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العدد الثالث

لسنة ٢٠٠٩

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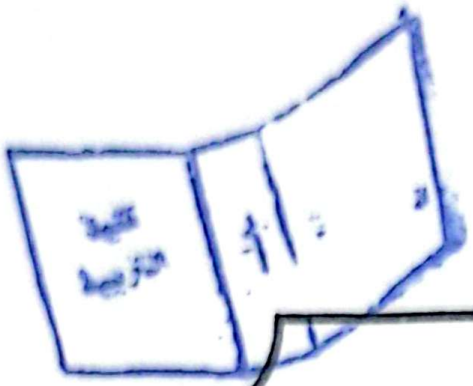

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On The Classical Groups

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Abstract

The main object of this work is to calculate the graded characters of classical groups A_n ,C_n ,D_n by using the basic spin characters .

الخلاصة

الهدف الأساسي لهذا البحث هو حساب الشخوص المتدرجة العقدية للزمر الكلاسيكية من النوع C_n ,A_nو D_n .

*1.*Introduction

Let V be an n dimensional real Euclidean space with inner product <,>.

For each $r \in V$,$r \neq 0$,let ω_r be the reflection in the hyperplane orthogonal to r , i.e. if

$$\omega_r(x) = x - \frac{2 < r,x >}{< r,r >} r.$$

In [5]Morris define the Weyl group $W=W(\Phi)$ is the group generated by the reflections ω_r ,$r \in \Phi$,where Φ is a finite subset of V which spans V such that Φ is the root system of a simple Lie algebra ,and he classified W into two kinds ,the first kind is called the classical

types which contain the types A_n ,B_n ,C_n and D_n .The second kind is called the exceptional Weyl groups which contain the type E_n (n=6,7,8) ,F_4 and G_2 ,and he studied their basic Spin characters .

In this paper we calculate explicitly **the graded character** of the Weyl groups of the type A_n ($n\geq 1$) ,C_n ($n\geq 3$)and D_n ($n\geq 4$) ,by using their basic Spin characters which are called **the Classical groups** .

2.On The Graded Character

In this section we give the definition of the graded character of the orthogonal group and calculate explicity the graded characters by using its basic spin characters.

Definition (2.1) **[1]** : Let V be an n-dimensional real Euclidean space with inner product $<,>$,and O(V) be the group of orthogonal linear transformation of V ,A subset Φ of V is called a root system in V if it satisfies the following conditions :

i- Φ is a finite set of non zero vectors, and Φ generates (spans) V .

ii- If $r\in\Phi$,then $\tau_r(\Phi)=\Phi$,where τ_r is an orthogonal map $\tau_r\in O(V)$.

iii- If $r,s\in\Phi$,then $\frac{2\langle r,s\rangle}{\langle r,r\rangle}\in Z$.

iv- If $r,\lambda_r\in\Phi$,and $\lambda\in R$ (real number),then $\lambda=\pm 1$.

Also the dimension of V is called the rank of the root system .

Definition (2.2) **[5]** : Let Φ be a root system in V ,then the Weyl group of Φ which denoted by $W(\Phi)$ is the group generated by the reflection τ_r, $r\in\Phi$.

We note that $W(\Phi)$ is a subgroup of O(V) .

Definition (2.3) [1] : Let O(n) be an orthogonal group of all n×n orthogonal matrices ,if $\tau \in O(n)$ then the basic Spin representation of O(n) is defined as follows :

$$\overline{P}(\tau) = P(\widetilde{P}^{-1}(\tau)),$$
$$\overline{P}'(\tau) = P'(\widetilde{P}'^{-1}(\tau)),$$

Where P *and* P' are the representation of the Clifford algebra [1] .

In particular, if $u=\sum_{i=1}^{n}\alpha_i e_i \in V$,and τ_u is the corresponding reflection in O(n) ,then :

$$\overline{P}(\tau_u) = P(\widetilde{P}^{-1}(\tau_u)) = P(\sum_{i=1}^{n}\alpha_i e_i) = \sum_{i=1}^{n}\alpha_i M_i,$$

Similariy $\overline{P}'(\tau_u) = \sum_{i=1}^{n}\alpha_i M_i',$

where M_i and M'_i (i=1,…,n) are defined in [4].then the representations $\overline{P}$*and* $\overline{P}'$ are the basic spin representation of O(n) of the degree $2^{n/2}$,and we compute its value by the following theorem .

