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On the Analysis of Variance for Symmetrical Triangular and Normal Fuzzy Observations in a Randomized Complete Block Design

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ABSTRACT

Fuzzy ANOVA has been successfully applied in one-way experimental designs, where fuzzy observations are modelled using symmetrical triangular fuzzy numbers (STFNs) and normal fuzzy numbers (NFNs). These approaches have proven helpful in various fields, such as environmental sciences, agriculture, and engineering, where data uncertainty is pervasive. However, previous work has focused mainly on Completely Randomized Designs (CRD), which assume homogeneity among experimental units. This assumption often oversimplifies real-world conditions where variability among experimental units cannot be ignored. Thus, this study proposed fuzzy Analysis of Variance (ANOVA) within a Randomised Complete Block Design (RCBD) framework using Symmetric Triangular Fuzzy Numbers (STFNs) and Normal Fuzzy Numbers (NFNs) to address measurement imprecision commonly encountered in experimental data. The methodology preserves the structural integrity of classical ANOVA while incorporating fuzziness to enhance data interpretation in the presence of uncertainty. Simulated datasets were analyzed to evaluate the performance of the proposed fuzzy ANOVA, and both treatment and block effects were found to be statistically significant, demonstrating the method's effectiveness in detecting variability and treatment differences. STFNs provided a simple approach for modeling symmetric uncertainty, whereas NFNs offered smoother, more continuous representations, including visual tools such as triangular and Gaussian membership function plots, which further improved result interpretation and accessibility. The findings affirm that fuzzy ANOVA is a robust and interpretable extension of classical ANOVA, suitable for various areas where uncertain data is prevalent.

1. Introduction

Analysis of variance (ANOVA) is a critical statistical method widely used in various fields for comparing the means of multiple groups to determine if there are significant differences between them [1]. While classical ANOVA assumes precise numerical data, real-world scenarios often involve imprecise, uncertain, or vague information, which cannot be adequately represented by crisp values. This has necessitated the extension of classical statistical methods to fuzzy environments, leading to the development of fuzzy ANOVA methods. The concept of fuzzy sets, introduced by [2], provides a mathematical framework for modelling uncertainty and imprecision. Fuzzy ANOVA leverages this concept to analyze data expressed in terms of fuzzy numbers or intervals, offering more flexibility in dealing with ambiguous observations. This approach is particularly useful in applications where data uncertainty arises due to subjective judgment, measurement errors, or inherent variability in the system being studied [3]. Early studies on fuzzy ANOVA explored the behaviour and

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foundational aspects of fuzzy statistical methods [4]. Subsequent advancements introduced methodologies for hypothesis testing in fuzzy environments, such as the fuzzy p-value approach [5] and the bootstrap-based techniques for fuzzy data [6]. These methods have been applied to various domains, including agriculture [7], engineering (Kaya & Kahraman, 2011), and process capability analysis [8].

One key development in this field is the use of fuzzy random variables, which generalize classical random variables to incorporate fuzziness in observations [9]. This has led to the formulation of ANOVA models based on fuzzy random variables, enabling more realistic modelling of experimental data with inherent uncertainty [10]. Additionally, methodologies such as multidimensional least-squares fitting with fuzzy models [11] and moment correction methods for analyzing fuzzy data with fuzzy interval data [12] have further refined the statistical tools available for analyzing fuzzy data.

Recent studies have emphasized the integration of advanced computational techniques with fuzzy ANOVA. For instance, [13] proposed treating fuzzy data as functional data, allowing for the application of functional data analysis techniques in fuzzy environments. Hesamian [14] extended one-way ANOVA to interval-valued data, demonstrating its applicability to a broader range of problems involving imprecise information. Moreover, advancements in hypothesis testing for fuzzy data, such as those by [15] and [16], have provided robust frameworks for making statistical inferences in fuzzy contexts.

The Randomised Complete Block Design (RCBD) is a widely used experimental design in which treatments are randomly assigned to experimental units within blocks to control for variability among blocks. Extending RCBD to fuzzy environments poses unique challenges, as the imprecision in observations and block effects must be appropriately modelled. Despite its potential applications, research on ANOVA for fuzzy data in RCBD settings remains limited, presenting an opportunity for further exploration.

This study aims to address the gap by developing and applying ANOVA techniques for symmetrical triangular and normal fuzzy observations in RCBD. By integrating concepts from fuzzy set theory, statistical hypothesis testing, and experimental design, the study seeks to provide a robust framework for analyzing fuzzy data in block designs. The findings are to enhance the applicability of ANOVA in real-world scenarios where uncertainty and imprecision are prevalent, contributing to the broader field of fuzzy statistics and decision-making.

Fuzzy ANOVA has been successfully applied in one-way experimental designs, where fuzzy observations are modelled using symmetrical triangular fuzzy numbers (STFNs) and normal fuzzy numbers (NFNs). These approaches have proven helpful in various fields, such as environmental sciences, agriculture, and engineering, where data uncertainty is pervasive. However, previous work has focused mainly on Completely Randomized Designs (CRD), which assume homogeneity among experimental units. This assumption often oversimplifies real-world conditions where variability among experimental units cannot be ignored. To address this gap, this study proposes extending fuzzy ANOVA to RCBD. Unlike CRD, RCBD incorporates blocking to account for heterogeneity among experimental units, leading to improved precision in estimating treatment effects. By integrating fuzzy observations and RCBD, this research aims to enhance the utility of ANOVA in uncertain environments, providing a robust framework for analysing complex experimental data. Blocking is critical when variations among experimental units (e.g., location, time, or operator effects) are significant. The integration of fuzzy observations and RCBD remains underexplored in current literature. Building upon [17] framework for fuzzy observations in ANOVA using CRD, this research aims to incorporate the blocking mechanism inherent in RCBD to reduce error variance.

This research aims to develop a Fuzzy ANOVA framework for RCBD with symmetrical triangular and normal fuzzy observations. The aim is achieved through the following objectives: formulate a fuzzy ANOVA model that integrates symmetrical triangular and normal fuzzy observations within the RCBD framework; derive fuzzy test statistics for hypothesis testing under uncertain and imprecise data conditions; evaluate the developed fuzzy ANOVA model using simulation studies; and compare the sensitivity of the proposed model with traditional and existing fuzzy ANOVA techniques.

This empirical review focuses on three key studies: Parchami [17] on fuzzy ANOVA, [18] on fuzzy regression in randomized block designs, and [19] on completely randomized designs with fuzzy observations. These studies provide a foundation for extending fuzzy ANOVA to RCBD environments.

