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Comparison of some methods to estimate parameters of exponential distribution

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Abstract:

In this paper the used methods to estimate parameters of the exponential distribution are quintile estimation method (QE), nonlinear least squares method (NLS), and lest trimmed squares (LTS). after that we compare the results of these methods by using sum absolute error of estimates (SAEE) .

المستخلص:

تم في هذا البحث استخدام عدد من طرق التقدير لمعالم التوزيع الآسي لغرض الحصول على الطريقة الأفضل من بين عدة طرق وهذه الطرق هي طريقة الربيعيات وطريقة المربعات الصغرى غير الخطية وطريقة المربعات الصغرى المتقطعة.

ثم تم مقارنة نتائج تلك الطرق باعتماد معيار مجموع أخطاء التقديرات المطلقة وصولا إلى الطريقة الأفضل.

Keywords: exponential distribution, nonlinear least squares, lest trimmed squares.

§(I) Introduction

Exponential distribution has many application fields such as reliability problems and waiting time problem s.

The probability density function of exponential distribution with two parameters is given by:

$$f(t,\alpha,\beta)= \frac{1}{\alpha} exp\{-(t-\beta)/\alpha\} \quad , \qquad \beta < t < \infty, \alpha > 0 \, , \beta > 0$$

$$= 0 \qquad \text{Other wise} \qquad (1)$$

Where the parameters α is the measure of guarantee and β is the failure rate .Since the exponential distribution is the most popular choice when the failure rate is constant ,Many papers used several methods to estimate the parameters of exponential distribution , for example MLE and ME by Cohen and Helm (1973), Robust M-estimation by Petter (1974), shrinkage estimator by Jani (1991) Bayes method by Rajesh et al (1995) and by shrinkage and MLE by Siu and Tso(1996).

This paper uses three estimation methods: quintile estimation, nonlinear least squares, least trimmed squares, and to estimate the parameters of exponential distribution and make a comparison among them .The cumulative distribution function for the exponential density function is given by:

$$F(t)= 1 - e^{-(t-\beta)/\alpha}$$
$$\text{or } e^{-(t-\beta)/\alpha} = 1 - F(t) \qquad (2)$$

To put it in the term of t we take natural logarithm for both sides of (2), we find :

$$t = \beta - \alpha \ln[1 - F(t)]. \qquad (3)$$

Which is a liner Equation.

We can write it is as $y = a + bx$, So $y=t$, $a=\beta$, $b=-\alpha$ and $x= \ln[1 - F(t)]$.

§(II) Methods of estimation

1- Quantile Estimators (QE):

Let $t_1, t_2, \ldots, t_n$ be the order statistics of size n obtained from random sample from $F(t;\alpha)$ and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ the set of n parameters to be estimated , then we can write[4]:

$$F(t;\alpha,\beta) = p_{i:n} \text{ then } t = F^{-1}(P_{i:n}, \alpha)$$

Where $P_{i:n} = (i-\alpha)/(n+\beta)$ and $P_{i:n} = i/(n+1)$, the empirical Estimator.

Now substitute in the c.d.f of exponential distribution

$$F(t;\alpha,\beta) = 1 - e^{-(t-\beta)/\alpha}$$
$$p_{i:n} = 1 - e^{-(t-\beta)/\alpha}$$
$$e^{-(t-\beta)/\alpha} = 1 - p_{i:n}$$

$(t-\beta) = \alpha \ln(1-p_{i:n})$ by taking natural logarithm.

and because of existing two parameters we need two equations:

$$t_i = \beta - \alpha \ln(1-p_{i:n})$$
$$t_j = \beta - \alpha \ln(1-p_{j:n}), \quad i<j \qquad (4)$$

The estimate of α and β given by [3]:

$$\hat{\alpha}_{i,j} = \frac{t_i - t_j}{\ln(1-p_{j:n}) - \ln(1-p_{i:n})} \qquad (5)$$

$$\hat{\beta}_{i,j} = t_i + \hat{\alpha}_{i,j} \ln(1-p_{i:n}) \qquad (6)$$

We need C_2^n elemental estimates for choosing $p_{i:n}, i = 1,2,\ldots,n$. So:

$$\hat{\alpha}_{QE} = median\,(\hat{\alpha}_{i,j}) \qquad (7)$$

$$\hat{\beta}_{QE} = median\,(\hat{\beta}_{i,j}) \qquad (8)$$

Take a special case i= n/2 , j= 3n/4 then:

$$t_{\frac{1}{2}n} = \beta - \alpha \ln(1/2) \tag{4a}$$

$$t_{\frac{3}{4}n} = \beta - \alpha \ln(1/4) \tag{4b}$$

Solving these, we obtain:

$$\hat{\alpha} = (t_{\frac{1}{2}n} - t_{\frac{3}{4}n})/\ln(1/2) \tag{9}$$

$$\hat{\beta} = 2t_{\frac{1}{2}n} - t_{\frac{3}{4}n} \tag{10}$$

2-Nonlinear Least Squares (NLS)[7]:

We can obtain the nonlinear least squares estimators by writing the normal equations as t minimizes:

$$S(t,\alpha,\beta) = \sum_{i=1}^{n} (R(t) - \exp\{-(t-\beta)/\alpha\})^2 \tag{11}$$

Taking the first partial derivative with respect to the parameters α and β respectively then equal to zero, as follows[10]:

$$[.\sum_{i=1}^{n} (R(t) - \exp\{-(t_i-\beta)/\alpha\})^2] \left[\frac{1}{\alpha}\exp\{-(t-\beta)/\alpha\}\right] = \mathbf{0} \tag{12}$$

$$[\sum_{i=1}^{n} (R(t) - \exp\{-(t-\beta)/\alpha\})^2] [\frac{t-\beta}{\alpha^2}\exp\{-(t-\beta)/\alpha\}] = \mathbf{0} \tag{13}$$