Theorem (2.1) **[4]**:Let $\overline{P}$*and* $\overline{P}'$ are the basic spin representation of the representation P and P' respectively ,then :

$$i - \overline{P}(\tau_1) = \overline{P}(\tau_2) = \begin{bmatrix} e^{\frac{1}{2}i\theta_1} & 0 \\ 0 & e^{-\frac{1}{2}i\theta_1} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e^{\frac{1}{2}i\theta_v} & 0 \\ 0 & e^{-\frac{1}{2}i\theta_v} \end{bmatrix},$$

$$ii - \overline{P}(\tau_3) = \begin{bmatrix} o & i \\ -i & 0 \end{bmatrix} \otimes \begin{bmatrix} e^{\frac{1}{2}i\theta_2} & 0 \\ 0 & e^{-\frac{1}{2}i\theta_2} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e^{\frac{1}{2}i\theta_v} & 0 \\ 0 & e^{-\frac{1}{2}i\theta_v} \end{bmatrix},$$

$$iii - \overline{P}(\tau_4) = \begin{bmatrix} e^{\frac{1}{2}i\theta_1} & 0 \\ 0 & -e^{-\frac{1}{2}i\theta_1} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e^{\frac{1}{2}i\theta_v} & 0 \\ 0 & -e^{-\frac{1}{2}i\theta_v} \end{bmatrix},$$

$$iv - \overline{P}'(\tau_1) = \overline{P}'(\tau_2) = \begin{bmatrix} e^{-\frac{1}{2}i\theta_1} & 0 \\ 0 & e^{\frac{1}{2}i\theta_1} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e^{-\frac{1}{2}i\theta_v} & 0 \\ 0 & e^{\frac{1}{2}i\theta_v} \end{bmatrix},$$

$$v - \overline{P}'(\tau_3) = \begin{bmatrix} o & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} e^{-\frac{1}{2}i\theta_2} & 0 \\ 0 & e^{\frac{1}{2}i\theta_2} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e^{-\frac{1}{2}i\theta_v} & 0 \\ 0 & e^{\frac{1}{2}i\theta_v} \end{bmatrix},$$

$$vi - \overline{P}'(\tau_4) = i\begin{bmatrix} e^{-\frac{1}{2}i\theta_1} & 0 \\ 0 & -e^{\frac{1}{2}i\theta_1} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e^{-\frac{1}{2}i\theta_v} & 0 \\ 0 & -e^{\frac{1}{2}i\theta_v} \end{bmatrix}.$$

Now we can obtain the basic spin characters by the following corollary.

Corollary (2.2) : Let $\bar{\chi}$ *and* $\bar{\chi}'$ be denote the characters of $\bar{P}$ *and* $\bar{P}'$ respectively ,then :

$$i-\bar{\chi}(\tau_1)=\bar{\chi}(\tau_2)=\prod_{j=1}^{\nu}2\cos\frac{1}{2}\theta_j \ ,$$

$$ii-\bar{\chi}(\tau_3)=0 \ ,$$

$$iii-\bar{\chi}(\tau_4)=i^{\nu}\prod_{j=1}^{\nu}2\sin\frac{1}{2}\theta_j \ ,$$

$$iv-\bar{\chi}'(\tau_1)=\bar{\chi}'(\tau_2)=\prod_{j=1}^{\nu}2\cos\frac{1}{2}\theta_j \ ,$$

$$v-\bar{\chi}'(\tau_3)=0 \ ,$$

$$vi-\bar{\chi}(\tau_4)=i^{\nu+1}\prod_{j=1}^{\nu}2\sin\frac{1}{2}\theta_j \ .$$

Now, we can see the following lemma and the two definitions in [3] .

Lemma (2.3) : Let SO(n) be the set of all the special orthogonal matrices forms a subgroup of O(n) ,then :

i- If $\tau\in SO(n)$ *then* $\bar{\chi}(\tau)\neq o$ *and* $\bar{\chi}'(\tau)\neq o$ iff -1 is not an eigen value of τ .

ii- If $\tau\in O(n)/SO(n)$ *then* $\bar{\chi}(\tau)\neq o$ *and* $\bar{\chi}'(\tau)\neq o$ iff 1 is not an eigen value of τ .

Definition (2.4) : Let $H\leq G$ be a finite group such that [G:H]=2 and T be irreducible projective representation of G then T is said to be split in H if $T=T_1\oplus T_2$ on H ,where T_1 and T_2 are irreducible projective representation of H .

