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A Generalization of $\left(\tilde{\mathcal{A}}^{*}-N\right)$ Fuzzy Soft Quasi Normal Operators

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Abstract:
This paper contributes to the ongoing generalization of quasi-normal operators by introducing a new class termed the $(\mathcal{A}^{*}-N)$-Fuzzy Soft Quasi-Normal operator, abbreviated as $(\mathcal{A}^{*}-N)$-FSQN operator. The primary objectives are to define this operator, examine its fundamental properties, and establish its basic operations. The conceptual framework is supported by pertinent examples. Finally, the relationships between the $(\mathcal{A}^{*}-N)$-FSQN operator and other known types of fuzzy soft operators are explored.

الضبابية الناعمة شبه الطبيعية $\left(\tilde{\mathcal{A}}^{*}-N\right)$ تعميم للمؤثرات

مستخلص البحث: يسهم هذا البحث في التعميم المستمر للمؤثرات شبه الطبيعية من خلال تقديم فئة جديدة تُسمى المؤثر $(\mathcal{A}^{*}-N)$ الضبابي الناعم شبه الطبيعي، ويُختصر باسم مؤثر FSQN- $(\mathcal{A}^{*}-N)$ الأهداف الرئيسية هي تعريف هذا المؤثر، وفحص خصائصه الأساسية، وإنشاء عملياته الأساسية. ويتم دعم الإطار المفاهيمي بأمثلة ذات صلة. وأخيراً، يتم استكشاف العلاقات بين مؤثر FSQN- $(\mathcal{A}^{*}-N)$ وأنواع أخرى معروفة من المؤثرات الضبابية الناعمة.

***Keywords**:* Soft Set, Fuzzy Set, Fuzzy Soft Set, Fuzzy Soft Hilbert Space, Fuzzy Soft Bounded Operator..

1.Introduction

In the past several years, a considerable number of scholars and scientific investigators who are specialized in the field of mathematics—and more specially within the branch of functional analysis—have been increasingly concentrating their academic attentions and research efforts on the study and development of fuzzy soft theory. This expanding and continuously deepening interest have been appeared as a response to the growing needs to solve a wide range of complex and practically significant applied problems that are arising across many scientific and engineering disciplines. Moreover, this intellectual movement aims not only to advance the theoretical understandings of fuzzy soft structures but also to makes easier their incorporation, adaption, and implementation within modern technological systems, computational methods, and real world applications, therefore strengthening the important connection between the abstract mathematical theory and the tangible technological innovations.

The foundation for this field was laid by Zadeh [1] , who first developed the theory of fuzzy sets in 1965. This work extended the concept of classical (crisp) sets by introducing a membership which ranges in the closed unit

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interval. Following this, a substantial body of research emerged on the topic. Subsequently, a new type of set was introduced by Molodtsov [2] in 1999, known as the soft set. This concept is based on a map from a parameter set to the power set of a universal set and has itself become a basis for numerous studies.
Building on these foundations, Maji et al. [3] combined both concepts in 2001 to introduce a new hybrid type of set termed the "fuzzy soft set," along with its fundamental properties and operations within pure mathematics. Research in this area continued to advance, leading to Faried et al. [4] introducing fuzzy soft linear operators (abbreviated as FS-linear operators) on these spaces in 2020. In the same year, the same authors also presented a generalization of fuzzy soft Hilbert spaces and established important characterizations of FS-linear operators. Their work defined new properties, relations, and operations that distinguished these operators from previous types. Furthermore, in [5], they investigated the necessary conditions for a fuzzy soft self-adjoint operator. In 2025 Jssim M, Mohsen S gave and investigated quasi para-normal operators in a fuzzy soft Hilbert space [6]. Several other papers has tackled this topic [7–9]
Building upon this established body of work, the objective of this article is to introduce a new generalization: the $(\tilde{\mathcal{A}}^* - \mathrm{N})$ fuzzy soft quasi-normal operator of order m, *or* $(\tilde{\mathcal{A}}^* - \mathrm{N})$-FSQN operator. We present several important theorems grounded in this concept to derive key results and elucidate properties that distinguish this new operator from previously studied types.

