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RESEARCH ARTICLE

Oscillation Conditions for Solutions of First Order Piecewise Constant Neutral Differential Equations

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ABSTRACT

The research seeks to examine a new form of equations as well as possible criteria for oscillation and non-oscillation for all solutions of first-order neutral differential equations with the piecewise constant, whether these oscillatory solutions are convergent or non-convergent, by finding sufficient conditions that guarantee the oscillation of all solutions of these equations. According to NDEPC, these equations contain three delays and three coefficients, which requires that part of the coefficients in this equation be piecewise continuous. In witch A function that assumes constant values over predetermined intervals is known as a piecewise constant argument (PCA). The biggest integer function $[\omega]$, which rounds down any real number ω to the closest integer, is a typical example. The oscillation criteria for this type of equation necessitate the acquisition of some essential and sufficient conditions to ensure the achievement of this objective. The biggest integer function represents the piecewise constant. All solutions to this type of problem must be oscillatory, and necessary and sufficient conditions have been devised to assure this. Based on these conditions, one can deduce that the technique used for producing results relied on classifying coefficients into two independent types.

Keywords: Asymptotic behavior, Differential equation, Delay, Oscillation, Piecewise constant

Introduction

The delay neutral equations are considered one of the most important types of differential equations due to their frequent use in various applications such as biology, ecology, engineering, communications, and others. [1–3] Oscillation about equilibrium, for Hematopoiesis Models and anemia were studied in females and males. First-order nonlinear neutral differential equations with multiple non-monotonic delays and several variable coefficients with influential terms are discussed. The asymptotic behavior of all solutions was also studied, and some conditions were obtained that guarantee the oscillation or convergence of these solutions. [4] As is the case in some types of differential equations, the equations of many of these models in applications contain some undefined points, while these points are treated in the delay equations with piecewise constants. [5,6] To ensure that the solutions to these kinds of equations oscillate, sufficient and necessary criteria were established. Oscillation criteria and periodic solutions of delay differential equations (DDE) with piecewise constant arguments are discussed, along with some conditions obtained to ensure that every solution of DDE oscillates. [7,8]

The existence of the almost periodic solutions of second-order neutral delay-differential equations with piecewise constant arguments of the form $(\varphi(\omega)+\varphi(\omega-1))''=qx([\omega])+f(\omega)$, was studied, where $f(\omega)$ is almost periodic. [9] The first order impulsive neutral differential equations were discussed and some impulsive

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conditions were obtained to ensure the oscillation of all solutions to these equations.[10] The oscillation criterion was investigated for all solutions of the first-order linear neutral differential equations with positive and negative coefficients.[11] A technique is also presented to obtain some oscillation conditions for the delay differential equation with piecewise constant arguments of the form $r'(\omega) + a(\omega)r(\omega) + b(\omega)r([\omega - k]) = 0$, where $a(\omega)$ and $b(\omega)$ are right continuous functions on $[-k, \infty)$, k is a positive integer, and [.] denotes the greatest integer function.[12,13] For first order delay differential equation $\varphi'(\omega) + \sum_{i=1}^{k} p_i(\omega)\varphi(\omega - \tau_i)) = 0,\ where\ p_i : [\omega_0, \infty) \to [0, \infty)$. Sufficient conditions are established under which some "limit inferior-type" conditions for oscillation of all solutions remain valid if one replaces the limit inferior by limit superior.[14] Some conditions to guarantee the existence of bounded solution to the second order multi delayed arguments differential equation are given.[15] An optimization relationship between the solution and its corresponding function for second and forth-order neutral differential equation can be notice.[16] Existence criteria for non-homogeneous differential equations with piecewise constant arguments that have two periodic solutions was done.[17]

More Exploring the asymptotic properties for a class of fourth-order neutral differential equations, based on a comparison with the differential inequality of the first-order.[18] The aim of this artcle is to study the oscilatory of solution to the first-order neutral differential equations with piecewise constants are discussed in the form of

$$\left[\varphi(\omega) + p\varphi(\omega - \tau)\right]' - \alpha(\omega)\varphi(\omega - \sigma) - \beta(\omega)\varphi([\omega - 1]) = 0. \tag{1}$$

Where three different delays are chosen to be more of a development equation, $p, \tau > 0$, $\alpha(\omega), \beta(\omega) \in C[[0, \infty); R]$, [.] denotes the greatest integer function.

