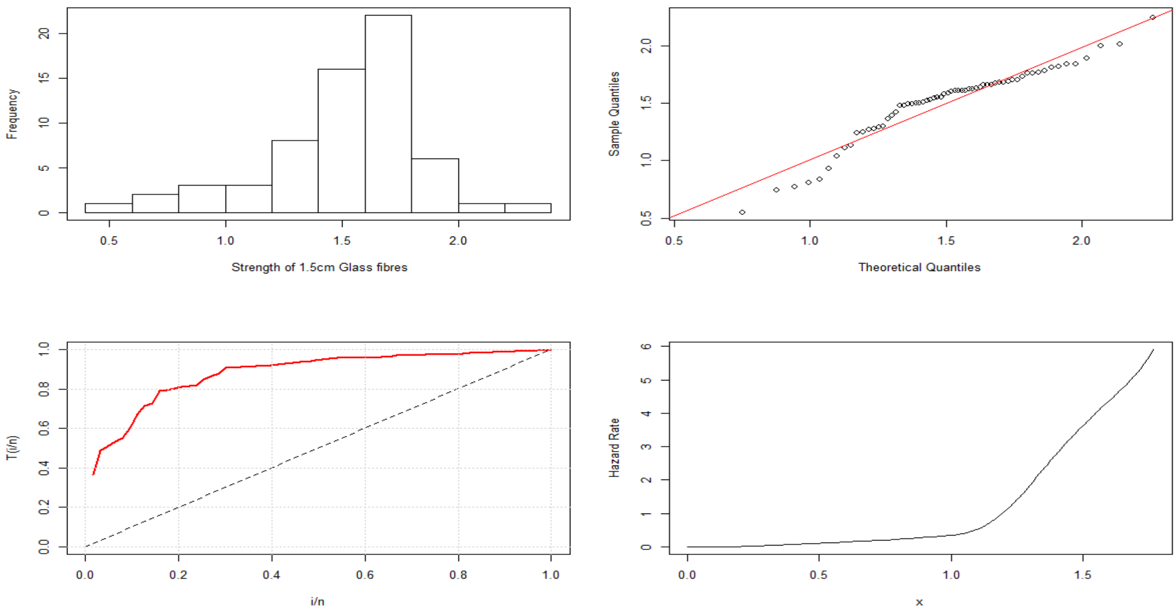

Figure 6: Descriptive plots for the second dataset.

The EGG_{ST} distribution is compared with skew-t (ST), type I half logistic skew-t ($TIHL_{ST}$), exponentiated half logistic skew-t (EHL_{ST}), Gompertz Lomax (GO_L) and Gompertz Weibull (GO_W) distributions. Selection of the competing models done based on the model's relationship for test statistics efficient conclusion. The MLEs and some statistical criteria of the fitted models for the first and second datasets are presented in Tables 3 and 4, respectively. The results in Table 3 and 4 show that the EGG_{ST} distribution has the lowest values for AIC, BIC, W*, A*, KS and highest p-value among the fitted models for the two datasets. Hence, the EGG_{ST} model best fit the stress strength datasets. Furthermore, a closer look at the estimated pdfs and cdfs curves in Figure 7 and 8, show the superiority of the new EGG_{ST} model over the other models for the two datasets.

Table 3: The MLEs and some statistical criteria for the first dataset.

Model	Estimates				-LL	AIC	BIC	W*	A*	KS	p-values
EGG_{ST} $(\alpha,\theta,\gamma,\lambda)$	3.911 (16.213)	0.184 (1.199)	1.683 (81.427)	11.319 (80.247)	**85.420**	**178.841**	**187.600**	**0.059**	**0.360**	**0.071**	**0.889**
EHL_{ST} (α,β,λ)	3.814 (0.867)	70.886 (43.992)	20.737 (9.503)	-	92.078	190.156	196.725	0.259	1.383	0.130	0.213
$TIHL_{ST}$ (α,λ)	0.714 (0.100)	5.211 (1.755)	-	-	151.941	307.881	312.261	0.209	1.108	0.422	1.2e-10
ST (λ)	14.912 (3.313)	-	-	-	177.063	356.125	358.315	0.140	0.735	0.617	2.2e-16
GO_L (α,γ,a,b)	0.013 (0.013)	8.002 (13.942)	0.898 (0.812)	0.605 (1.160)	85.672	179.345	188.103	0.068	0.441	0.085	0.722
GO_W (α,γ,a,b)	1.048 (0.012)	0.020 (0.001)	0.321 (0.118)	3.399 (0.577)	86.064	180.127	188.886	0.093	0.528	0.082	0.761

Table 4: The MLEs and some statistical criteria for the second dataset.