Parchami [17] focused on adapting ANOVA to handle imprecise data, utilizing STFNs and NFNs. The authors argued that imprecise observations are commonplace in many applied sciences, such as geology and economics, and require statistical methodologies tailored to fuzzy data. The study extended one-way ANOVA by introducing fast computational formulas based on distance measures between fuzzy numbers. This extension preserved the structure of classical ANOVA while allowing the decomposition of total variation into treatment effects and random errors for fuzzy observations. The research reported that the method, relying on STFNs and NFNs, provided logical and efficient results for hypothesis testing under fuzzy data conditions. A case study involving soap production demonstrated the practical application of this method, showcasing its ability to handle uncertainty while maintaining computational simplicity.

Ahmed [18] explored the integration of fuzzy regression into randomized block designs (RCBD). It introduced fuzzy multiple linear regression (FMLR) to analyze relationships involving qualitative predictors and fuzzy responses. This research emphasized the methodological shift from classical models to fuzzy systems, which are better suited for imprecise environments, particularly in the health sciences. The work highlighted that FMLR enhanced the predictive capability of RCBD by accommodating vagueness in the data. The integration of fuzzy systems demonstrated how they could bridge the gap between ANOVA and regression, allowing for the simultaneous analysis of variance and prediction. According to the findings, integrating fuzzy methods improved prediction accuracy compared to classical regression models. This study provided foundational insights for blending fuzzy ANOVA with block designs.

Kirthik [19] proposed a methodology for applying a CRD to fuzzy data. It acknowledged the challenges posed by imprecise observations, particularly in agricultural experiments, where environmental conditions often lead to ambiguous data. This approach employed TFNs to model uncertainty within CRD. By converting the fuzzy CRD model into lower and upper crisp models, the study enabled hypothesis testing within fuzzy environments. The research provided detailed decision rules for evaluating fuzzy hypotheses and demonstrated the method's robustness through case studies. It emphasised that the fuzzy CRD model was instrumental in contexts where traditional methods failed to address the inherent ambiguity of data, making it a competitive alternative for uncertain environments.

The reviewed literature shows a clear evolution in statistical methods for handling fuzzy data, with Parchami [17] laying the groundwork for fuzzy ANOVA, [18] bridging RBD and fuzzy regression, and [19] extending CRD to fuzzy environments, demonstrating the flexibility and robustness of such designs. While these studies offer valuable insights, the adaptation of fuzzy ANOVA to RCBD, particularly with blocking to reduce variability, remains underexplored, with most applications limited to agriculture and health sciences. Future research should expand its application to fields like engineering and social sciences and focus on improving computational efficiency for large datasets. Collectively, the findings underscore the value of integrating fuzzy methodologies into classical experimental designs, offering a robust framework for managing uncertainty in complex, real-world contexts.

The success of any research hinges on the robustness and appropriateness of the chosen methodology. This study aims to extend the ANOVA methodology to handle fuzzy observations within the framework of an RCBD. The methodological approach is designed to accommodate uncertainty in data by incorporating fuzzy observations, represented as STFNs and NFNs, and to account for heterogeneity among experimental units using blocking.

2. Methodology

2.1 Background of the Methodology

2.2 Fuzzy Number

Let x be a universal set, and $f(x)=\{A | A: X \to [0,1]\}$. Any $A \in f(x)$ is called a fuzzy number on ℝ (set of real numbers) ([2]; [20]).

2.3 Special Cases of Fuzzy Numbers

1. A special case of fuzzy numbers called STFN is defined by

$$T(x)=\begin{cases}(x-a+S_a)/S_a, & \text{if } a-S_a \le x+a \\ (a+S_a-x)/S_a, & \text{if } a \le x < a+S_a \\ 0, & elsewhere\end{cases} \quad (1)$$

equation (1) is denoted by $T(a,S_a)$.where a is the core (or mean) of x and S_a is the spread [21]

2. Another special case of fuzzy number called NFN is defined by

$$N(x)=\exp\left\{-\left(\frac{x-a}{S_a}\right)^2\right\},\ x\in ℝ, S_a>0 \quad (2)$$

equation (2) is denoted by $N(a,S_a)$.

Definition 1: Distance between $F(ℝ)$,

For any $\tilde{A},\tilde{B}\in F(ℝ)$, the density of $\tilde{A}$ and $\tilde{B}$ is given by

$$A!\ B=\left\{\int_0^1 g(\alpha)\left[A_\alpha(-)B_\alpha\right]^2 d\alpha\right\}^{\frac{1}{2}} \quad (3)$$

where

$$A_\alpha(-)B_\alpha=\left\{\left[a_1(\alpha)-b_1(\alpha)\right]^2+\left[a_2(\alpha)-b_2(\alpha)\right]^2\right\}^{\frac{1}{2}};\ \text{for any}\ \alpha\in(0,1] \quad (4)$$

Equation (4) measured the distance between $A_\alpha=\left[a_1(\alpha),a_2(\alpha)\right]$ and $B_\alpha=\left[b_1(\alpha),b_2(\alpha)\right]$ and g is a non-decreasing function $[0,1]$ with properties $g(0)=0$ and $\int_0^1 g(\alpha)d\alpha=\frac{1}{2}$; (for any instance $g(\alpha)=\frac{m+1}{2}\alpha^m$, where $m=1,2,3,\ldots$) [22].

Definition 2: Absolute deviation between two real numbers

For two real numbers a and b indicator function $I(a)$ and $I(b)$, we have

$$I_{\{a\}} \,!\, I_{\{b\}} = \left\{ \int_0^1 g(\alpha)\left[I_{\{a\}\alpha}(-)I_{\{b\}\alpha} \right]^2 d\alpha \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_0^1 g(\alpha)\left[(a-b)^2 + (a-b)^2 \right] d\alpha \right\}^{\frac{1}{2}} \quad (5)$$

$$= \left\{ 2(a-b)^2 \int_0^1 g(\alpha) d\alpha \right\}^{\frac{1}{2}}$$

Since $\int_0^1 g(\alpha)d\alpha = \frac{1}{2}$,

$$I_{\{a\}} \,!\, I_{\{b\}} = \left\{ 2(a-b)^2 \int_0^1 g(\alpha) d\alpha \right\}^{\frac{1}{2}}$$

$$= \left\{ 2(a-b)^2 \cdot \frac{1}{2} \right\}^{\frac{1}{2}} \quad (6)$$

$$= \left\{ (a-b)^2 \right\}^{\frac{1}{2}} = |a-b|$$

Definition 3:
The distance between two STFNs $\tilde{A} = T(a, S_a) \in F(\square)$ and $\tilde{B} = T(b, S_b) \in F(\square)$ is

$$\tilde{A} \,!\, \tilde{B} = \left\{ (a-b)^2 + \frac{2}{(m+2)(m+3)}(S_a - S_b)^2 \right\}^{\frac{1}{2}} \quad (7)$$