A function $\varphi(\omega)$ is said to be a solution of Eq. (1), if $\varphi(\omega) + p\varphi(\omega - \tau) \in C^1[[\omega_0 - \tau, \infty); R]$, with the possible exception of the points $[\omega] \in [0, \infty)$ where one-sided derivatives exist; and $\varphi(\omega)$ satisfies Eq. (1).[19] If there exists a sequence $\{\omega_m\}$, $\omega_m \to \infty$ such that $\varphi(\omega_m) = 0$ as $m \to \infty$, then the solution $\varphi(\omega)$ is said to be oscillate.[2]

Let

$$G(\omega) = \varphi(\omega) + p\varphi(\omega - \tau). \tag{2}$$

Then Eq. (1) reduce to

$$G'(\omega) = \alpha(\omega)\varphi(\omega - \sigma) + \beta(\omega)\varphi([\omega - 1]). \tag{3}$$

This rest of the this article is organized as follows: section two, introduced a new theorems to get the oscillatory of Eq. (1). In the third section, some examples to illustrate the theorems in the section two were given. Section four discuss the results in which obtained in section three and the last section notify the conclusions where gained in this paper.

Materials and methods

In this section four results obtained for oscillation of every solution of Eq. (1).

Theorem 1: *Suppose that $\alpha(\omega) \geq 0$, $\beta(\omega) \geq 0$, $\tau - \sigma \geq 1$, $\sigma \geq 1$ and there is a continuous function $\lambda(\omega) \geq 1$ such that for $\omega \geq 0$:*

$$\lambda\left(k+1\right) + p < \int_{k}^{k+1} \left(\alpha(\omega) + \beta(\omega)\right) d\omega,\ k = 0, 1, 2, \ldots. \tag{4}$$

Then there are only possible oscillatory solutions in Eq. (1).

Proof: Consider the case where Eq. (1) has an eventually positive solution. For any $k = 0, 1, 2, \ldots,$ and $\omega \in [k, k+1)$, Eq. (3) becomes

$$G'(\omega) = \alpha(\omega)\varphi(\omega - \sigma) + \beta(\omega)\varphi\left(k - 1\right),\ k = 0, 1, 2, \ldots. \tag{5}$$

Hence $G'(\omega) \geq 0$, so $G(\omega)$ is nondecreasing for any $k = 1, 2, \ldots,$ and $\omega \in [k, k+1)$. One can Note that $\varphi(\omega)$ must be nondecreasing in $[k, k+1)$, otherwise if $\varphi(\omega)$ is decreasing function that is

$$\varphi(\omega - \tau) > \varphi(\omega - \tau + 1) \text{ for } \omega \in [k, k+1).$$

On the other side, since $G(\omega)$ is nondecreasing, so $G(\omega + 1) \geq G(\omega)$ that is

$$0 < \varphi(\omega + 1) + p\varphi(\omega + 1 - \tau) \geq \varphi(\omega) + p\varphi(\omega - \tau), \ t \in [k, k+1).$$

$$0 < \varphi(\omega + 1) \geq \varphi(\omega) + p[\varphi(\omega - \tau) - \varphi(\omega + 1 - \tau)].$$

But $\varphi(\omega - \tau) - \varphi(\omega - \tau + 1) > 0$, then $\varphi(\omega + 1) > \varphi(\omega)$, which means that $\varphi(\omega)$ is an increasing function in $[k, k+1)$, in a contradiction, hence $\varphi(\omega)$ is nondecreasing function for $\omega \in [k, k+1)$. Let

$$\psi(\omega) \geq \frac{\varphi(\omega)}{\varphi(\omega - \tau)} \geq 1, \ \omega \in [k, k+1), \ k = 0, 1, 2, \ldots. \tag{6}$$

By integrating Eq. (5) from k to $k+1$, one can obtain

$$\varphi(k+1) + p\varphi(k+1-\tau) - \varphi(k) - p\varphi(k-\tau) = \int_k^{k+1} \alpha(t)\varphi(\omega - \tau)\,d\omega + \varphi(k-1)\int_k^{k+1} \beta(\omega)\,d\omega.$$

$$\varphi(k+1) + p\varphi(k+1-\tau) \geq \int_k^{k+1} \alpha(\omega)\varphi(\omega - \sigma)\,d\omega + \varphi(k-1)\int_k^{k+1} \beta(\omega)\,d\omega.$$

By applying Eq. (6), the last inequality leads to

$$\varphi(k+1-\tau)\psi(k+1) + p\varphi(k+1-\tau) \geq \varphi(k-\sigma)\int_k^{k+1} \alpha(\omega)\,d\omega + \varphi(k-1)\int_k^{k+1} \beta(\omega)\,d\omega.$$

Since $\sigma \geq 1$, and $\tau - \sigma \geq 1$, then $\varphi(k-1) \geq \varphi(k-\sigma) \geq \varphi(k+1-\tau)$,

$$\varphi(k-\sigma)[\psi(k+1) + p] \geq \varphi(k+1-\tau)[\psi(k+1) + p] \geq \varphi(k-\sigma)\int_k^{k+1} (\alpha(\omega) + \beta(\omega))\,d\omega.$$

$$\varphi(k-\sigma)[\psi(k+1) + p] \geq \varphi(k-\sigma)\int_k^{k+1} (\alpha(\omega) + \beta(\omega))\,d\omega.$$

hence

$$\psi(k+1) + p \geq \int_k^{k+1} (\alpha(\omega) + \beta(\omega))\,d\omega, \ k = 0, 1, 2, \ldots \tag{7}$$

That means 7 is a contradiction with 4. In the case where the solution is negative, the proof can be treated in the same way.