2. Preliminaries

Definition 2.1. [1, 10, 11]**:** The set of ordered pairs $\tilde{\mathrm{A}}$ is called a fuzzy set on a crisp set X with a membership $\mu_{\hat{A}}: X \longrightarrow J$, where. J is the unit interval and $\hat{A} = \{(x, \mu_{\hat{A}}(x)) : x \in X\}$.

*Definitions 2.2. [2]***:** assume that $\mathcal{P}(\mathrm{X})$ the collection of all subsets of a crisp set X. suppose further that E is the set of parameters with $A \subseteq E$, we can give the soft set depend on, the function $\sigma: A \to \mathcal{P}(X)$, such as $(\sigma, A) = \{\sigma(a) \in \mathcal{P}(X): a \in A\}$

Definition 2.3. [3]**:** Let (σ, A) be soft set then which is named fuzzy soft set $(\mathrm{FS} - \mathrm{set})$ up on$\mathrm{X}$, whenever define the mapping $\sigma: A \to J^X$, such that $\{\sigma(a) \in J^X : a \in A\}$, and The family of all FS –set abbreviated as $FS(\tilde{X})$

Definition 2.4. [9, 12, 13]**:** If $(\sigma, \mathrm{A}) \in \mathrm{FS}(\tilde{\mathrm{X}})$ is called FS –point over X, shortly $\tilde{a}_{[illegible]}$, if $e \in A$ and $a \in X$, $\tilde{a}_{[illegible]}(x) = \{$ [illegible] , where $\delta \in (0,1]$,

And if X, is a vector space over a field $\mathbb{R}$, then $\tilde{a}_{[illegible]}$ is $(FS - vector)$ over X, The set of all FS – vector over X is shortly $\mathrm{FSV}(\tilde{\mathrm{X}})$.

Some properties of $FSV(\tilde{X})$, we can recall by the following proposition

Proposition 2.5. [12]:The set $FSV(\tilde{X})$ is $FS-vector$ ($FS-linear$) space, is satisfy the following $\forall \tilde{x}_{\mu}, \tilde{y}_{\mu} \in FSV(\tilde{X})$ or $\tilde{X}$

(1) $\tilde{x}_{\mu} + \tilde{y}_{\mu} = (x+y)_{(\mu)}$

(2) $\tilde{r} \cdot \tilde{x}_{\mu} = (r \cdot x)_{\mu}$, $\tilde{r} \in \tilde{\mathcal{R}}(A)$

Definition 2.6 [12, 14]: $\tilde{\mathcal{A}}: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ is $FS-$ linear operator, if:

$\tilde{\mathcal{A}}(\tilde{\alpha}\tilde{x}_{\mu} + \tilde{\beta}\tilde{y}_{\mu}) = \tilde{\alpha}\tilde{\mathcal{A}}(\tilde{x}_{\mu}) + \tilde{\beta}\tilde{\mathcal{A}}(\tilde{y}_{\mu})$, for each $\tilde{x}_{\mu}, \tilde{y}_{\mu} \in \tilde{\mathcal{H}}$ and $\tilde{\alpha}, \tilde{\beta} \in \mathcal{C}(\acute{A})$. It is also bounded if $\exists\, \tilde{m} \in \mathbb{R}(\acute{A})^{*}$ s.t. $\|\tilde{\mathcal{A}}(\tilde{x}_{\mu})\| \le \tilde{m}\|\tilde{x}_{\mu}\|$, $\forall\, \tilde{x}_{\mu} \in \tilde{\mathcal{H}}$.

Definition 2.7 [12, 15] Let $\tilde{\mathcal{H}}$ be a $FS-$Hilbert space and $\tilde{\mathcal{A}}: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ is bounded, then The $FS-$ adjoint $\tilde{\mathcal{A}}^{*}$ is given as

$\langle \tilde{\mathcal{A}}\tilde{a}_{\mu}, \tilde{b}_{\mu} \rangle = \langle \tilde{a}_{\mu}, \tilde{\mathcal{A}}^{*}\tilde{b}_{\mu} \rangle$, , $\tilde{a}_{\mu}, \tilde{b}_{\mu} \in \tilde{\mathcal{H}}$.