Definition (2.5) : Let [G:H]=2 and an irreducible projective representation P of G split in H ,then

$\tilde{P}(g) \cong (-1)^{v(g)} P(g)$,$\forall g \in G$ *where*

$$v(g) = \begin{cases} o & \text{if } g \in H \\ 1 & \text{other wise} \end{cases}$$,*is called the projective representation associate to P.*

Thus, if $\overline{\chi}$ is the character of P and δ is the character of $\tilde{P}$ then δ is an associate character to $\overline{\chi}$.

Theorem (2.4) : Let $\overline{\chi}$ *and* $\overline{\chi}'$ are the basic spin characters of $\overline{P}$ *and* $\overline{P}'$ respectively ,and let δ and δ' denote the associate character to $\overline{\chi}$ *and* $\overline{\chi}'$ respectively ,then :

$$i - \delta(\tau_1) = \delta(\tau_2) = \prod_{j=1}^{v} 2\cos\frac{1}{2}\theta_j \ ,$$

$$ii - \delta(\tau_3) = 0 \ ,$$

$$iii - \delta(\tau_4) = -i^v \prod_{j=1}^{v} 2\sin\frac{1}{2}\theta_j \ ,$$

$$iv - \delta'(\tau_1) = \delta'(\tau_2) = \prod_{j=1}^{v} 2\cos\frac{1}{2}\theta_j \ ,$$

$$v - \delta'(\tau_3) = 0 \ ,$$

$$vi - \delta'(\tau_4) = -i^{v+1} \prod_{j=1}^{v} 2\sin\frac{1}{2}\theta_j \ .$$

Proof: By def.2.5. , $\tilde{\overline{P}}(\tau_1) \cong (-1)^{v(\tau_1)} \overline{P}(\tau_1)$, since $\det(\tau_1)=1$ then $\tau_1 \in SO(n)$,thus $v(\tau_1)=0$,hence $\tilde{\overline{P}}(\tau_1) \cong \overline{P}(\tau_1)$, but $trace(\tilde{\overline{P}}(\tau_1)) = trace(\overline{P}(\tau_1))$,then

$\delta(\tau_1) = \overline{\chi}(\tau_1) = \prod_{j=1}^{v} 2\cos\frac{1}{2}\theta_j$, *hence* $\delta(\tau_1) = \delta(\tau_2) = \prod_{j=1}^{v} 2\cos\frac{1}{2}\theta_j$.

To prove $\delta(\tau_3)=0$, by def.2.5. , $\tilde{\bar{P}}(\tau_3)\cong(-1)^{v(\tau_3)}\bar{P}(\tau_3)$, since $\det(\tau_3)$=-1 ,then $\tau_3\in O(n)/SO(n)$,thus $v(\tau_3)$=1 ,hence $\tilde{\bar{P}}(\tau_3)\cong-\bar{P}(\tau_3)$, but $trace(\tilde{\bar{P}}(\tau_3))=-trace(\bar{P}(\tau_3))$, then $\delta(\tau_3)=-\bar{\chi}(\tau_3)=0,\ since\ \ \bar{\chi}(\tau_3)=0\cdot$

Also, since $\det(\tau_4)$=-1 ,then $v(\tau_4)$=1 ,hence $\tilde{\bar{P}}(\tau_4)\cong-\bar{P}(\tau_4)$, but $trace(\tilde{\bar{P}}(\tau_4))=-trace(\bar{P}(\tau_4))$, then $\delta(\tau_4)=-\bar{\chi}(\tau_4)=-i^{v}\prod_{j=1}^{v}2\sin\frac{1}{2}\theta_j$.

Similarly of (iv),(v) and (vi) .

Definition (2.6) **[4]**: Let $\bar{\chi}$ be the basic spin character ,and δ be an associate character to $\bar{\chi}$,then for all $\tau\in O(n)$,we define a graded character $\chi(\tau)$ by

$$\chi(\tau)=\begin{cases}\bar{\chi}(\tau) & if\ \ n\ \ is\ \ even\\ \bar{\chi}(\tau)+\delta(\tau) & if\ \ n\ \ is\ \ odd\end{cases}.$$

Theorem (2.5) : Let χ *and* χ' be the graded characters of $\bar{P}$ *and* $\bar{P'}$ respectively ,then :

$$i-\chi(\tau_1)=\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j\ ,$$

$$ii-\chi(\tau_2)=2\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j\ ,$$

$$iii-\chi(\tau_3)=\chi(\tau_4)=0\ ,$$

$$iv-\chi'(\tau_1)=\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j\ ,$$

$$v-\chi'(\tau_2)=2\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j\ ,$$

$$vi-\chi'(\tau_3)=\chi'(\tau_4)=0\ .$$

Proof : By coro.(2.2) ,we get : $\bar{\chi}(\tau_1)=\bar{\chi}(\tau_2)=\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j$,*and* $\delta(\tau_1)=\delta(\tau_2)=\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j$ by th.(2.3) ,and if n is even ,then $\chi(\tau_1)=\bar{\chi}(\tau_1)=\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j$,*and* if n is odd ,then

$$\chi(\tau_2)=\bar{\chi}(\tau_2)+\delta(\tau_2)=\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j+\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j=2\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j\cdot$$

Now, since $\bar{\chi}(\tau_3)=0$ by cor.(2.2) ,then $\chi(\tau_3)=0$.