Theorem 2: *Assume that $\alpha(\omega) \leq 0$, $\beta(\omega) < 0$, $\sigma \geq 1$, and there is a continuous function $\lambda(\omega) \geq 1$, such that for $t \geq 0$:*

$$\int_k^{k+1} (|\beta(\omega)| + |\alpha(\omega)|)d\omega > p\lambda(\omega) + 1, \ k = 0, 1, 2, \ldots. \tag{8}$$

Then Eq. (1) possesses only an only an oscillatory solution.

Proof: Consider the case where Eq. (1) has an eventually positive solution $\varphi(\omega)$. For any $k = 0, 1, 2, \ldots,$ and $\omega \in [k, k+1)$, from Eq. (5), it follows that $G'(\omega) \leq 0$, so $G(\omega)$ is nonincreasing for any $k = 1, 2, \ldots,$ and $\omega \in [k, k+1)$. Note that $\varphi(\omega)$ is must be nonincreasing, otherwise if $\varphi(\omega)$ is increasing function that is

$$\varphi(\omega - \tau) < \varphi(\omega - \tau + 1) \text{ for } \omega \in [k, k+1).$$

On the other side, since $G(\omega)$ is nonincreasing, so $G(\omega+1) \leq G(\omega)$ that is

$$0 < \varphi(\omega+1) + p\varphi(\omega+1-\tau) \leq \varphi(\omega) + p\varphi(\omega-\tau), \ \omega \in [k, k+1).$$

$$0 < \varphi(\omega+1) \leq \varphi(\omega) + p[\varphi(\omega-\tau) - \varphi(\omega+1-\tau)].$$

But $\varphi(\omega-\tau) - \varphi(\omega-\tau+1) < 0$, then $\varphi(\omega+1) < \varphi(\omega)$, which means that $\varphi(\omega)$ is decreasing function in $[k, k+1)$, in a contradiction, hence $\varphi(\omega)$ a nonincreasing function for $\omega \in [k, k+1)$. Let

$$\psi(\omega) \geq \frac{\varphi(\omega-\tau)}{\varphi(\omega)} \geq 1, \ \omega \in [k, k+1), \ k = 0, 1, 2, \ldots. \tag{9}$$

By integrating Eq. (5) from k to $k+1$, one can obtain

$$\varphi(k+1) + p\varphi(k+1-\tau) - \varphi(k) - p\varphi(k-\tau) = \int_k^{k+1} \alpha(\omega)\varphi(\omega-\tau)\,d\omega + \varphi(k-1)\int_k^{k+1}\beta(\omega)\,d\omega.$$

$$-\varphi(k) - p\varphi(k-\tau) \leq \int_k^{k+1} \alpha(\omega)\varphi(\omega-\tau)\,d\omega + \varphi(k-1)\int_k^{k+1}\beta(\omega)\,d\omega.$$

$$\varphi(k) + p\varphi(k-\tau) \geq \int_k^{k+1} |\alpha(\omega)|\varphi(\omega-\sigma)\,dt + \varphi(k-1)\int_k^{k+1}|\beta(\omega)|\,d\omega.$$

By applying in Eq. (9), the last inequality leads to

$$\varphi(k) + p\varphi(k)\psi(k) \geq \varphi(k+1-\sigma)\int_k^{k+1} |\alpha(\omega)|\,d\omega + \varphi(k-1)\int_k^{k+1}|\beta(\omega)|\,d\omega. \tag{10}$$

Since $\sigma \geq 1$, then $\varphi(k+1-\sigma) \geq \varphi(k)$ and $\varphi(k-1) \geq \varphi(k)$, thus in Eq. (10) leads to

$$\varphi(k)[1 + p\psi(k)] \geq \varphi(k)\int_k^{k+1}(|\alpha(\omega)| + |\beta(\omega)|)\,d\omega.$$

Taking into account the positivity of $\varphi(k)$, yields

$$1 + p\psi(k) \geq \int_k^{k+1}(|\alpha(\omega)| + |\beta(\omega)|)\,d\omega, \ k = 0, 1, 2, \ldots. \tag{11}$$

From in Eq. (11) , one got a contradiction with in Eq. (8). In the case where the solution is negative, the proof can be treated in the same way.

Proposition 1: Assume that $\alpha(\omega) = 0$, $\beta(\omega) \leq 0$ for Eq. (1), If

$$1 + p \leq \int_k^{k+1} |\beta(\omega)|\,d\omega, \ k = 0, 1, 2, \ldots. \tag{12}$$

Then every solution of Eq. (1) oscillates.