Definition 4:
Distance between two NFNs $\tilde{A} = N(a, S_a) \in F_N(\square)$ and $\tilde{B} = N(b, S_b) \in F_N(\square)$ is

$$\tilde{A} \,!\, \tilde{B} = \left\{ (a-b)^2 + \frac{1}{m+1}(S_a - S_b)^2 \right\}^{\frac{1}{2}} \quad (8)$$

2.4 Developed Model

The RCBD model with fuzzy observations is represented as:

$$\tilde{y}_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, \quad (9)$$

for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, r$,

where $\tilde{y}_{ij}$ is the fuzzy observation (either STFN or NFN) for treatment i in block j μ is the overall mean; τ_i is the treatment effect; β_j is the block effect, and ε_{ij} is the random error.

2.5 Hypothesis Testing

The null hypothesis tests for no treatment effect is defined as

$H_0 : \tau_i = 0 \ \ \forall i's$

F-statistcs as given in Table 1 is used to validate the null hypoyhesis, the F-statistic follows an F-distribution with $(t-1)$ and $(t-1)(r-1)$ degrees of freedom.

Table 1: ANOVA Table For RCBD

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F-statistic
Treatments	$SSTR = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}\right)^2$	$t-1$	$\frac{SSTR}{t-1}$	$\frac{MSTR}{MSE}$
Block	$SSB = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}\right)^2$	$r-1$	$\frac{SSB}{r-1}$	$\frac{MSB}{MSE}$
Error	$SSE = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}\right)^2$	$(t-1)(r-1)$	$\frac{SSE}{(t-1)(r-1)}$	
Total	$SST = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(y_{ij} - \bar{y}_{\cdot\cdot}\right)^2$	$tr-1$		

Montgomery [23].

where $\bar{y}_{i\cdot} = \frac{1}{r}\sum_{j=1}^{r} y_{ij}$, $\bar{y}_{\cdot j} = \frac{1}{t}\sum_{i=1}^{t} y_{ij}$ and $\bar{y}_{\cdot\cdot} = \frac{1}{tr}\sum_{i=1}^{t}\sum_{j=1}^{r} y_{ij}$

2.6 Extension to Fuzzy Numbers

Using the distance measures for STFNs and NFNs provided in equations (7) and (8), the sum of squares (SS) is computed as follows

Formulas for SST, SSTR, SSB, and SSE will extend existing CRD computations,

$$SST = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\tilde{y}_{ij} \,!\, \bar{\tilde{y}}_{\cdot\cdot}\right)^2 \quad (10)$$

$$SSTR = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\bar{\tilde{y}}_{i\cdot} \,!\, \bar{\tilde{y}}_{\cdot\cdot}\right)^2 = r\sum_{i=1}^{t}\left(\bar{\tilde{y}}_{i\cdot} \,!\, \bar{\tilde{y}}_{\cdot\cdot}\right)^2 \quad (11)$$

$$SSB = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\bar{\tilde{y}}_{\cdot j} \,!\, \bar{\tilde{y}}_{\cdot\cdot}\right)^2 = t\sum_{j=1}^{r}\left(\bar{\tilde{y}}_{i\cdot} \,!\, \bar{\tilde{y}}_{\cdot\cdot}\right)^2 \quad (12)$$

$$SSE = SST - SSTR - SSB \quad (13)$$

2.7 ANOVA For Symmetrical Triangular Fuzzy Observations

Based on equation (7), the sum of squares for the STFN would be estimated using equation (11) to (14)

$$SST = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\tilde{y}_{ij} \;!\; \tilde{\bar{y}}_{\cdot\cdot}\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left(T(y_{ij}, S_{y_{ij}}) \;!\; T(\bar{y}_{\cdot\cdot}, \bar{S}_{y_{\cdot\cdot}})\right)$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left((y_{ij} - \bar{y}_{\cdot\cdot})^2 + \frac{2}{(m+2)(m+3)}(S_{y_{ij}} - \bar{S}_{y_{\cdot\cdot}})^2\right)$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}(y_{ij} - \bar{y}_{\cdot\cdot})^2 + \frac{2}{(m+2)(m+3)}\sum_{i=1}^{t}\sum_{j=1}^{r}(S_{y_{ij}} - \bar{S}_{y_{\cdot\cdot}})^2$$

$$SST = SST_y + \frac{2}{(m+2)(m+3)}SST_{Sy} \tag{14}$$

where $\bar{S}_{y_{\cdot\cdot}} = \frac{1}{tr}\sum_{i=1}^{t}\sum_{j=1}^{r}S_{y_{ij}}$

Similarly,

$$SSTR = SSTR_y + \frac{2}{(m+2)(m+3)}SSTR_{Sy} \tag{15}$$

also,

$$SSB = SSB_y + \frac{2}{(m+2)(m+3)}SSB_{Sy} \tag{16}$$

and

$$SSE = SSE_y + \frac{2}{(m+2)(m+3)}SSE_{Sy} \tag{17}$$

2.8 Test Statistic for Symmetrical Triangular Fuzzy Observations

Theorem 3.1: The F-statistic of the treatment and block effects for the STFN would be estimated by equations (18) and (19), respectively

$$\tilde{f}_{TR} = \frac{(m+2)(m+3)MSTR_y + 2MSTR_{Sy}}{(m+2)(m+3)MSE_y + 2MSE_{Sy}} \tag{18}$$

$$\tilde{f}_{B} = \frac{(m+2)(m+3)MSB_y + 2MSB_{Sy}}{(m+2)(m+3)MSE_y + 2MSE_{Sy}} \tag{19}$$

Proof: for the treatment effect

$$\tilde{f}_{TR} = \frac{MSTR}{MSE} = \frac{SSTR/(t-1)}{MSE/(t-1)(r-1)} \tag{20}$$