And, since $\bar{\chi}(\tau_4)=i^v\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j$,*and* $\delta(\tau_4)=-i^v\prod_{j=1}^{v}2\cos\frac{1}{2}\theta_j$ by theorem.(2.4) ,and if n is odd ,then

$$\chi(\tau_4)=\bar{\chi}(\tau_4)+\delta(\tau_4)=i^v\prod_{j=1}^{v}2\sin\frac{1}{2}\theta_j-i^v\prod_{j=1}^{v}2\sin\frac{1}{2}\theta_j=0\cdot$$

Similarly of (iv) ,(v) and (vi) .

3. The Classical Group A_n, when n≥1

In this section we shall calculated the value of the graded character of $W(A_n)$,when n≥1 by using the proof of theorem (3.1) .

We note that $W(A_n)$ is the Symmetric group S_{n+1} of the degree n+1 ,and the conjugacy classes are parameterized by the partition $(\lambda_1,\lambda_2,...,\lambda_s)$ of n+1 or by Carter diagrams of the form $Ai_1+Ai_2+...+Ai_k$,where $\sum_{r=1}^{k}(i_r+1)=n+1$,[2] .

Theorem (3.1): If $\bar{\chi}$ is the basic spin character of $W(A_n)$,then:

$$\bar{\chi}(\sigma)=\begin{cases} 2^{(k-1)/2} & \text{if } \sigma\in A_{i_1}+A_{i_2}+...+A_{i_k} \\ & \text{where } i_r=r=1,...,k \text{ are all even} \\ i^{(n-1)/2}\sqrt{(n+1)/2} & \text{if } \sigma\in A_n \text{ ,n is odd} \\ o & \text{if } \sigma\in W(A_n) \end{cases} .$$

Proof : If $\sigma \in (1)$,then the characteristic polynomial of σ is $(1-t)^{-1}\prod_{r=1}^{k}(1-t^{i_r+1})$,and the eigenvalues of σ are the (i_r+1)th roots of unity (r=1,…,k) without 1 deleted , and $\overline{\chi}$ is the character of the basic spin representation , $\overline{\chi} \neq 0$ iff $\sigma \in (1)$,where all the i_rs are even ,or $\sigma \in A_n$,n is odd .

Note that if n is even ,then k is odd ,and if n is odd ,then k is even .

Now ,if $\sigma \in W(A_n)$ we calculate the value of $\overline{\chi}$ of σ :

i) If $\sigma \in (1)$, where all the i_rs are even then σ is of type τ_2 , τ_2 is the canonical of O(n) ,then by cor.(2.2) we have

$$\overline{\chi}(\sigma) = 2^{k-1/2}\left(\prod_{r_1=1}^{i_1/2} 2\cos\frac{r_1\pi}{i_1+1}\right)\ldots\left(\prod_{r_k=1}^{i_k/2} 2\cos\frac{r_k\pi}{i_k+1}\right)$$

$$= 2^{k-1/2}\prod_{s=1}^{k}\prod_{r_s}^{i_s/2} 2\cos\frac{r_s\pi}{i_s+1}$$

$$= 2^{k-1/2} \ .$$

ii) If n is odd , $\sigma \in A_n$ then by cor.(2.2) :

$$\overline{\chi}(\sigma) = i^{n-1/2}\left(\prod_{r=1}^{n-1/2} 2\sin\frac{r_k\pi}{n+1}\right)$$

$$= 2^{n-1/2}\sqrt{n+1/2} \ .$$

Theorem (3.2) : If χ is the graded character of $W(A_n)$ and if $\sigma = A_{i_1} A_{i_2} \dots A_{i_k}$ ...(1) ,where $i_r = r = 1,\dots,k$ are all even,then :

$$\chi(\sigma) = \begin{cases} 2^{k-1/2} & \text{if } k \text{ is odd } (n \text{ is even}) \\ 2^{k-1/2} & \text{if } k \text{ is even } (n \text{ is odd}) \\ 0 & \text{for all other } \sigma \in W(A_n) \end{cases}$$

Proof : If $\sigma \in (1)$, where all the i,s are even that is Ai, corresponds to a cycle of the length (i,+1),then Ai, corresponds to even cycles ,then $\delta(\sigma) = \bar{\chi}(\sigma) = 2^{k-1/2}$,*for all* $\sigma \in (1)$.