Proof: Since $\alpha(\omega) = 0$, Eq. (1) becomes

$$[\varphi(\omega) + p\varphi(\omega-\tau)]' - \beta(\omega)\varphi([\omega-1]) = 0.$$

Suppose for a contradiction there is a nonoscillatory solution of Eq. (1). Let $\varphi(\omega) > 0$ and $\varphi(\omega-\tau) > 0, \ k = 0, 1, \ 2, \ \ldots$. For $\omega \in [k, k+1)$, Eq. (1) reduces to

$$[\varphi(\omega) + p\varphi(\omega-\tau)]' = \beta(\omega)\varphi(k-1) \leq 0.$$

So $\varphi(\omega) + p\varphi(\omega-\tau)$ is nonincreasing. One can claim that $\varphi(k-\tau) - \varphi(k+1-\tau) \geq 0, \ k = 0, 1, 2, \ldots$. Otherwise, if $\varphi(k-\tau) - \varphi(k+1-\tau) \leq 0$, or $\varphi(k-\tau) \leq \varphi(k+1-\tau)$ hence $\varphi(k)$ is nondecreasing $k = 0, 1, 2, \ldots$.

On the other side $\varphi(k) + p\varphi(k-\tau) > \varphi(k+1) + p\varphi(k+1-\tau)$ this implies that $\varphi(k) > \varphi(k+1)$, this mean $\varphi(k)$ is decreasing function, a contradiction, hence $\varphi(k-\tau) - \varphi(k+1-\tau) \geq 0$ or $\varphi(k-\tau) \geq \varphi(k+1-\tau)$

That φ is nonincreasing function in $[k-\tau, k+1-\tau)$.

By integrating Eq. (7) from $k \; to \; k+1$, it becomes

$$\varphi\left(k+1\right) + p\varphi\left(k+1-\tau\right) - \varphi\left(k\right) - p\varphi\left(k-\tau\right) = -\varphi\left(k-1\right)\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega.$$

$$-\varphi\left(k\right) - p\varphi\left(k-\tau\right) \leq -\varphi\left(k-1\right)\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega.$$

Since $\varphi(k)$ is a nonincreasing function, the final inequality mentioned above becomes:

$$-\varphi\left(k-1\right) - p\varphi\left(k-\tau\right) \leq -\varphi\left(k-1\right)\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega.$$

Since $\tau > 1$,

$$\varphi\left(k-\tau\right)\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega - \varphi\left(k-\tau\right) \leq p\varphi\left(k-\tau\right).$$

$$\varphi\left(k-1\right)\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega - \varphi\left(k-1\right) \leq p\varphi\left(k-1\right).$$

$$\varphi\left(k-1\right)\left(\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega - 1\right) \leq p\varphi\left(k-1\right).$$

$$\int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega - 1 \leq p.$$

$$1 + p \geq \int_k^{k+1} \left|\beta\left(\omega\right)\right| d\omega.$$

This is a contradiction with condition 12.

Theorem 3: *Suppose that* $0 \leq p < 1, \; \alpha(\omega) \geq 0, \; \beta(\omega) \geq 0, \; \tau + \sigma \geq 1$, *and*

$$\lim_{\omega\to\infty} \inf \int_\omega^{\omega+1} \left(p\alpha\left(s\right) - \beta\left(s\right)\right) ds > \frac{1}{1-p}. \tag{13}$$

Then every bounded solution of Eq. (1) oscillate.

Proof: Assume that there is a nonoscillatory bounded solution $\varphi(\omega)$ to Eq. (1). Let $\varphi(\omega) > 0$ and $\varphi(\omega - \tau) > 0$ for $\omega > T$. From Eq. (1), it follows that $G(t)$ is nondecreasing bounded function, hence $\varphi(\omega)$ is nondecreasing function for $\omega \in [k, k+1)$.

From Eq. (2) , one can get

$$\varphi\left(\omega\right) = G\left(\omega\right) - p\varphi\left(\omega - \tau\right).$$

$$\varphi\left(\omega\right) \geq G\left(\omega\right) - pG\left(\omega - \tau\right) \geq G\left(\omega\right)\left(1-p\right).$$

Hence, Eq. (5) leads to

$$G'\left(\omega\right) - \alpha\left(\omega\right)G\left(\omega - \sigma\right) + p\alpha\left(\omega\right)\varphi\left(\omega - \sigma - \tau\right) - \beta\left(\omega\right)\varphi\left(k-1\right) = 0,$$

$$G'\left(\omega\right) - \alpha\left(\omega\right)G\left(\omega\right) \leq \beta\left(\omega\right)\varphi\left(k-1\right) - p\alpha\left(\omega\right)\varphi\left(\omega - \sigma - \tau\right),$$

$$G'(\omega)e^{-\int_k^{\omega}\alpha(s)ds}-\alpha(\omega)G(\omega)e^{-\int_k^{\omega}\alpha(s)ds}\le\left(\beta(\omega)\varphi(k-1)-p\alpha(\omega)\varphi(\omega-\sigma-\tau)\right)e^{-\int_k^{\omega}\alpha(s)ds},$$

$$\left(G(\omega)e^{-\int_k^{\omega}\alpha(s)ds}\right)'\le\varphi(k-1)\left(\beta(\omega)-p\alpha(\omega)\right)e^{-\int_k^{\omega}\alpha(s)ds}.$$