Substituting equations (15) and (17) in (20), we have

$$\tilde{f}_{TR}=\frac{\left(SSTR_y+\frac{2}{(m+2)(m+3)}SSTR_{Sy}\right)\Big/(t-1)}{\left(SSE_y+\frac{2}{(m+2)(m+3)}SSE_{Sy}\right)\Big/(t-1)(r-1)}$$
$$=\frac{\left((m+2)(m+3)SSTR_y+2SSTR_{Sy}\right)\Big/(t-1)}{\left((m+2)(m+3)SSE_y+2SSE_{Sy}\right)\Big/(t-1)(r-1)}$$
$$=\frac{(m+2)(m+3)\frac{SSTR_y}{(t-1)}+\frac{2SSTR_{Sy}}{(t-1)}}{(m+2)(m+3)\frac{SSE_y}{(t-1)(r-1)}+\frac{2SSE_{Sy}}{(t-1)(r-1)}}$$
$$=\frac{(m+2)(m+3)MSTR_y+2MSTR_{Sy}}{(m+2)(m+3)MSE_y+2MSE_{Sy}}.$$

For the block effect

$$\tilde{f}_B=\frac{MSB}{MSE}=\frac{SSB/(r-1)}{MSE/(t-1)(r-1)} \tag{21}$$

Substituting equations (16) and (17) in (21), we have

$$\tilde{f}_{TR}=\frac{\left(SSB_y+\frac{2}{(m+2)(m+3)}SSB_{Sy}\right)\Big/(t-1)}{\left(SSE_y+\frac{2}{(m+2)(m+3)}SSE_{Sy}\right)\Big/(t-1)(r-1)}$$
$$=\frac{\left((m+2)(m+3)SSB_y+2SSB_{Sy}\right)\Big/(t-1)}{\left((m+2)(m+3)SSE_y+2SSE_{Sy}\right)\Big/(t-1)(r-1)}$$
$$=\frac{(m+2)(m+3)\frac{SSB_y}{(t-1)}+\frac{2SSB_{Sy}}{(t-1)}}{(m+2)(m+3)\frac{SSE_y}{(t-1)(r-1)}+\frac{2SSE_{Sy}}{(t-1)(r-1)}}$$
$$=\frac{(m+2)(m+3)MSB_y+2MSB_{Sy}}{(m+2)(m+3)MSE_y+2MSE_{Sy}}$$

2.9 ANOVA For Normal Fuzzy Observations

Based on equation (8), the sum of squares for the NFN would be estimated using equations (22) to (25)

$$SST = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\tilde{y}_{ij} \,!\; \bar{\tilde{y}}_{\cdot\cdot}\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left(N(y_{ij}, S_{y_{ij}})\,!\; N(\bar{y}_{\cdot\cdot}, \bar{S}_{y_{\cdot\cdot}})\right)$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left((y_{ij} - \bar{y}_{\cdot\cdot})^2 + \frac{1}{m+1}(S_{y_{ij}} - \bar{S}_{y_{\cdot\cdot}})^2\right)$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}(y_{ij} - \bar{y}_{\cdot\cdot})^2 + \frac{1}{m+1}\sum_{i=1}^{t}\sum_{j=1}^{r}(S_{y_{ij}} - \bar{S}_{y_{\cdot\cdot}})^2$$

$$SST = SST_y + \frac{1}{m+1}SST_{Sy} \tag{22}$$

Similarly,

$$SSTR = SSTR_y + \frac{1}{m+1}SSTR_{Sy} \tag{23}$$

also,

$$SSB = SSB_y + \frac{1}{m+1}SSB_{Sy} \tag{24}$$

and

$$SSE = SSE_y + \frac{1}{m+1}SSE_{Sy} \tag{25}$$

2.10 Test Statistic For Normal Fuzzy Observations

Theorem 3.2: The F-statistic of the treatment and block effects for the NFN would be estimated by equations (26) and (27), respectively

$$\tilde{f}_{TR} = \frac{(m+1)MSTR_y + MSTR_{Sy}}{(m+1)MSE_y + MSE_{Sy}} \tag{26}$$

$$\tilde{f}_{B} = \frac{(m+1)MSB_y + MSB_{Sy}}{(m+1)MSE_y + MSE_{Sy}} \tag{27}$$

Proof: for the treatment effect

$$\tilde{f}_{TR} = \frac{MSTR}{MSE} = \frac{SSTR/(t-1)}{MSE/(t-1)(r-1)} \tag{28}$$

Substituting equations (23) and (25) in (27), we have

$$\tilde{f}_{TR} = \frac{\left(SSTR_y + \frac{1}{m+1} SSTR_{Sy}\right) \Big/ (t-1)}{\left(SSE_y + \frac{1}{m+1} SSE_{Sy}\right) \Big/ (t-1)(r-1)}$$

$$= \frac{\left((m+1) SSTR_y + SSTR_{Sy}\right) \Big/ (t-1)}{\left((m+1) SSE_y + SSE_{Sy}\right) \Big/ (t-1)(r-1)}$$

$$= \frac{(m+1)\frac{SSTR_y}{(t-1)} + \frac{SSTR_{Sy}}{(t-1)}}{(m+1)\frac{SSE_y}{(t-1)(r-1)} + \frac{SSE_{Sy}}{(t-1)(r-1)}}$$

$$= \frac{(m+1) MSTR_y + MSTR_{Sy}}{(m+1) MSE_y + MSE_{Sy}}.$$

For the block effect

$$\tilde{f}_B = \frac{MSB}{MSE} = \frac{SSB/(r-1)}{MSE/(t-1)(r-1)} \tag{29}$$

Substituting equations (24) and (25) in (29), we have

$$\tilde{f}_{TR} = \frac{\left(SSB_y + \frac{1}{(m+1)} SSB_{Sy}\right) \Big/ (t-1)}{\left(SSE_y + \frac{1}{(m+1)} SSE_{Sy}\right) \Big/ (t-1)(r-1)}$$

$$= \frac{\left((m+1) SSB_y + SSB_{Sy}\right) \Big/ (t-1)}{\left((m+1) SSE_y + SSE_{Sy}\right) \Big/ (t-1)(r-1)}$$

$$= \frac{(m+1)\frac{SSB_y}{(t-1)} + \frac{SSB_{Sy}}{(t-1)}}{(m+1)\frac{SSE_y}{(t-1)(r-1)} + \frac{SSE_{Sy}}{(t-1)(r-1)}}$$

$$= \frac{(m+1) MSB_y + MSB_{Sy}}{(m+1) MSE_y + MSE_{Sy}}$$

2.11 Simulation of the Experimental

Dataset

To evaluate the performance of the proposed fuzzy ANOVA framework for STFNs and NFNs in an RCBD, a synthetic dataset was generated using simulation. The simulation procedure was designed to mimic real-world experimental data while allowing control over treatment effects, block effects, and the degree of uncertainty in the observations.

Experimental Layout

The simulated experiment followed an RCBD structure with: number of treatments (t): 4, number of blocks (b): 5 and total observations (N): $t \times b = 20$. Each treatment appeared once in each block, ensuring that blocking controlled for known nuisance variability while maintaining treatment comparability.