Now,

i) If k is odd n is even ,then by (th.3.1) , $\chi(\sigma) = \bar{\chi}(\sigma) = 2^{k-1/2}$.

ii) If k is even, n is odd ,then by (th.3.1), $\chi(\sigma) = \delta(\sigma) + \bar{\chi}(\sigma)$

$$= 2^{k-1/2} + 2^{k-1/2} = 2 \cdot 2^{k-1/2} = 2^{k-1/2}.$$

iii) if $\sigma \in A_n$, n *is odd*, then A_n corresponds to a cycle of the length (n+1),i.e A_n corresponds to an odd cycle .

Thus the associate character of the class A_n ,n is odd is $\delta(\sigma) = -\bar{\chi}(\sigma) = -i^{n-1/2}\sqrt{n+1/2}$, then by (th .3.1)

$$\chi(\sigma) = \delta(\sigma) + \bar{\chi}(\sigma)$$

$$= -i^{n-1/2}\sqrt{n+1/2} + i^{n-1/2}\sqrt{n+1/2} = 0.$$

And for all other $\sigma \in W(A_n)$. $\bar{\chi}(\sigma) = 0$,hence $\chi(\sigma) = 0$.

4. The Classical Group Cn, when n≥3

The conjugacy classes of $W(C_n)$ are in a one-to-one correspondence with admissible diagrams [2] of the form $Ai_1+Ai_2+\ldots+Ai_r+\ Cj_1+Cj_2+\ldots+Cj_s$ where $\sum_{k=1}^{r}(i_k+1)+\sum_{k=1}^{s}j_k=n$,or equivalently to pairs of partition (λ,μ),where $\lambda=i_1+1,\ldots,i_r+1$, $\mu=j_1,\ldots,j_s$.

In this section we calculate the value of the graded character of C_n depended on the following theorem.

Theorem (4.1): If $\overline{\chi}$ is the basic spin character of $W(C_n)$,then:

1- If $\sigma\in Ai_1+Ai_2+\ldots+Ai_r+\ Cj_1+Cj_2+\ldots+Cj_s$ ... (2) ,where $i_1,\ldots,i_r$, $j_1,\ldots,j_s$ are all even ,then

$$\overline{\chi}(\sigma)=\begin{cases}2^{1/2} & \text{if } n \text{ is even,}\\ 2^{t-1/2} & \text{if } n \text{ is odd , } t=r+s \text{ is the number of parts in the partition of } n\end{cases}.$$

2- If $\sigma\in Cj_1+Cj_2+\ldots+Cj_s$ ... (3) ,then

$$\overline{\chi}(\sigma)=\begin{cases}i^{(n-1)/2}2^{(s-1)/2} & \text{if } n \text{ is even ,}\\ o & \text{for all other } \sigma\in W(C_n)\end{cases}.$$

Proof : If $\sigma \in (2)$,then the characteristic polynomial of σ is $\prod_{k=1}^{r}(1-t^{j_k+1})\prod_{k=1}^{s}(1+t^{j_k})$,and the eigenvalues of an element in this class are $(t_k+1)$the roots of unity (k=1,…,r) and the $j_k 2_{jk}$ the roots of unity which are not j_k the roots of unity (k=1,…,s).

If $\overline{\chi}$ is the basic spin character, then by lemma(2.3)
$\overline{\chi} \neq 0$ iff $\sigma \in (2)$,where $i_1,\ldots,i_r$, $j_1,\ldots,j_s$ are all even,
or $\sigma \in (3)$,when n is odd .

Now , we calculate the value of $\overline{\chi}$ of σ :

1- If $\sigma \in (2)$, then by cor.(2.2) we have

$$\overline{\chi}(\sigma) = 2^{r/2}\left(\prod_{k_1=1}^{r}\left(\prod_{r_k=1}^{i_k/2} 2\cos\frac{r_k\pi}{j_k+1}\right)\right)\prod_{k=1}^{s}\left(\left(\prod_{r_k=1}^{j_k-1} 2\cos\frac{r_s\pi}{2j_k}\right)\left(\prod_{r_k=1}^{i_k/2} 2\cos\frac{r_k\pi}{j_k}\right)^{-1}\right)$$

$$= 2^{r/2}\prod_{k=1}^{s}\left(\sqrt{j_k}\left(\sqrt{j_k/2}\right)^{-1}\right) = 2^{r/2}.2^{s/2} \quad .$$

Hence , $\overline{\chi}(\sigma) = \begin{cases} 2^{1/2} & \text{if } n \text{ is even,} \\ 2^{t-1/2} & \text{if } n \text{ is odd } ,t=r+s \end{cases}$.