Taking into account condition 13, we conclude that $p\alpha(\omega)-\beta(\omega)\ge 0$, for $\omega\ge k\ge k_1$ where k_1 large enough. Integrate the last inequality from k to $k+1$

$$G(k+1)e^{-\int_k^{k+1}\alpha(s)ds}-G(k)\le\varphi(k-1)\int_k^{k+1}\left(\beta(\omega)-p\alpha(\omega)\right)e^{-\int_k^{\omega}\alpha(s)ds}d\omega,$$

$$\frac{G(k+1)}{G(k)}\frac{G(k)}{G(k-1)}e^{-\int_k^{k+1}\alpha(s)ds}-\frac{G(k)}{G(k-1)}\le\frac{\varphi(k-1)}{G(k-1)}\int_k^{k+1}\left(\beta(\omega)-p\alpha(\omega)\right)e^{-\int_k^{\omega}\alpha(s)ds}d\omega,$$

$$\le\frac{\varphi(k-1)}{G(k)}\frac{G(k)}{G(k-1)}\int_k^{k+1}\left(\beta(\omega)-p\alpha(\omega)\right)e^{-\int_k^{\omega}\alpha(s)ds}d\omega,$$

Let $H(\omega)=\frac{G(\omega)}{G(\omega-1)}\ge 1$, since $\varphi(\omega)$ is bounded, and $G(\omega)$ is also bounded, hence $\lim_{\omega\to\infty}H(\omega)=1$, and $H(\omega)$ is bounded.

$$H(k+1)H(k)e^{-\int_k^{k+1}\alpha(s)ds}-H(k)\le-\frac{\varphi(k-1)}{G(k)}H(k)\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)e^{-\int_k^{\omega}\alpha(s)ds}d\omega,$$

$$H(k)\left[H(k+1)e^{-\int_k^{k+1}\alpha(s)ds}+\frac{\varphi(k-1)}{G(k)}\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)e^{-\int_k^{\omega}\alpha(s)ds}d\omega-1\right]\le 0$$

$$H(k)e^{-\int_k^{k+1}\alpha(s)ds}\left[H(k+1)+\frac{\varphi(k-1)}{G(k)}\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)d\omega-1\right]\le 0$$

$$\frac{\varphi(k-1)}{G(k)}\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)d\omega\le 1$$

$$\frac{G(k-1)(1-p)}{G(k)}\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)d\omega\le 1$$

$$\frac{(1-p)}{H(k)}\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)d\omega\le 1$$

$$\int_k^{k+1}\left(p\alpha(\omega)-\beta(\omega)\right)d\omega\le\frac{H(k)}{1-p}$$

As $k\to\infty$, due to condition 13, the last inequality amounts to a contradiction.

Applications (section three)

In this section, some illustrative examples of the results obtained are presented.

Example 1: *Consider the following neutral differential equation with piecewise constant argument:*

$$\frac{d}{d\omega}\left[\varphi(\omega)+\frac{1}{2}\varphi(\omega-2\pi)\right]-\left(3-\sin(2k-2)\cos 2\omega\right)\varphi\left(\omega-\frac{3\pi}{4}\right)-\cos^2 2\omega\varphi(k-1)=0,$$

$$k=1,2,\ldots. \tag{14}$$

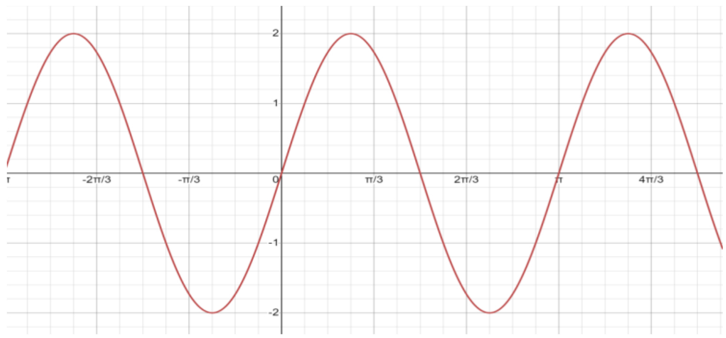

Fig. 1. $\varphi(\omega) = 2sin2\omega$ is an oscillatory solution of Eq. (14).