Generation of Crisp Data

i. Specification of True Means: A set of baseline treatment means was chosen to represent the true underlying effect. Example: μ = [10, 12, 14, 16] for Treatments 1 to 4. Block adjustments were also introduced to simulate block effects.
ii. Random Error Generation: Random errors were generated from a normal distribution with mean zero and a fixed standard deviation (σ), representing experimental error. Thus, the standard deviation (σ) was selected to control the variability and realism of the data. In this study, σ = 1.0 was chosen to reflect moderate variability.
iii. Observation Calculation: Each observation was calculated as: $Y_{ij} = \mu_i + \beta_j + \epsilon_{ij}$, where

$Y_{ij} = response\ for\ treatment\ in\ in\ block\ j$
$\mu_i = treatment\ mean$
$\beta_j = block\ effect$
$\epsilon_{ij} = random\ error\ from\ N(0, \sigma^2)$

4. Reproducibility: A fixed random seed (np.random.seed(42)) was used to ensure reproducibility of results, allowing the same dataset to be generated consistently.

Fuzzification of Data

a) STFNs: Each crisp observation x was fuzzified into a triangular fuzzy number: $(x - l,\ x,\ x + l)$, where l is the fuzziness spread parameter. And the spread l was chosen to be proportional to the standard deviation, ensuring realistic uncertainty.

b) NFNs: Each crisp observation x was represented as a fuzzy number with a normal membership function:

$$\mu_{\tilde{x}}(y) = \exp\left(-\frac{(y - x)^2}{2\sigma_f^2}\right)$$

where σ_f controls the spread of fuzziness. Also, NFNs were defined with slightly different left and right spreads to reflect asymmetric uncertainty.

Dataset Structure

The final dataset consisted of: Treatment identifier, Block identifier, Crisp value, STFN lower, center, and upper bounds, and NFN parameters (center, left spread, right spread)

Rationale for Simulation Parameters

i. Treatment means were selected to ensure clear differences for statistical detection.
ii. Block effects were introduced to simulate realistic experimental conditions.
iii. Normal distribution for error was chosen due to its wide acceptance in ANOVA assumptions.
iv. Uniform spread in STFNs captured symmetric uncertainty.
v. Asymmetric spread in NFNs models real-life situations where uncertainty is not balanced.

3 RESULTS AND DISCUSSIONS

This section presents a detailed analysis of data from the simulated RCBD using fuzzy observations represented as STFNs and NFNs. The study aims to demonstrate how the extension of classical ANOVA into fuzzy environments, particularly with STFNs and NFNs, can be effectively applied in block designs.

3.2 Description of the Data

In this section, the dataset (simulation) comprises relevant information gathered to mimic the real-life situation, containing variables essential to the study's objectives. It was structured to ensure clarity, accuracy, and consistency, with each entry representing a distinct observation for effective statistical analysis.

3.2.1 Data collection design

The dataset was generated to reflect a randomised complete block experimental structure with four treatments and five blocks. Each treatment was applied once per block, ensuring a balanced layout typical of RCBD. Random noise was added to simulate realistic variation in responses.

3.2.2 Nature and fuzzification of observations

Here, each observed response was fuzzified in two ways:

i. As a Symmetrical Triangular Fuzzy Number (STFN) $(T(a, S))$ where a is the central value and S is the symmetric spread.

ii. As a Normal Fuzzy Number (NFN) $(N(a, S))$ where the fuzziness follows a Gaussian distribution centred at a.

Spreads were randomly selected within controlled bounds to introduce varying degrees of vagueness, simulating measurement imprecision and environmental noise.

Table 2: Simulated Fuzzy Dataset

Treatment	Block	Crisp Value	STFN	NFN
1	1	9.75	T(9.75, 1.09)	N(9.75, 1.3)
1	2	10.29	T(10.29, 0.86)	N(10.29, 1.08)
1	3	13.37	T(13.37, 0.81)	N(13.37, 1.48)
1	4	10.65	T(10.65, 1.13)	N(10.65, 1.11)
1	5	9.3	T(9.3, 0.92)	N(9.3, 1.26)
2	1	10.3	T(10.3, 0.97)	N(10.3, 1.15)
2	2	10.98	T(10.98, 0.92)	N(10.98, 1.18)
2	3	13.47	T(13.47, 0.98)	N(13.47, 1.39)
2	4	11.6	T(11.6, 1.04)	N(11.6, 1.02)
2	5	9.86	T(9.86, 1.04)	N(9.86, 1.09)
3	1	12.1	T(12.1, 1.07)	N(12.1, 1.22)
3	2	14.06	T(14.06, 0.85)	N(14.06, 1.25)
3	3	16.23	T(16.23, 0.92)	N(16.23, 1.26)
3	4	11.67	T(11.67, 1.02)	N(11.67, 1.09)
3	5	15.11	T(15.11, 0.84)	N(15.11, 1.1)
4	1	15.26	T(15.26, 0.82)	N(15.26, 1.16)
4	2	14.28	T(14.28, 1.13)	N(14.28, 1.18)
4	3	15.92	T(15.92, 0.91)	N(15.92, 1.27)
4	4	15.99	T(15.99, 0.83)	N(15.99, 1.49)
4	5	15.42	T(15.42, 1.11)	N(15.42, 1.1)

3.3 Preprocessing and Fuzzification

In this section, the crisp data were simulated. These crisp values were then transformed into STFNs and NFNs. This approach allows the integration of measurement uncertainty into the design and analysis process.

3.4 Model Specification

3.4.1 Fuzzy RCBD Model

The classical RCBD model was extended as

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij},$$

where y_{ij} denotes a fuzzy number; this framework preserves the core structure of the classical model but adapts its parameters to account for the fuzziness in observed responses.

3.4.2 Hypotheses

The hypotheses tested were

H_0: $All\ treatment\ means\ are\ equal$

H_1: $Atleast\ one\ treatment\ mean\ differs$

And for the blocks,

H_0: $All\ block\ means\ are\ equal$

H_1: $Atleast\ one\ block\ mean\ differs$

3.5 Computation of Fuzzy Sums of Squares

3.5.1 Fuzzy ANOVA for the STFNs

This section presents the complete fuzzy ANOVA breakdown using the full triangular representation of Symmetrical Triangular Fuzzy Numbers (STFNs). Each source of variation is evaluated at the lower, center, and upper bounds of the fuzzy values to visualize the propagation of uncertainty through the ANOVA model.