2- If $\sigma \in (3)$,let p be the number of j_k which are odd, then by cor.(2.2) :

$$\overline{\chi}(\sigma) = 2^{p-1/2}\left(\prod_{k=1}^{p}\left(\prod_{r_k=1}^{i_k-1/2} 2\cos\frac{r_k\pi}{j_k}\right)\right)\prod_{k=p+1}^{s}\left(\left(\prod_{r_k=1}^{j_k-1} 2\sin\frac{r_s\pi}{2j_k}\right)\left(\prod_{r_k=p+1}^{i_k-1/2} 2\sin\frac{r_k\pi}{j_k}\right)^{-1}\right)$$

$$= 2^{p-1/2} j^{n-1/2}\prod_{k=p+1}^{s}\left(\sqrt{j_k}\left(\sqrt{j_k/2}\right)^{-1}\right) = j^{n-1/2}.2^{p-1/2}.2^{s-p/2} \quad = i^{n-1}.2^{s-1/2}.$$

Theorem (4.2) : If χ is the graded character of $W(C_n)$, then :

1- If $\sigma \in Ai_1+Ai_2+\ldots+Ai_r+\ Cj_1+Cj_2+\ldots+Cj_s$ ... (2) ,where $i_1,\ldots,i_r$, $j_1,\ldots,j_s$ are all even ,then

$$\chi(\sigma)=\begin{cases} 2^{1/2} & \text{if } n \text{ is even,} \\ 2^{t+1/2} & \text{if } n \text{ is odd , } s \text{ is odd} \\ 0 & \text{if } n \text{ is odd, } s \text{ is odd, } t=r+s \end{cases} .$$

2- If $\sigma \in Cj_1+Cj_2+\ldots+Cj_s$ ... (3) ,n is odd ,then

$$\chi(\sigma)=\begin{cases} i^{(n-1)/2}2^{s+1} & \text{if } p \text{ is even ,} \\ 0 & \text{if } p \text{ is odd ,} \\ o & \text{for all other } \sigma \in W(C_n) \end{cases}$$. where p is the number of j_k which are odd .

Proof :

1- If $\sigma \in (2)$, where all the i_ks and j_ms are even that is Ai_k corresponds to odd cycles ,and C_{jm} corresponds to acycle of the length j_m ,then C_{jm} ,m=1,...,s corresponds to odd cycle,thus (2)is an even class if s is even ,and (2) is an odd class if s is odd ,then

$$\delta(\sigma)=\begin{cases} \overline{\chi}(\sigma) & \text{if } s \text{ is even,} \\ -\overline{\chi}(\sigma) & \text{if } s \text{ is odd} \end{cases} .$$

Now if n is even ,then we have $\chi(\sigma)=\overline{\chi}(\sigma)$ *and* $\chi(\sigma)=2^{1/2}$,by th.(4.1) ,and if n is odd ,then by th.(4,1) :

$$\chi(\sigma)=\begin{cases} 2^{l+1/2} & \text{if } s \text{ is even,} \\ 0 & \text{if } s \text{ is odd} \end{cases} .$$

2- If $\sigma \in (3)$,and n is odd ,then if p is the number of j_k which are odd ,we get

$$\chi(\sigma)=\begin{cases} i^{(s-1)/2} & \text{if } p \text{ is even ,} \\ -i^{(s-1)/2} & \text{if } p \text{ is odd ,} \end{cases} .$$

Thus if p is even ,then by th.(4.1) :

$$\chi(\sigma)=i^{n-1/2}2^{s-1/2}+i^{n-1/2}2^{s-1/2}$$
$$=2i^{n-1/2}2^{s-1/2}=i^{n-1/2}2^{s+1/2},$$

And if p is odd , $\chi(\sigma)=-i^{n-1/2}2^{s-1/2}+i^{n-1/2}2^{s-1/2}=0$,

Hence ,

$$\chi(\sigma)=\begin{cases} i^{(n-1)/2}2^{s+1/2} & \text{if } p \text{ is even ,} \\ 0 & \text{if } p \text{ is odd ,} \end{cases} .$$

And for all other $\sigma \in W(C_n)$, $\overline{\chi}(\sigma)=0$, *then* $\chi(\sigma)=0$.