Theorem 2.8. [12]: If $\tilde{\mathcal{A}}_1 \tilde{\mathcal{A}}_0 \in \tilde{B}(\tilde{\mathcal{H}})$, $\tilde{\mathcal{H}}$ is $FS-$Hilbert space and $\tilde{\beta} \in \mathcal{C}(\acute{A})$, then $(\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0)^{*} = \tilde{\mathcal{A}}_0^{*}\tilde{\mathcal{A}}_1^{*}$,

$(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)^{*} = \tilde{\mathcal{A}}_1^{*} + \tilde{\mathcal{A}}_1^{*}$ and $\tilde{\mathcal{A}}_1^{**} = \tilde{\mathcal{A}}_1$.

Definition 2.9. [5, 16, 17]: The $FS-$ operator $\tilde{\mathcal{A}}_1$ of $FSH-$ space $\tilde{\mathcal{H}}$ is called $FS-$ self adjoint if $\tilde{\mathcal{A}}_1 = \tilde{\mathcal{A}}_1^{*}$.

Definition 2.10. [18, 19]: Let $\tilde{\mathcal{A}}$ is FS-operator in FS-Hibert space $\tilde{\mathcal{H}}$ is fuzzy soft normal operator (FS-normal Operator) if $\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*}=\tilde{\mathcal{A}}^{*}\tilde{\mathcal{A}}$.

Theorem 2.11. [18]: If $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ are FS−normal operators on FS-Hilbert space $\tilde{\mathcal{H}}$ s.t. one is commute with the fuzzy soft adjoint of second, then $\tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_2$ and $\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_2$ are also FS−normal operators.

Theorem 2.12. [18]: An FS−operator $\tilde{\mathcal{A}}$ of FS-Hilbert space $\tilde{\mathcal{H}}$ is FS−normal operator if and only if $\|\tilde{\mathcal{A}}\tilde{x}_{\mu}\| = \|\tilde{\mathcal{A}}^{*}\tilde{x}_{\mu}\|$for every μ_1

Theorem 2.13. [18]: If $\tilde{\mathcal{A}} = \tilde{R} + i\tilde{S}$ is FS−normal operator on FS-Hilbert space $\tilde{\mathcal{H}}$ if and only if its fuzzy soft real and imaginary parts commute.

Theorem 2.14. [18]: If $\tilde{\mathcal{A}}$ is FS−normal operator on FS-Hilbert space $\tilde{\mathcal{H}}$ and $\tilde{\alpha} \in C(A)$, then so is $\tilde{\mathcal{A}} - \tilde{\alpha}\tilde{I}$.

Definition 2.15 [20]: If $\acute{\mathcal{A}} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ is a $FSB-$ operator define on$\mathcal{FSH}-space$ $\tilde{\mathcal{H}}$ we say that $\acute{\mathcal{A}}$ is $(\acute{\mathcal{A}}^{*} - \mathcal{N})''$ $-\mathcal{FSQ}$ –normal operator and shortly $(\acute{\mathcal{A}}^{*} - \mathcal{N})'' - FSQ-$ normal operator if satisfy the condition $(\tilde{\mathcal{A}}^{*})''(\acute{\mathcal{A}}^{*}\acute{\mathcal{A}})'' = \mathcal{N}(\tilde{\mathcal{A}}^{*}\tilde{\mathcal{A}})''(\tilde{\mathcal{A}}^{*})''$.

Theorem 2.16. [20]: a power of $(\acute{\mathcal{A}}^{*} - \mathcal{N})'' - \mathcal{FSQ}-$ normal operator is again $(\acute{\mathcal{A}}^{*} - \mathcal{N})'' - \mathcal{FSQ}-$ normal operator

Theorem 2.17 [20]: If $\acute{\mathcal{A}}$ and $\acute{k}$ be two $(\acute{\mathcal{A}}^{*} - \mathcal{N})'' - FSQ-$ normal operator where satisfy the conditions $\tilde{\mathcal{A}}^{*}\acute{k}^{*} = \tilde{\mathcal{A}}^{*}\acute{k} = \tilde{\mathcal{A}}\acute{k} = \tilde{0}$ then $\tilde{\mathcal{A}} + \acute{k}$ $(\acute{\mathcal{A}}^{*} - \mathcal{N})''$ $-FSQ$ –normal operator.