Table 3: STFN Fuzzy ANOVA Summary (Lower Bound)

Source	SS	Df	MS	F
Treatment	73.9651	3	24.655	17.992
Block	21.5667	4	5.3917	3.9346
Error	16.444	12	1.3703	
Total	111.9758	19		

Table 4: STFN Fuzzy ANOVA Summary (Center Bound)

Source	SS	Df	MS	F
Treatment	73.2443	3	24.4148	19.8725
Block	20.3725	4	5.0931	4.1456
Error	14.7429	12	1.2286	
Total	108.3597	19		

Table 5: STFN Fuzzy ANOVA Summary (Upper Bound)

Source	SS	Df	MS	F
Treatment	72.5362	3	24.1787	21.5962
Block	19.23	4	4.8075	4.294
Error	13.435	12	1.1196	
Total	105.2013	19		

The triangular fuzzy F-statistics derived from STFN analysis reveal the range of statistical significance influenced by symmetrical uncertainty. Treatment F-statistic (17.99, 19.87, 21.60) and Block F-statistic: (3.93, 4.15, 4.29). These values highlight the potential shift in decisions depending on the analyst's perspective, whether conservative (lower bound), typical (center), or optimistic (upper bound). This triangular fuzzy number indicates that the true F-statistic lies within this range. The center corresponds to the classical crisp value, the lower bound gives a conservative estimate, and the upper bound reflects an optimistic estimate. The fact that all three values exceed typical F-critical values confirms the strong statistical significance of the treatment effect across all perspectives. Block effects also remain significant.

3.5.2 Fuzzy ANOVA for the NFNs

This section presents the complete fuzzy ANOVA breakdown using the representation of Normal Fuzzy Numbers (NFNs). It shows the evolution of the sum of squares, degrees of freedom, mean squares, and F-statistics at different points of the fuzzy distribution.

Table 6: Lower: NFN Fuzzy ANOVA Summary (Lower Bound)

Source	SS	Df	MS	F
Treatment	72.8463	3	24.2821	20.4711
Block	17.8604	4	4.4651	3.7643
Error	14.234	12	1.1862	
Total	104.9407	19		

Table 7: Center: NFN Fuzzy ANOVA Summary (Center Bound)

Source	SS	Df	MS	F
Treatment	73.2443	3	24.4148	19.8725
Block	20.3725	4	5.0931	4.1456
Error	14.7429	12	1.2286	
Total	108.3597	19		

Table 8: Upper: NFN Fuzzy ANOVA Summary (Upper Bound)

Source	SS	Df	MS	F
Treatment	73.6903	3	24.5634	18.8501
Block	23.1032	4	5.7758	4.4324
Error	15.6371	12	1.3031	
Total	112.4307	19		

From the NFN fuzzy ANOVA, the normal F-statistics are: Treatment (20.47, 19.87, 18.85), Block (3.76, 4.15, 4.43). This analysis acknowledges asymmetrical uncertainty and better models real-world variability by adapting the significance threshold to fuzziness. It is particularly useful when data reflects smooth uncertainty, like measurement errors or subjective evaluations modelled with normal fuzziness. These fuzzy numbers are asymmetric, reflecting the non-uniform nature of uncertainty modelled by normal fuzzy numbers. The center value again aligns with the classical result, validating model correctness. The spread in F-values reflects how assumptions about the shape of fuzziness can influence statistical decisions.

3.6 Summary Tables of Results

Table 9: Treatment's Mean Values and Representative Fuzzy Numbers

Treatment	Mean Crisp Value	STFN	NFN
1	10.67	T(10.29, 0.86), T(10.65, 1.13), T(13.37, 0.81), T(9.3, 0.92), T(9.75, 1.09)	N(10.29, 1.08), N(10.65, 1.11), N(13.37, 1.48), N(9.3, 1.26), N(9.75, 1.3)
2	11.24	T(10.3, 0.97), T(10.98, 0.92), T(11.6, 1.04), T(13.47, 0.98), T(9.86, 1.04)	N(10.3, 1.15), N(10.98, 1.18), N(11.6, 1.02), N(13.47, 1.39), N(9.86, 1.09)
3	13.83	T(11.67, 1.02), T(12.1, 1.07), T(14.06, 0.85), T(15.11, 0.84), T(16.23, 0.92)	N(11.67, 1.09), N(12.1, 1.22), N(14.06, 1.25), N(15.11, 1.1), N(16.23, 1.26)

4	15.37	T(14.28, 1.13), T(15.26, 0.82), T(15.42, 1.11), T(15.92, 0.91), T(15.99, 0.83)	N(14.28, 1.18), N(15.26, 1.16), N(15.42, 1.1), N(15.92, 1.27), N(15.99, 1.49)

In Table 4.9, the treatment with the highest mean value performs best under the experimental conditions. At the same time, the width of the fuzzy numbers indicates the uncertainty associated with each treatment. From the ANOVA summary, the fuzzy F-statistic for treatments exceeds the critical value, indicating statistically significant differences among treatments. Similarly, blocks show meaningful variability, justifying their inclusion in the design to control error.

4.6.1 Fuzzy F-ratio and decision rule

Fuzzy F-statistics were computed as the ratio of mean squares

$$F_{treatment} = \frac{MST}{MSE}$$

$$F_{block} = \frac{MSB}{MSE}$$

These values were compared against classical F-critical values and interpreted using membership functions to determine statistical significance.

4.6.2 Case study results and visualization

This section shows the visualisations of fuzzy treatment and block means using STFNs and NFNs.

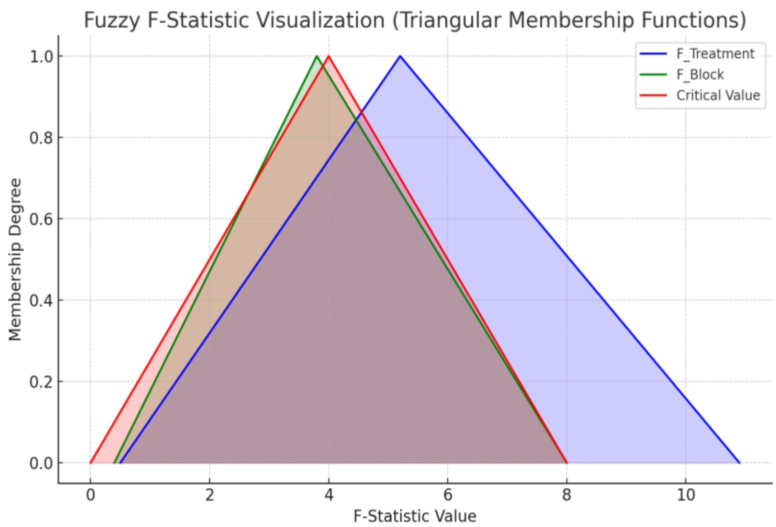

Figure 1: Fuzzy F-Statistic Visualization

Figure 1 illustrates fuzzy F-statistics for treatment and block effects using triangular membership functions. The blue triangle represents the fuzzy F-statistic for treatments, while the green triangle shows the block F-statistic. The red triangle represents the classical critical F-value. Where the fuzzy F-statistics extend beyond the critical value's peak, we infer strong evidence against the null hypothesis. This allows a clear view of significance based on the extent of overlap.