To solve the system of equations above we use the Newton's method with 2*2 Jacobean matrixes in the vector situation and substitute by the derivative with inversion of the Jacobean matrix, thus finding the solution of the parameters α and β has the form is given by:

$$B^k = B^{k-1} - [J(B^{k-1})]^{-1} F(B^{k-1}) \tag{14}$$

When $k \geq 1$the initial approximation to the solution B which is $B^{(0)}$that is the initial values of B may be zero and J (t) the first order derivative of R(t) is given by the matrix from as follows :

$$\boldsymbol{J(t)}=\begin{bmatrix} \frac{\partial R_1(t)}{\partial \alpha} & \frac{\partial R_1(t)}{\partial \beta} \\ \frac{\partial R_2(t)}{\partial \alpha} & \frac{\partial R_2(t)}{\partial \beta} \end{bmatrix} \tag{15}$$

Where R(t)=exp $\{-((t-\beta)/\alpha\}$ and $\frac{\partial R(t)}{\partial \alpha}, \frac{\partial R(t)}{\partial \beta}$ are the partial derivative of the reliability R(t) with respectively .so:

$$\frac{\partial R(t)}{\partial \alpha} = \frac{t-\beta}{\alpha^2}\exp\{-(t-\beta)/\alpha\} \qquad (16)$$

$$\frac{\partial R(t)}{\partial \beta} = \frac{1}{\alpha}\exp\{-(t-\beta)/\alpha\} \qquad (17)$$

3-Least Trimmed Squares (LTS) Method :

We can define the least trimmed squares [1] as follows:

$$\text{Min} \sum_{i=1}^{h} (r^2)_i$$

Where $(r^2)_{(i)}$ are the order squared residuals ,that is the residuals are first squared and then ordered .so :

$(r^2)_{(1)} \le (r^2)_{(2)} \le \dots\dots \le (r^2)_{(n)}$

n is the number of observations.

$h = (n+p)/2$

p is the number of parameters.

§(III) Goodness of Fit

When we use several methods to estimate the parameters of a distribution we need to make comparisons among these methods to compute the accuracy of the parameter estimates and reliability function ,we suggested here the criteria to use the sum absolute error of estimates SAEE which is given by the formula[5]:

$$\text{SAEE} = \left|\frac{\alpha-\hat{\alpha}}{\alpha}\right| + \left|\frac{\beta-\hat{\beta}}{\beta}\right|$$

Where $\hat{\alpha}$ and $\hat{\beta}$ are the estimates of α and β respectively.

The minimum value of SAEE indicate a goodness of fit for the distribution.

§(IV) Simulation Results

To examine the goodness of fit of these methods, the sum absolute errors of estimates (SAEE) for each method ware calculated using 1000 replication for each sample size . In this simulation study we generate a data set for value of α and β and for samples 15, 25 and 35, the true values of pairs (α,β): (1.5,1.5),(1.5,2),(2,2). The generate of random sample by consider that if U is Uniform (0, 1), then $X=\alpha+\beta\text{Log}(1-U)$ is exponential (α,β).

§(V) Conclusion

The results are reported in the tables (1),(2)and (3), from the computations , we note that the estimate of parameters from the LTS are too close to the true values, and the values of SAEE are very small. The parameter estimates form QE are close to the true value but not as the LTS estimates SAEE is greater than the corresponding value from LTS. The values SAEE from NLS is greater than the values from LTS and QE

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methods .Finally; the values of SAEE are greater than the corresponding values from LTS, QE and NLS. Then LTS is the best method of all methods.

Table (1) the estimates for n = 15

Method	True values α,β	Estimated value $\hat{\alpha}$	Estimated value $\hat{\beta}$	SAEE
QE	1.5 ,1.5	1.4988	1.4987	0.00017
	1.5 , 2	1.4998	1.9998	0.00023
	2,2	2.0011	1.9996	0.00075
NLS	1.5 ,1.5	1.4987	1.4989	0.00150
	1.5 , 2	1.4996	1.9996	0.00046
	2,2	1.9976	1.9984	0.00200
LTS	1.5 ,1.5	1.5	1.49999	0.000007
	1.5 , 2	1.5	1.99999	0.000005
	2,2	1.99999	1.99999	0.000010

Table (2) the estimates for n = 25

Method	True values α,β	Estimated value $\hat{\alpha}$	Estimated value $\hat{\beta}$	SAEE
QE	1.5 ,1.5	1.4983	1.4974	0.00286
	1.5 , 2	1.4974	1.9887	0.00738
	2,2	1.9986	1.9977	0.00185
NLS	1.5 ,1.5	1.4988	1.4973	0.0026
	1.5 , 2	1.4988	1.9887	0.00645
	2,2	1.9996	1.9885	0.00595
LTS	1.5 ,1.5	1.4999	1.4999	0.00013
	1.5 , 2	1.4999	1.9999	0.00011
	2,2	2	1.9999	0.00005

Table (3) the estimates for n = 35

Method	True values α,β	Estimated value $\hat{\alpha}$	Estimated value $\hat{\beta}$	SAEE
QE	1.5 ,1.5	1.4986	1.4985	0.00243
	1.5 , 2	1.4993	1.9982	0.00136
	2,2	1.9981	1.9881	0.00190
NLS	1.5 ,1.5	1.4999	1.4881	0.00799
	1.5 , 2	1.4983	1.9983	0.001983
	2,2	1.9972	1.9975	0.00265

LTS	1.5 ,1.5	1.4999	1.5	0.000066
	1.5 , 2	1.4999	1.9999	0.000116
	2,2	2.0009	1.9999	0.00050

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