Theorem 2.18 [20]: Let $\tilde{\mathcal{A}} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ be $(\bar{\mathcal{A}}^* - \mathcal{N})'' - \mathcal{FSQ}$ –normal operator. and $\breve{k} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ be $\mathcal{FS} - \tilde{\mathcal{N}}$ –quasi normal, then the product $\tilde{\mathcal{A}}\breve{k}$ is $(\bar{\mathcal{A}}^* - \mathcal{N})''$ f–$\mathcal{FSQ}$ –normal operator., if the following conditions are had $\tilde{\mathcal{A}}\breve{k} = \breve{k}\tilde{\mathcal{A}}$ and $\breve{k}^*\tilde{\mathcal{A}} = \tilde{\mathcal{A}}\breve{k}^*$.

Theorem 2.19 [20]: Let $\tilde{\mathcal{A}} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ is $(\bar{\mathcal{A}}^* - \mathcal{N})'' - \mathcal{FSQ}$ –normal operators then $\mathcal{A}/_{\tilde{\mathcal{M}}}$ is $(\bar{\mathcal{A}}^* - \mathcal{N})'' - \mathcal{FSQ}$ –normal operator such that $\mathcal{N} = \mathcal{N}/_{\tilde{\mathcal{M}}}$ where $\tilde{\mathcal{M}}$ is closed subspace.

3. Main Findings

This particular section is mainly devoted to presenting and explaining the definition of a completely new and distinct type of mathematical operator, together with several of its important and interesting properties that are related with it.

Definition 3.1: A $FS -$ bounded linear operator on a FS –Hilbert space $\tilde{\mathcal{A}} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ is order m $(\tilde{\mathcal{A}}^* - N)$ $-FSQN-$ operator if satisfy the condition $\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m}) = \tilde{\mathcal{N}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})\tilde{\mathcal{A}}$.

Theorem 3.2: If $\tilde{\mathcal{A}} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ is $(\tilde{\mathcal{A}}^* - N) - FSQN -$ operator of order m then $\tilde{\mathcal{A}}^n$ is also $(\tilde{\mathcal{A}}^* - N) - FSQN -$ operator of order m where $n \geq 1$

Proof :

By using mathematical induction

if $n = 1$,

$\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m}) = |\tilde{\mathcal{N}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})\tilde{\mathcal{A}}|$.

if it is true when $n = k$

$|\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})|^k = |\tilde{\mathcal{N}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})\tilde{\mathcal{A}}|^k$

To prove it when $n = k + 1$

$|\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})|^{k+1} = |\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})|^k |\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})|$,

So that

$$|\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})|^k |\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})| = [\tilde{\mathcal{N}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})\tilde{\mathcal{A}}]^k [\tilde{\mathcal{N}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})\tilde{\mathcal{A}}] = [\tilde{\mathcal{N}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*m})\tilde{\mathcal{A}}]^{k+1}.$$

Theorem 3.3: If $\bar{\mathcal{A}}_0$ and $\hbar$ be two $(\bar{\mathcal{A}}_0^* - N) - FSQN -$ operator of order m where satisfy $\bar{\mathcal{A}}_0\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_0^* \bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_0^* \bar{\mathcal{A}}_1^* = \tilde{0}$ then $\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1$ is $(\tilde{\mathcal{A}}^* - N)$ $-FSQN -$ operator of order m.

Proof:

$$\begin{aligned}
&(\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)\ (\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)\ (\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)^{*m} \\
&= (\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)((\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)(\bar{\mathcal{A}}_0^* + \bar{\mathcal{A}}_1^*)^m) \\
&= (\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)(\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)(\bar{\mathcal{A}}_0^{*m} + m(\bar{\mathcal{A}}_0^{*m-1})\bar{\mathcal{A}}_1^* + \cdots + \bar{\mathcal{A}}_1^{*m}) \\
&= (\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)(\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)\ (\bar{\mathcal{A}}_0^{*m} + \bar{\mathcal{A}}_1^{*m}) \\
&= (\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_1)(\bar{\mathcal{A}}_0\bar{\mathcal{A}}_0^{*m} + \bar{\mathcal{A}}_1\bar{\mathcal{A}}_0^{*m} + \bar{\mathcal{A}}_0\bar{\mathcal{A}}_1^{*m} + \bar{\mathcal{A}}_1\bar{\mathcal{A}}_1^{*m})
\end{aligned}$$