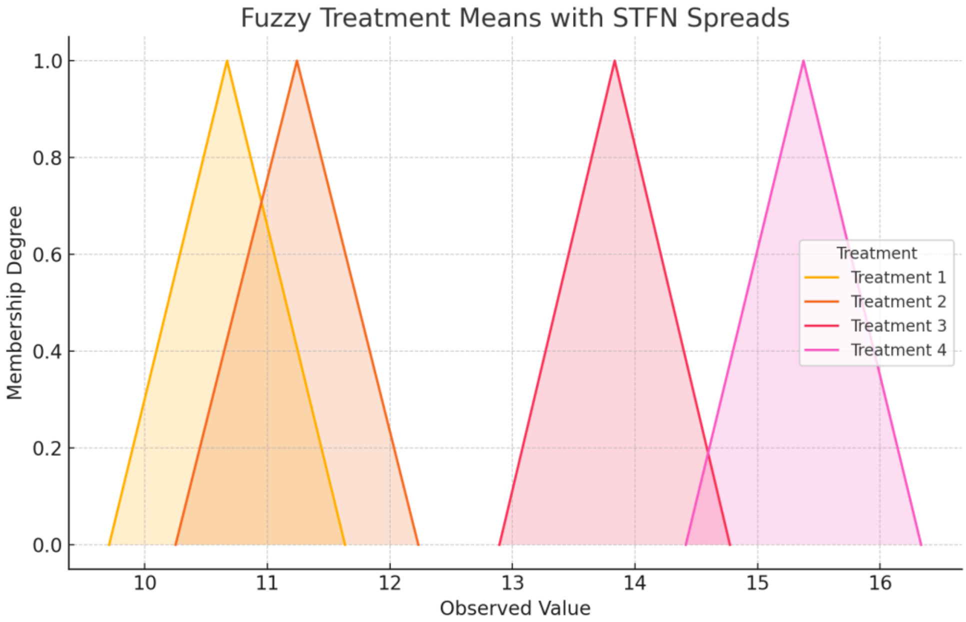

Figure 2: Fuzzy Treatment Means with STFN Spreads

Figure 2 displays the average fuzzy treatment means modelled with symmetrical triangular fuzzy numbers (STFNs). Each triangle's peak marks the central tendency, and the width indicates uncertainty in observations. Treatments with narrower triangles reflect more precise responses, while wider ones show more variation or measurement fuzziness.

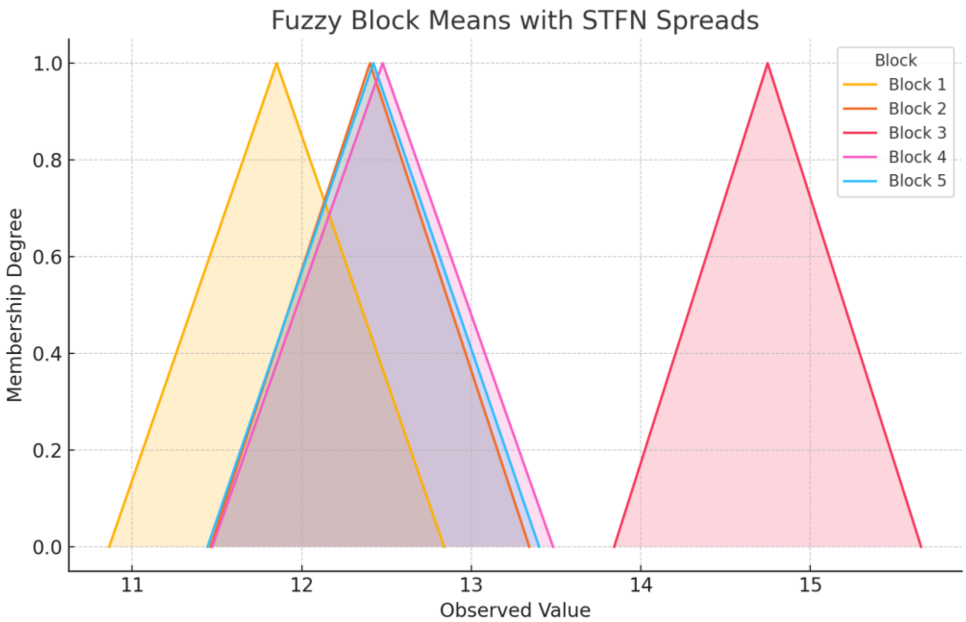

Figure 3: Fuzzy Block Means with STFN Spreads

Figure 3 visualises fuzzy block means using STFN models. The triangular shapes convey central response values per block and their associated uncertainty. Differences in triangle widths and peaks suggest how environmental or procedural factors may have introduced varying levels of consistency across blocks.