Note that $\tau = 2\pi$, $\sigma = \frac{3\pi}{4} > 1, p = \frac{1}{2}$, $\alpha(\omega) = 3 - \sin(2k-2)\cos 2\omega > 0$, $\beta(\omega) = \cos^2 2\omega \geq 0$. Let $\lambda(\omega) = 2$, by checking condition 4, one can see that

$$\frac{5}{2} < \int_k^{k+1} \left(3 - \sin\left(2k-2\right)\cos 2\omega + \cos^2 2\omega\right) d\omega, \quad k = 0, 1, 2, \ldots$$

So, all conditions of Theorem 1 are satisfied. Hence according to Theorem 1, every solution of Eq. (14) oscillate, the solution $\varphi(\omega) = 2\sin 2\omega$ is such an oscillatory solution. The shape of the solution of Eq. (14) can be seen in Fig. 1.

Example 2: *Consider the first order neutral differential equation with piecewise constant argument:*

$$\frac{d}{d\omega}\left[\varphi\left(\omega\right) + 2e^{-\pi}\varphi\left(\omega - 2\pi\right)\right] + \left(\sqrt{8}e^{-\frac{5\pi}{4}} + 2e^{-\frac{k}{2}+\frac{5\pi+2}{4}}\sin\frac{k-1}{2}\left(\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\right)\right)\varphi\left(\omega - \frac{5\pi}{2}\right)$$
$$+\left(\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\right)^2\varphi\left(k-1\right) = 0, \quad k = 1, 2, \ldots. \tag{15}$$

Note that $\tau = 2\pi$, $\sigma = \frac{5\pi}{2} > 1, p = 2e^{-\pi}, \alpha(\omega) = -(\sqrt{8}e^{-\frac{5\pi}{4}} + 2e^{-\frac{k}{2}+\frac{5\pi+2}{4}}\sin\frac{k-1}{2}(\cos\frac{\omega}{2} - \sin\frac{\omega}{2})) < 0$, $\beta(\omega) = -(\cos\frac{\omega}{2} - \sin\frac{\omega}{2})^2 \leq 0$.
Let $\lambda(\omega) = \frac{1}{2}$, let's check condition 8, one can see that

$$\int_k^{k+1}(|\beta\left(\omega\right)| + |\alpha\left(\omega\right)|)d\omega = \int_k^{k+1}\left(\left(\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\right)^2 + \left(\sqrt{8}e^{-\frac{5\pi}{4}} + 2e^{-\frac{k}{2}+\frac{5\pi+2}{4}}\sin\frac{k-1}{2}\left(\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\right)\right)\right)d\omega$$
$$> e^{-\pi} + 1.$$

So, all conditions of Theorem 2 are satisfied. Hence, according to Theorem 2, every solution of Eq. (15) oscillates, the solution $\varphi(\omega) = 2e^{-\frac{\omega}{2}}\sin\frac{\omega}{2}$ is such an oscillatory solution, and $\lim_{\omega\to\infty}\varphi(\omega) = 0$. The shape of the solution of Eq. (15) can be seen in Fig. 2.

Example 3: *Consider the first order neutral differential equation with piecewise constant argument:*

$$\left[\varphi\left(\omega\right) + 2e^{-\pi}\varphi\left(\omega - 2\pi\right)\right]' - \left(\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} - e^{-\frac{\omega}{2}-\frac{k-1}{2}+10}\left(\sin\frac{\omega}{2} - \cos\frac{\omega}{2}\right)\sin\left(\frac{k-1}{2}\right)\right)\varphi\left(\omega - \frac{\pi}{2}\right)$$
$$-\frac{1}{\sqrt{2}}e^{-\frac{\omega}{2}+\frac{\pi}{4}+10}\left(\sin\frac{\omega}{2} - \cos\frac{\omega}{2}\right)^2\varphi\left(k-1\right) = 0, \quad k = 1, 2, \ldots. \tag{16}$$

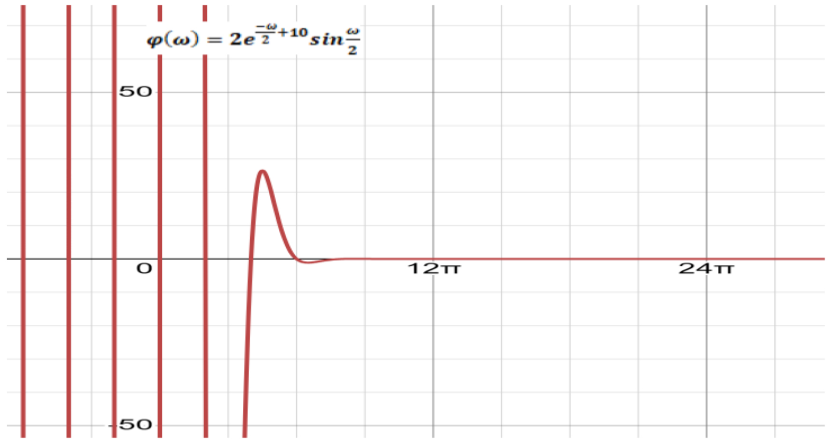

Fig. 2. $\varphi(\omega) = 2e^{\frac{\omega}{2}} sin\frac{\omega}{2}$ is an oscillatory solution of Eq. (15) and $\lim_{\omega\to\infty}\varphi(\omega) = 0$.