$= \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1^{*m} + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m}$

$= \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m}$

$= \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_0 + \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})\tilde{\mathcal{A}}_1 \ldots.(*)$

$\tilde{\mathcal{N}}\left((\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)^{*m}\right)(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}[(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)\ (\tilde{\mathcal{A}}_0^{*} + \tilde{\mathcal{A}}_1^{*})^{m}]\ (\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0^{*m} + m(\tilde{\mathcal{A}}_0^{*m-1})\tilde{\mathcal{A}}_1^{*} + \ldots. + \tilde{\mathcal{A}}_1^{*m})(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_1^{*m})(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m} + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1^{*m} + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}\left(\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1^{*m}\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m}\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1^{*m}\tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m}\tilde{\mathcal{A}}_1\right)$

$= (\tilde{\mathcal{N}}\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_0 + \tilde{\mathcal{N}}\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m}\tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_0 + \tilde{\mathcal{N}}(\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})\tilde{\mathcal{A}}_1 \ldots.(**)$

$= \tilde{\mathcal{N}}\left((\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)^{*m}\right)(\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1)$

Therefore; $\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1$ is a $(\tilde{\mathcal{A}}^* - N)$ $-FSQN-$ operator order m.

Theorem 3.4: Let $\tilde{\mathcal{A}}_0 : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ be $(\tilde{\mathcal{A}}^* - N)$ $-FSQN-$ operator order m. and $\tilde{\mathcal{A}}_1 : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ be $FS-\tilde{\mathcal{N}}$ –quasi normal operator, then the product $\tilde{N}_0\tilde{N}_1$ is $(\tilde{\mathcal{A}}^* - N)$ fuzzy soft quasi normal operator order m., if it satisfied $\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1^{*} = \tilde{\mathcal{A}}_1^{*}\tilde{\mathcal{A}}_0$

Proof:

$(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)^{*m} = (\tilde{\mathcal{A}}_1.\tilde{\mathcal{A}}_0)((\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_1.\tilde{\mathcal{A}}_0)^{*m}$

$= (\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_1^{*m})$

$= (\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0)(\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_1^{*m})$

$= (\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})$

$= (\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m})(\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})$

$= \tilde{\mathcal{N}}_0((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_0)\tilde{\mathcal{N}}_1((\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})\tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_0)((\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_1^{*m})\tilde{\mathcal{A}}_1)]$

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0)((\tilde{\mathcal{A}}_1^{*m})\tilde{\mathcal{A}}_1$

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_0)((\tilde{\mathcal{A}}_1^{*m})\tilde{\mathcal{A}}_1$

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m})\tilde{\mathcal{A}}_1^{*m}\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1$

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_0^{*m}\tilde{\mathcal{A}}_1^{*m})(\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1)$

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1)(\mathcal{A}_1.\tilde{\mathcal{A}}_0)^{*m})(\tilde{\mathcal{A}}_1.\tilde{\mathcal{A}}_0)]$,

$= \tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1[((\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)^{*m})(\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1.)]$

By choosing $\tilde{\mathcal{N}}_0\tilde{\mathcal{N}}_1 = \tilde{\mathcal{N}}$, then, we have $(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)((\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)^{*m}$ $= \tilde{\mathcal{N}}[(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)(\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)^{*m}](\tilde{\mathcal{A}}_0.\tilde{\mathcal{A}}_1)$

Therefore; $\tilde{\mathcal{A}}_0\tilde{\mathcal{A}}_1.$ is a $(\tilde{\mathcal{A}}^* - N)$ $-FSQN-$ operator order m.