Model	Estimates				-LL	AIC	BIC	W*	A*	KS	p-values
EGG_{ST} $(\alpha,\theta,\gamma,\lambda)$	2.718 (4.780)	0.012 (0.009)	4.535 (8.275)	6.760 (35.985)	**14.415**	**36.830**	**45.402**	**0.188**	**1.041**	**0.143**	**0.154**
EHL_{ST} (α,β,λ)	4.001 (0.747)	180.007 (97.365)	4.319 (1.768)	-	31.442	68.884	75.313	0.734	4.030	0.212	0.007
$TIHL_{ST}$ (α,λ)	0.731 (0.103)	1.682 (0.551)	-	-	106.172	216.344	220.630	0.549	3.024	0.452	1.3e-11
ST (λ)	4.504 (1.003)	-	-	-	130.259	262.518	264.661	0.467	2.571	0.649	2.2e-16
GO_L (α,γ,a,b)	0.005 (0.002)	8.179 (2.298)	0.507 (0.153)	1.516 (0.450)	14.503	37.005	45.578	0.168	0.946	0.154	0.099
GO_W (α,γ,a,b)	0.223 (0.175)	0.009 (0.001)	0.798 (0.346)	5.615 (1.055)	15.188	38.377	46.949	0.233	1.283	0.152	0.109

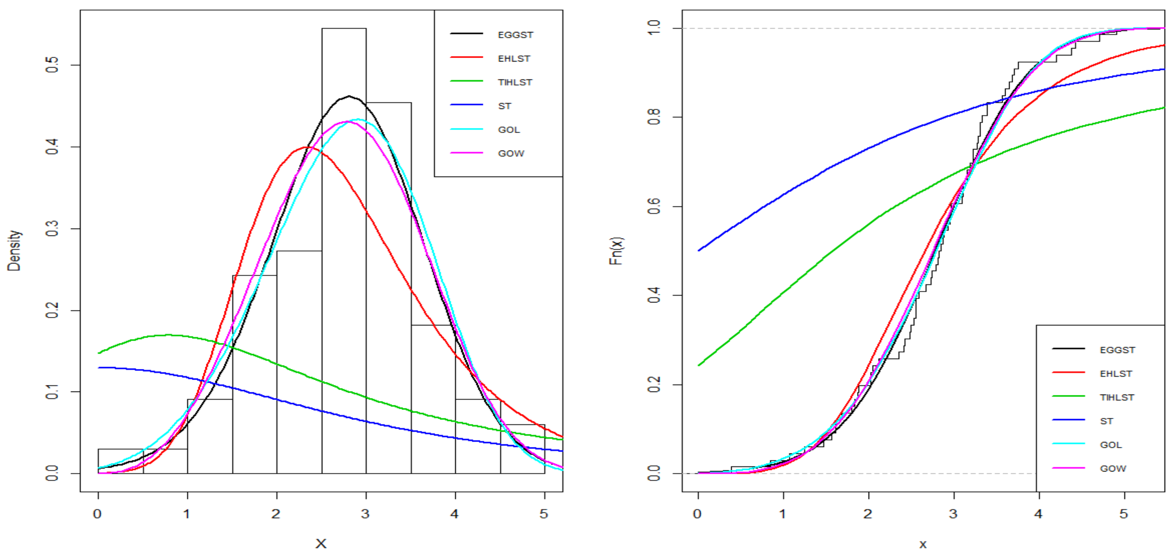

Figure 7: Estimated pdfs and cdfs plots for the first dataset

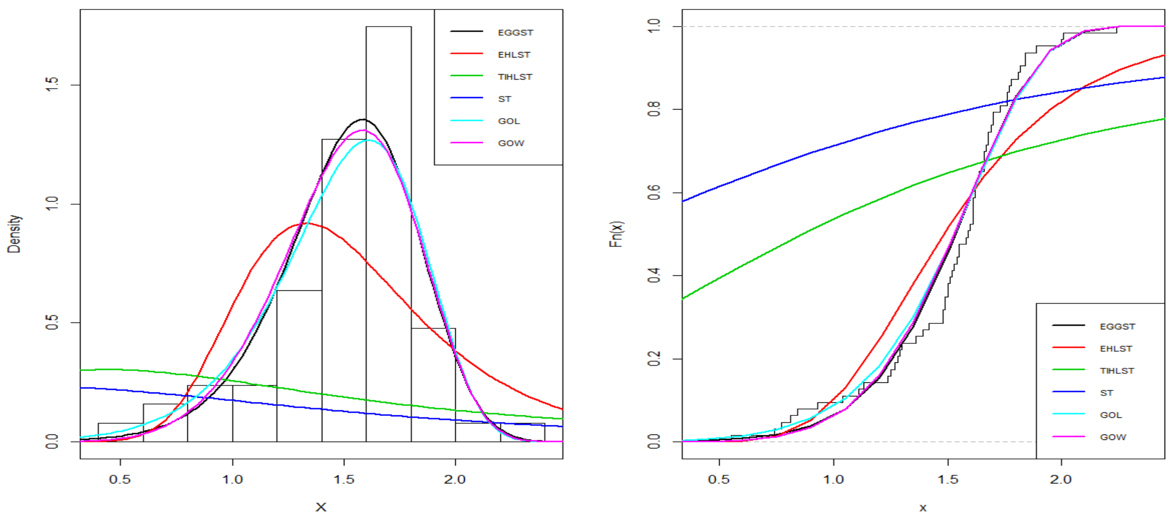

Figure 8: Estimated pdfs and cdfs plots for the second dataset.

7. Conclusion

We propose a new continuous distribution called the exponentiated Gompertz generated skew-t (EGG_{ST}) distribution which extends the applicability of the skew-t distribution by Jones (2001) and Jones and Faddy (2003). Some important structural properties of the new EGG_{ST} model were derived. The EGG_{ST} model provides more flexibility in modelling left skewed, long and heavy-tailed datasets than the skew-t model. Maximum likelihood estimation used to estimate the parameters of the model and simulation study performed to assess the finite sample performance of the estimates. Applicability of the new EGG_{ST} model was demonstrated using two well-known breaking stress and strength datasets.

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