Note that $\tau = 2\pi$, $\sigma = \frac{\pi}{2} > 1$, $p = 2e^{-\pi}$, $\alpha(\omega) = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} - e^{-\frac{\omega}{2}-\frac{k-1}{2}+10}(\sin\frac{\omega}{2} - \cos\frac{\omega}{2})\sin(\frac{k-1}{2}) > 0$, $\beta(\omega) = \frac{1}{\sqrt{2}}e^{-\frac{\omega}{2}+\frac{\pi}{4}+10}(\sin\frac{\omega}{2} - \cos\frac{\omega}{2})^2 \geq 0$. to check condition 13, one can see that

$$\liminf_{t\to\infty}\int_{\omega}^{\omega+1}(p\alpha(s) - \beta(s))\,ds = \liminf_{\omega\to\infty}\int_{\omega}^{\omega+1}\left(2e^{-\pi}\left(\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} - e^{-\frac{s}{2}-\frac{k-1}{2}+10}\left(\sin\frac{s}{2} - \cos\frac{s}{2}\right)\right)\right.$$
$$\left. - \frac{1}{\sqrt{2}}e^{-\frac{s}{2}+\frac{\pi}{4}+10}\left(\sin\frac{s}{2} - \cos\frac{s}{2}\right)^2\right) ds = \infty.$$

So, all conditions of Theorem 4 are satisfied. Hence, according to Theorem 4, every bounded solution of Eq. (16) oscillates, the solution $\varphi(\omega) = e^{-\frac{\omega}{2}+10}\sin\frac{\omega}{2}$ is an oscillatory solution, and $\lim_{\omega\to\infty}\varphi(\omega) = 0$. the shape of the solution of Eq. (16) can be seen in Fig. 3:

Fig. 3. $\varphi(\omega) = 2e^{\frac{\omega}{2}+10} sin\frac{\omega}{2}$ is oscillatory solution of Eq. (16) and $\lim_{\omega\to\infty}\varphi(\omega) = 0$.

Results and discussion

The results that were reached required finding sufficient conditions for the solutions of neutral equations to oscillate, the findings showed that the behavior of the solution is dependent on the coefficient p; if p is greater than zero, the oscillation conditions are different from those for p between zero and one. Additionally, the oscillation is influenced by the delays τ and σ, which led to classifying the space of the coefficients into two separate classes, for each class, the appropriate conditions were obtained, and all the solutions of the given examples to the first-order neutral differential equation with piecewise constant argument are oscillatory and approaches to zero as the independent variable approaches to infinity.

Conclusion

Understanding and computing all possible solutions to the first-order neutral differential equation with piecewise constant argument. Moreover, oscillation is trend focuses on researching and determining the necessary and sufficient conditions for the oscillation of solutions to the first-order neutral differential equation with piecewise constant argument.

Adequate conditions were drawn in this way to produce the convergence to zero or divergence of all solutions for the first-order neutral differential equation with piecewise constant argument for $t \longrightarrow \infty$. All of the acquired results are presented with illustrative examples.

Future work: Finding the oscillations to the second and third-order neutral differential equation with piecewise constant argument.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Technology.