Proposition 3.5: Let $\tilde{\mathcal{A}}:\tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ is $(\tilde{\mathcal{A}}^{*}-N)-FSQN-$ operators of order m then

i) $\beta\tilde{\mathcal{A}}$ is $(\tilde{\mathcal{A}}^{*}-N)-FSQN-$ operator of order m, where $\beta \in \mathcal{R}(\mathrm{A})$.

ii) $\mathcal{A}/_{\tilde{\mathcal{M}}}$ is $(\tilde{\mathcal{A}}^{*}-N)-FSQN-$ operator of order m s.t. $N/_{\tilde{\mathcal{M}}}$ where $\tilde{\mathcal{M}}$ is closed subspace.

Proof i):

$(\ddot{\lambda}\tilde{\mathcal{A}})(\ddot{\lambda}\tilde{\mathcal{A}}(\ddot{\lambda}\tilde{\mathcal{A}}^{*^{m}}) = \ddot{\lambda}\ddot{\lambda}\,\ddot{\lambda}^{m}(\tilde{\mathcal{A}}(\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*^{m}}))$

$= \ddot{\lambda}\ddot{\lambda}\,\ddot{\lambda}^{m}\tilde{\mathcal{N}}((\tilde{\mathcal{A}}\tilde{\mathcal{A}}^{*^{m}})\tilde{\mathcal{A}})$

$= \tilde{\mathcal{N}}((\ddot{\lambda}\tilde{\mathcal{A}}\ddot{\lambda}^{m}\tilde{\mathcal{A}}^{*^{m}})\ddot{\lambda}\tilde{\mathcal{A}})$

Therefore; $\ddot{\lambda}\tilde{\mathcal{A}}$ is a $(\tilde{\mathcal{A}}^{*}-N)-FSQN-$ operator order m

ii)

$\mathcal{A}/_{\tilde{\mathcal{M}}}(\mathcal{A}/_{\tilde{\mathcal{M}}}\mathcal{A}/_{\tilde{\mathcal{M}}}$ [illegible] $) =$ [illegible] $/_{\tilde{\mathcal{M}}}$

$= \tilde{\mathcal{N}}[$ [illegible] $]/_{\tilde{\mathcal{M}}}$

$= \tilde{\mathcal{N}}(\mathcal{A}/_{\tilde{\mathcal{M}}}\mathcal{A}/_{\tilde{\mathcal{M}}}$ [illegible] $)\mathcal{A}/_{\tilde{\mathcal{M}}}$

Therefore; $\mathcal{A}/_{\tilde{\mathcal{M}}}$ is a $(\tilde{\mathcal{A}}^{*}-N)-FSQN-$ operator order m.

4. Conclusions

This paper builds upon the foundational concepts of fuzzy and soft sets, key components within the field of fuzzy pure mathematics. We have advanced this area by introducing the more generalized framework of FS-sets. From this foundation, we have developed new mathematical structures, including FS-normed spaces and a generalization of inner product spaces, leading to the concepts of FS-inner product spaces, fuzzy soft Hilbert spaces, and fuzzy soft bounded linear operators. A central contribution is the introduction of a novel operator class: the $(\tilde{\mathcal{A}}^{*}-N)$ fuzzy soft quasi-normal operator of order m, or $(\tilde{\mathcal{A}}^{*}-N)$-FSQN operator. We have established its fundamental properties, defined essential operations, and elucidated its relationships with other established types of fuzzy soft operators.

Reference

[1] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965, https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] D. Molodtsov, “Soft set theory—First results,” *Computers & Mathematics with Applications*, vol. 37, no. 4–5, pp. 19–31, Feb. 1999, https://doi.org/10.1016/S0898-1221(99)00056-5.

[3] P. K. Maji, R. Biswas, and A. R. Roy, “Fuzzy soft sets,” *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602, 2001.

[4] N. FARIED, M. S.S. ALI, and H. SAKR, “FUZZY SOFT HERMITIAN OPERATORS,” *Advances in Mathematics: Scientific Journal*, pp. 73–82, Feb. 2020, https://doi.org/10.37418/amsj.9.1.7.