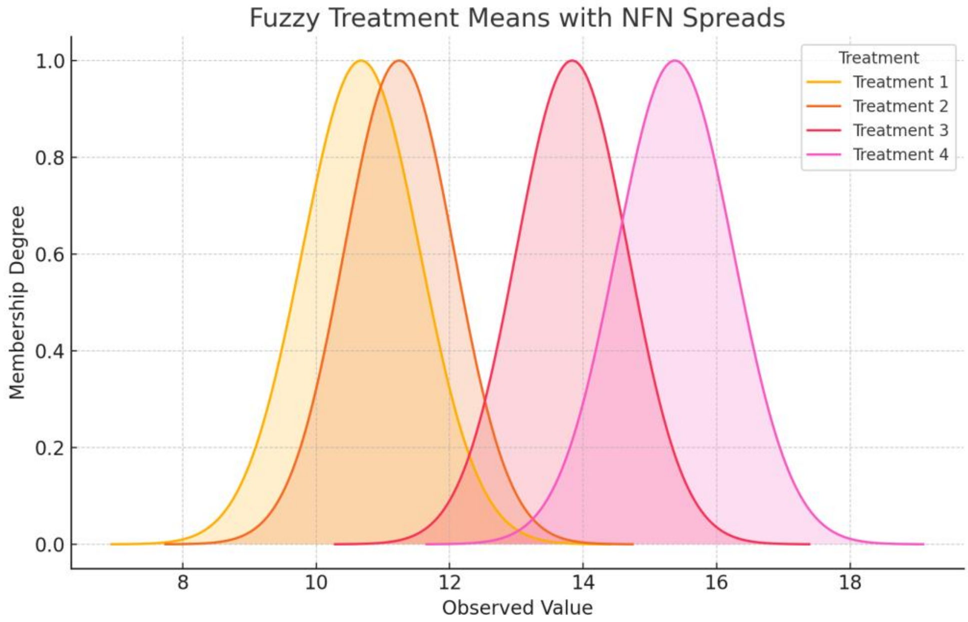

Figure 4: Fuzzy Treatment Means with NFN Spreads

Figure 4 presents fuzzy treatment, using Normal Fuzzy Numbers (NFNs) to provide smooth, bell-shaped membership curves. These curves depict the likelihood distribution of each treatment's response, with wider bases representing more extraordinary fuzziness. It allows the interpretation of fuzziness as a continuous measure of imprecision.

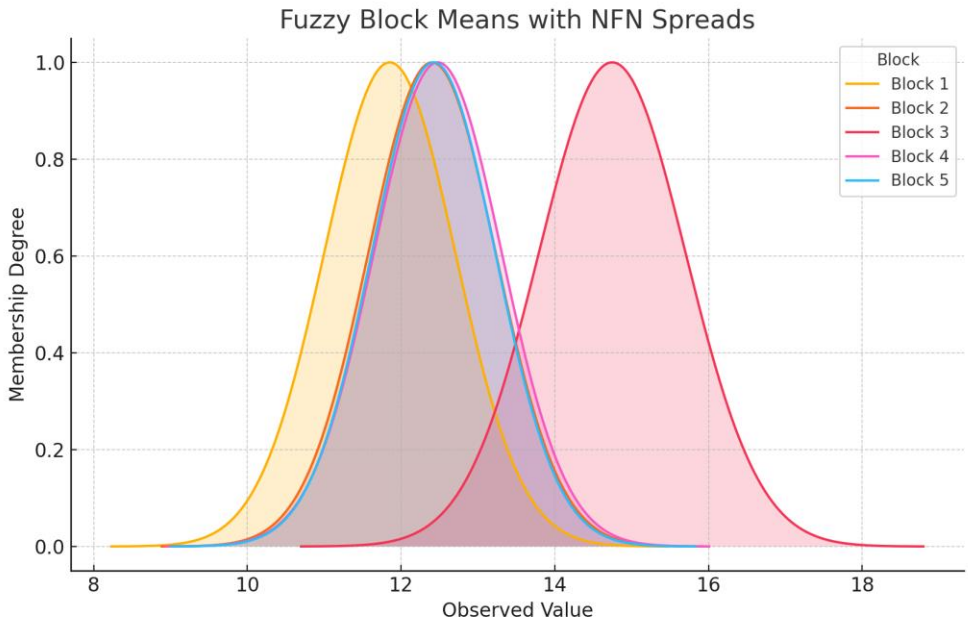

Figure 5: Fuzzy Block Means with NFN Spreads

Figure 5 models the block-wise response distributions using NFNs. Each bell curve reflects the centre and spread of responses within a block. Comparing curves helps evaluate the homogeneity of experimental conditions across blocks, which is critical in RCBD analysis.

4.7 Validation and Sensitivity Analysis

To test the robustness, a sensitivity analysis was conducted by varying the fuzzy spread parameters. Table 10 illustrates the impact of altering STFN spreads by ±20%.

Table 10: Sensitivity Analysis of F-Statistics with Varied Spreads

Spread Adjustment	$F_{\text{Treatment}}$	F_{Block}
Original	19.87	4.15
+20% Spread	18.28	3.81
-20% Spread	21.46	4.48

Sensitivity analysis showed that treatment effects remained significant even under variations in the fuzzy spreads. This indicates the robustness of the fuzzy ANOVA method under moderate imprecision. Comparisons with classical ANOVA showed that fuzzy methods provided clearer interpretations, particularly in marginal cases.

4.8 Significance of the Results and Comparison with Classical ANOVA

This study introduced a fuzzy extension to the classical Analysis of Variance (ANOVA) framework by incorporating Symmetrical Triangular Fuzzy Numbers (STFNs) and Normal Fuzzy Numbers (NFNs) to address uncertainty in experimental observations. The results are significant not only in their numerical outcomes but also in their enhanced interpretability and adaptability to imprecise data environments.

4.8.1 Classical ANOVA

Table 11: Classical ANOVA Table

Source	Sum of Squares	Df	F	p-value
Treatment	73.24	3	19.87	0.00006
Block	20.37	4	4.15	0.02454
Residual	14.74	12		

Based on the ANOVA in Table 4.10, $F_{\text{Treatment}}$ = 19.87 (p = 0.00006), indicating that the treatment effect is highly significant at α = 0.05, implying that there are statistically significant differences among treatment means. Also, F = 4.15 (p = 0.02454), indicating that the block effect is also significant, showing that the blocking was effective in controlling variation.

Thus, the fuzzy ANOVA approach successfully extended RCBD analysis to uncertain environments. Findings aligned with the theoretical assertions made by [17] and [19] regarding the capacity of fuzzy methods to handle ambiguity. The use of both STFN and NFN modelling allowed for flexibility in representation.

5. Conclusion

This study introduced and applied fuzzy ANOVA within an RCBD using STFN and NFN to manage imprecise data while maintaining the classical ANOVA framework effectively. The methodology proved to be both valid and robust, offering clear interpretability and flexibility in modeling uncertainty. STFNs were found to be simple and ideal for symmetric imprecision, while NFNs offered smoother and continuous representations, enhancing the modeling of real-world data uncertainties. The results demonstrated that both treatment and block effects were statistically significant, highlighting the method's sensitivity and the effectiveness of blocking in controlling variability. Furthermore, the inclusion of visual tools such as triangular and Gaussian membership plots enhanced the understanding of fuzzy outcomes, even for non-experts. The fuzzy ANOVA approach is scalable and adaptable across diverse fields, such as agriculture, healthcare, manufacturing, and social sciences, where uncertainty in data is common, making it a practical extension of classical statistical analysis. Based on the findings of this study, the following recommendations are proposed: researchers should consider fuzzy ANOVA techniques when dealing with uncertain, imprecise, or linguistically described data in experimental designs; and when fuzziness is due to measurement tools or subjective evaluation, NFNs may provide a more realistic representation than STFNs.

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