Authors' contribution statement

S. A. and H. A. found the oscillation of neutral differential equation with piecewise constant argument.

References

1. Aftabizadeh AR, Wiener J, Xu J-M. Oscillatory and Periodic Solutions of Delay Differential Equations with Piecewise Constant Argument. Amer Math Soc. 1987;99(4):673–679. https://doi.org/10.2307/2046474.
2. Cooke KL. Retarded Differential Equations with Piecewise Constant Delays. J Math Anal Appl. 1984;99(1):265–297. https://doi.org/10.1016/0022-247X(84)90248-8.
3. Qaraad B, Bazighifan O, Nofal TA, Ali AH. Neutral differential equations with distribution deviating arguments: Oscillation conditions. J Oceen Eng Sci. Published online 2022;19:35. https://doi.org/10.1016/j.joes.2022.06.032.
4. Hadeed IS, Mohamad HA. Oscillation of the Solutions for Hematopoiesis Models. Iraqi J Sci. 2023;64(10):5165–5172. https://doi.org/10.24996/ijs.2023.64.10.24.
5. Li H-X. Almost periodic solutions of second-order neutral delay–differential equations with piecewise constant arguments. J Math Anal Appl. 2004;298(2):693–709. https://doi.org/10.1016/j.jmaa.2004.05.034.
6. Agwo HA. Necessary and sufficient conditions for the oscillation of delay differential equation with piecewise constant argument. Int J Math Math Sci. 1998;21(3):493–498. http://dx.doi.org/10.1155/S0161171298000702.
7. Lakshmikantharn V. Stability and Oscillation for Linear and Nonlinear Scalar Equations with Piecewise Constant Argument. Appl Anal. 1992;1(4):113–125. https://doi.org/10.1080/00036819208840071
8. Papaschinopoulos G. Schinas J. Existence stability and oscillation of the solutions of first order neutral delay differential equations with piecewise constant argument. Appl Anal. 2015;44(1-2):99–111. https://doi.org/10.1080/00036819208840070.
9. Wang G-q, Cheng SS. Existence and Uniqueness of Periodic Solutions for a Second-Order Nonlinear Differential Equation with Piecewise Constant Argument. Int J Math Math Sci. 2009;4:14. Article ID 950797. https://doi.org/10.1155/2009/950797.
10. Mohamad HA, Jaddoa AF. Oscillation Criteria for Solutions of Neutral Differential Equations of Impulses Effect with Positive and Negative Coefficients. Baghdad Sci J. 2020;17(2):537–544. https://doi.org/10.21123/bsj.2020.17.2.0537.
11. Agarwal RP, Saker SH. Oscillation of Solutions to Neutral Delay Differential Equations with Positive and Negative Coefficients. Sci Dir Work Paper. 2001;2(4):449–465.
12. Bazighifan O, Grace SR, Alzabut J, Özbekler A. New results for oscillatory properties of neutral differential equations with a p-Laplacian like operator. Miskolc Math Notes. 2020;21(2):631–640. http://dx.doi.org/10.18514/MMN.2020.3322
13. Qaraad B, Bazighifan O, Nofal TA, Ali HA. Neutral differential equations with distribution deviating arguments: Oscillation conditions. J Ocean Eng Sci. 2022;2022;19:35. https://doi.org/10.1016/j.joes.2022.06.032.
14. Garab A, Stavroulakis IP. Oscillation criteria for first order linear delay differential equations with several variable delays. Appl Math Lett. 2020;106(1063662020):1–9. https://doi.org/10.1016/j.aml.2020.106366.
15. Jaddoa AF, Sharba BAJ Existence. Qualitative Property of Differential Equation with Delayed Arguments. Iraqi J Sci. 2023;64(9):4669–4686. https://doi.org/10.24996/ijs.2023.64.9.30.

16. Moaaz O, Albalawi W. Differential equations of the neutral delay type: More efficient conditions for oscillation. AIMS Math. 2023;8:12729–12750. https://doi.org/10.3934/math.2023641.
17. Muminov MI, Radjabov TA. Existence conditions for 2-periodic solutions to nonhomogeneous differential equations with piecewise constant argument. Ex Counterexamples. 2024;5(100145):1–3. https://doi.org/10.1016/j.exco.2024.100145.
18. Bazighifan O, Alotaibi H, Mousa AA. Neutral Delay Differential Equations: Oscillation Conditions for the Solutions. Symmetry. 2021;13(1):1–11. https://doi.org/10.3390/sym13010101.
19. Zhiguo L, Jianhua S. New Results on Oscillation for Delay Differential Equations with Piecewise Constant Argument. J Comp Math Appl. 2003;45:1841–1848. https://doi.org/10.1016/S0898-1221(03)90005-8.

شروط التذبذب لحلول المعادلات التفاضلية المحايدة الثابتة من الدرجة الأولى

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الخلاصة

سعى البحث إلى دراسة شكل جديد من المعادلات وكذلك المعايير الممكنة للتذبذب وعدم التذبذب لجميع حلول المعادلات التفاضلية المحايدة من الدرجة الأولى ذات الثابت القطعي، سواء كانت هذه الحلول التذبذبية متقاربة أو غير متقاربة، من خلال إيجاد شروط كافية تضمن تذبذب جميع حلول هذه المعادلات. وفقا لمعادلات تفاضلية محايدة مع جزء ثابت مقطعيا، تحتوي هذه المعادلات على ثلاثة تأخيرات وثلاثة معاملات، مما يتطلب أن يكون جزء من المعاملات في هذه المعادلة متصلًا قطعيًا. في ذلك تُعرف الدالة التي تفترض قيمًا ثابتة على فترات زمنية محددة مسبقًا باسم وسيطة الثابت القطعي. تعد أكبر دالة صحيحة [ω]، التي تقرب أي عدد حقيقي ω إلى أقرب عدد صحيح، مثالًا نموذجيًا. تتطلب معايير التذبذب لهذا النوع من المعادلات اكتساب بعض الشروط الأساسية والكافية لضمان تحقيق هذا الهدف. تمثل أكبر دالة صحيحة الثابت القطعي. يجب أن تكون جميع الحلول لهذا النوع من المشاكل متذبذبة، وقد تم ابتكار الشروط الضرورية والكافية لضمان ذلك. وبناءً على هذه الشروط، يمكننا أن نستنتج أن التقنية المستخدمة لإنتاج النتائج تعتمد على تصنيف المعاملات إلى نوعين مستقلين.

الكلمات المفتاحية: السلوك المقارب ، المعادلة التفاضلية ، التأخير ، التذبذب ، ثابت مقطعيا.