[5] N. Faried, M. S. S. Ali, and H. H. Sakr, "On Fuzzy Soft Linear Operators in Fuzzy Soft Hilbert Spaces," *Abstract and Applied Analysis*, vol. 2020, pp. 1–13, Jul. 2020, https://doi.org/10.1155/2020/5804957.
[6] M. Jssim and S. dawood Mohsen, "Investigating the Traits on a New Quasi Paranormal Operator Created by Fuzzy Soft Hilbert Description," *Mustansiriyah Journal of Pure and Applied Sciences*, vol. 3, no. 4, pp. 189–195, Sep. 2025, https://doi.org/10.47831/mjpas.v3i4.412.
[7] Alaa Jumaah Edan, "Some results on co fuzzy metric space," *Mustansiriyah Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 88–95, Mar. 2025, https://doi.org/10.47831/mjpas.v3i2.210.
[8] Salim Dawood Mohsen, "Novel Results of K Quasi (ℷ-M)-hyponormal Operator," *Iraqi Journal of Science*, pp. 374–380, Jan. 2024, https://doi.org/10.24996/ijs.2024.65.1.30.
[9] B. S. Reddy, S. Vijayabalaji, K. Punniyamoorthy, and A. V. R. Rao, "Soft \(n\)-Normed Linear Spaces: Generalizations and Extensions from Soft Normed Spaces," *Communications in Mathematics and Applications*, vol. 15, no. 1, pp. 265–277, May 2024, https://doi.org/10.26713/cma.v15i1.2427.
[10] H. Sadiq and M. A. Khalik, "Some Results on Fuzzy w-Local Covering Dimension Function in Fuzzy Topological Space," *Mustansiriyah Journal of Pure and Applied Sciences*, vol. 1, no. 2, pp. 11–20, Aug. 2023, https://doi.org/10.47831/mjpas.v1i2.26.
[11] B. Ahmad and A. Kharal, "On Fuzzy Soft Sets," *Advances in Fuzzy Systems*, vol. 2009, no. 1, Jan. 2009, https://doi.org/10.1155/2009/586507.
[12] N. Faried, M. S.S. Ali, and H. H. Sakr, "Fuzzy soft Hilbert spaces," *Journal of Mathematics and Computer Science*, vol. 22, no. 02, pp. 142–157, Jul. 2020, https://doi.org/10.22436/jmcs.022.02.06.
[13] Dalia raad abd and Gazwan Haider Abdulhusein, "Fuzzy Nano - Generalized Closed Set in Fuzzy Nano Symmetric Topology," *Mustansiriyah Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 169–177, Mar. 2025, https://doi.org/10.47831/mjpas.v3i2.353.
[14] N. Faried, M. S. S. Ali, and H. H. Sakr, "On Fuzzy Soft Linear Operators in Fuzzy Soft Hilbert Spaces," *Abstract and Applied Analysis*, vol. 2020, pp. 1–13, Jul. 2020, https://doi.org/10.1155/2020/5804957.
[15] A. Zahedi, A. Kilicman, and A. Salleh, "Parameterized Norm and Parameterized Fixed-Point Theorem by Using Fuzzy Soft Set Theory," Sep. 2013.
[16] S. Das and S. Samanta, "On soft inner product spaces," *Annals of Fuzzy Mathematics*, vol. 6, pp. 151–170, Jul. 2013.

[17] J. Mahanta and P. K. Das, “Fuzzy soft topological spaces,” *Journal of Intelligent & Fuzzy Systems*, vol. 32, no. 1, pp. 443–450, Jan. 2017, https://doi.org/10.3233/JIFS-152165.
[18] S. Dawood and A. Q. Jabur, “On fuzzy soft normal operators,” *J Phys Conf Ser*, vol. 1879, no. 3, p. 032002, May 2021, https://doi.org/10.1088/1742-6596/1879/3/032002.
[19] S. Dawood and A. Q. Jabur, “On fuzzy soft normal operators,” *J Phys Conf Ser*, vol. 1879, no. 3, p. 032002, May 2021, https://doi.org/10.1088/1742-6596/1879/3/032002.
[20] S. Abdulsada Assi, “On ((T ̃)^*-N)^η-Fuzzy Soft Quasi Normal Operators,” *Journal of Al-Qadisiyah for Computer Science and Mathematics*, vol. 16, no. 3, Sep. 2024, https://doi.org/10.29304/jqcsm.2024.16.31670.