5. The Classical Group Dn, when n≥4

The conjugacy classes are parameterized by certain admissible diagrams [2] of the form $Ai_1+Ai_2+\ldots+Ai_{r+}\ Dj_1(a_{k1})+\ldots+Dj_s(a_{ks})$ where $\sum_{k=1}^{r}(i_k+1)+\sum_{k=1}^{s}j_k=n$,and for convenience we have put $D_j= D_j\ (a_0)$,$0\le k_i\le (ji/2)-1$.

Furthermore ,there are two conjugacy classes when i_k (k=1,2,…r) are all odd and $j_k=0$,k=(1,…,s) .

In this section we calculated the value of the graded character of D_n .

Theorem (5.1): If $\overline{\chi}$ is the basic spin character of $W(D_n)$,then:

1- If $\sigma \in Ai_1+Ai_2+\ldots+Ai_r+ Dj_1(a_{k1})+\ldots+Dj_s(a_{ks})\ldots$ (4) ,where $i_1,\ldots,i_r$, $j_1,\ldots,j_s$ are all even ,and $k_1,\ldots,k_s$ are all odd ,then :

$$\overline{\chi}(\sigma)=\begin{cases} 2^{t+s/2} & \text{if } n \text{ is even,} \\ 2^{t+s-1/2} & \text{if } n \text{ is odd } ,t=r+s \end{cases} .$$

2- If $\sigma \in Dj_1(a_{k1})+\ldots+Dj_s(a_{ks})$ ... (5) ,then

$$\overline{\chi}(\sigma)=\begin{cases} i^{(n-1)/2}2^{(s-1)/2} & \text{if } n \text{ is odd } , \\ o & \text{for all other } \sigma \in W(D_n) \end{cases} .$$

Proof : If $\sigma \in W(D_n)$,then the characteristic polynomial of σ is $\prod_{p=1}^{r}(1-t^{i_p+1})\prod_{p=1}^{s}(1+t^{j_k-k_p-1})\prod_{p=1}^{s}(1+t^{k_p+1})$,

If $\overline{\chi}$ is the basic spin character, then by lemma(2.3)

$\overline{\chi} \neq 0$ iff

i-$\sigma \in (4)$,where $i_1,\ldots,i_r$, $j_1,\ldots,j_s$ are all even,and $k_1,\ldots,k_s$ are all odd

ii-$\sigma \in (5)$,when n is odd ,and s also odd.

Now , we calculate the value of $\overline{\chi}$ of σ :

1- If $\sigma \in (4)$,then by th.(4.1) ,[6],and cor.(2.2) we have :

$$\overline{\chi}(\sigma) = 2^{r/2}\Big(\prod_{t=1}^{r}\Big(\prod_{r_t=1}^{j_t/2} 2\cos\frac{r_t\pi}{j_t+1}\Big)\Big)\prod_{t=1}^{s}\Big(\Big(\prod_{r_t=1}^{k_t-1} 2\cos\frac{r_t\pi}{2t_k}\Big)\Big(\prod_{r_k=1}^{j_k/2} 2\cos\frac{r_t\pi}{k_t}\Big)^{-1}\Big)$$

$$\Big(\prod_{r_t=1}^{j_t-k_t-2} 2\cos\frac{r_t\pi}{j_k+1}\Big)\Big(\prod_{rr_t=1}^{j_t/2-k_t/2-1} 2\cos\frac{r_t\pi}{j_t-k_t-1}\Big)^{-1}$$ Hen

$$= 2^{r/2}\prod_{t=1}^{s}\Big(\sqrt{k_t}\Big(\sqrt{k_t/2}\Big)^{-1}\Big(\sqrt{j_t-k_t-1}\Big)\Big(\sqrt{j_t-k_t-1/2}\Big)^{-1} = 2^{r/2}.2^{s} \quad .$$

ce , $\overline{\chi}(\sigma) = \begin{cases} 2^{t+s/2} & \text{if } n \text{ is even,} \\ 2^{t+s-1/2} & \text{if } n \text{ is odd } ,t=r+s \end{cases}$.

2- If $\sigma \in (5)$,and n is odd ,then if p is the number of the j_k ,then by cor.(2.2), and [6], we have :

$$\overline{\chi}(\sigma) = i^{n-1}2^{p-1/2}\Big(\prod_{t=1}^{p}\Big(\prod_{r_t=1}^{j_t-1/2-k_t-1/2} 2\cos\frac{r_t\pi}{j_t-k_t-1}\Big)\Big)\Big(\prod_{r_t=1}^{k_t-1} 2\sin\frac{k_t\pi}{2k_t}\Big)\Big(\prod_{r_t=1}^{k_t/2-1} 2\cos\frac{k_t\pi}{k_t}\Big)^{-1}\Big)$$

$$\prod_{t=p+1}^{s}\Big(\prod_{rr=1}^{k_t-1} 2\cos\frac{r_t\pi}{2k_t}\Big)\Big(\prod_{rr_t=1}^{k_t/2-1} 2\cos\frac{r_t\pi}{2k_t}\Big)^{-1}\Big(\prod_{r_t=1}^{j_t-k_t-2} 2\cos\frac{r_t\pi}{2(j_tk_t-1)}\Big)\Big(\prod_{r_t=1}^{j_t/2-k_t/2-1} 2\cos\frac{r_t\pi}{j_t-k_t-1}\Big)^{-1}$$

$$= i^{n-1/2}2^{p/2}2^{p-1/2}2^{s-p} = i^{n-1/2}.2^{s-1/2} \quad .$$

And for all other $\sigma \in W(D_n)$, $\overline{\chi}(\sigma) \neq 0$.

Theorem (5.2) : If χ is the graded character of $W(D_n)$, then :

1- If $\sigma \in Ai_1+Ai_2+\ldots+Ai_r+ Dj_1(a_{k1})+\ldots+Dj_s(a_{ks})\ldots$ (4),where $i_1,\ldots,i_r$, $j_1,\ldots,j_s$ are all even , and $k_1,\ldots,k_s$ are all odd ,then

$$\chi(\sigma)=\begin{cases} 2^{t+s} & \text{if } n \text{ is even,} \\ 2^{t+s+1/2} & \text{if } n \text{ is even, } s \text{ is odd}, t=r+s \\ 0 & \text{if } n \text{ is odd}, s \text{ is odd}, t=r+s \end{cases}.$$

2- If $\sigma \in Dj_1(a_{k1})+\ldots+Dj_s(a_{ks})$ ... (5),n is odd ,then

$$\chi(\sigma)=\begin{cases} i^{(n-1)/2}\,2^{r+1/2} & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd,} \\ o & \text{for all other } \sigma \in W(D_n) \end{cases}$$. where p is the number of j_k

which are odd .

Proof :

1- If $\sigma \in (4)$, where all the i_ks and j_ms are even that is Ai_k corresponds to a cycle of the length (i_k+1) ,then Ai_k ,k=1,..r corresponds to even cycles,and Dj_m corresponds to a cycle of length j_m ,then Dj_m ,m=1,..,s ,thus (4)is an even class iff s is even ,and (4) is an odd class iff s is odd then :

$$\delta(\sigma)=\begin{cases} \bar{\chi}(\sigma) & \text{if } s \text{ is even,} \\ -\bar{\chi}(\sigma) & \text{if } s \text{ is odd} \end{cases}.$$

And if n is even , $\chi(\sigma)=2^{t+1}$, and if n is odd ,s is even ,

$$\chi(\sigma)=\bar{\chi}(\sigma)+\delta(\sigma)$$

$$=2^{t+s-1/2}+2^{t+s-1/2}=2^{t+s+1/2}$$, by lemm.(2.3) .

If s is odd ,

$$\chi(\sigma)=\bar{\chi}(\sigma)+\delta(\sigma)=2^{t+s-1/2}-2^{t+s-1/2}=0\ ,$$

2- If $\sigma \in (5)$,and n is odd ,then if p is the number of j_k ,then :

$$\delta(\sigma)=\begin{cases} i^{(n-1)/2}2^{s-1/2} & \text{if } p \text{ is even}\ , \\ -i^{(n-1)/2}2^{s-1/2} & \text{if } p \text{ is odd}\ , \end{cases}\ .$$

Thus if p is even ,then:

$$\chi(\sigma)=i^{n-1/2}2^{s-1/2}+i^{n-1/2}2^{s-1/2}=i^{n-1/2}2^{s+1/2}\ ,$$

And if p is odd ,

$$\chi(\sigma)=\bar{\chi}(\sigma)+\delta(\sigma)=i^{n-1/2}2^{s-1/2}-i^{n-1/2}2^{s-1/2}=0\ ,$$

And for all other $\sigma \in W(D_n)$, $\bar{\chi}(\sigma)\neq 0$, *then* $\chi(\sigma)=0$.

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