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The Length-Biased Weighted Kpendidum Distribution: Properties and Some Real-Life Applications

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Abstract

The main aim of this paper is to introduce a new class of one-parameter statistical models namely the length-biased weighted Kpendidum distribution. This distribution is a convex combination of three component of Gamma distribution. Some statistical and reliability properties of this distribution are discussed. The maximum likelihood estimation is used to estimate the one-parameter of the presented distribution. Moreover, five real life applications are provided to investigate the effectiveness of the new model in comparison with the Kpendidum distribution and some other one-parameter length-biased weighted models. Additionally, it is shown that in some real-life applications the proposed model gives a better results than some of the two-parameter models. Finally, different statistical measures are used to guarantee the accuracy of this model.

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1. Introduction

In statistical modeling and data analysis, it is often implicitly assumed that observations are collected via simple random sampling, where each unit in the population has an equal chance of being selected. However, in many real-life applications, this assumption does not hold. Instead, the probability of an item or event being observed or sampled is often proportional to some intrinsic characteristic of the item itself, such as its "length," size, duration, or magnitude. This non-random sampling mechanism leads to what are known as weighted distributions, with length-biased distributions being a particularly common and significant special case. The concept of weighted distributions was formalized to provide a unified approach for analyzing data arising from various non-random sampling schemes. If X a non negative random variable with the probability density function g(x). a weighted version of X, denoted as X_W, has a PDF $f_w(x)$ given by:

$$f_w(x) = \frac{w(x)g(x)}{E[w(X)]}$$

where w(x) is a non negative weight function and $E\big(w(x)\big) = \int_0^\infty w(x)g(x)dx < \infty$ is the normalizing constant, representing the expected value of the weight function with respect to the original distribution of X. This ensures that $f_w(x)$ integrates to unity. If $w(x) = x$, then the resulting distribution is named as the length-biased weighted distribution or the size-biased weighted distribution of order one,[1]. In this case, the pdf of the length-biased weighted random variable X_w takes the form $f_W(x) = \frac{xg(x)}{E(X)}$. Cox in 1962 was first introduced the concept of the length-biased weighted distribution [2]. Then many works were done on this subject. In recent years, research interest in LB distributions has increased, with several new models and inference methods being proposed. For instance, Hosseini et al. in [3] introduced a new weighted-Lindley distribution and studied both classical and Bayesian estimation methods, showing its competitiveness in lifetime data analysis. Andure and Ade in [4] proposed the length-biased weighted quasi-Lindley distribution, demonstrating its superior result compared with related models in reliability applications. Similarly, Ganaie and

Rajagopalan in [5] developed the length-biased weighted new quasi-Lindley distribution, highlighting its flexibility in capturing various hazard rate shapes. More recently, Shen et al. in reference [6] extended maximum likelihood estimation procedures to length-biased and interval-censored data with a non susceptible fraction, while Qiu et al. in [7] addressed the challenges of length-biased and partly interval-censored survival data in the presence of mismeasured covariates. These works reflect a broader trend of extending LB models to handle increasingly complex data structures. For more details see [3, 4, 8–22]. The Length-Biased Weighted Kpendidum Distribution, on the other hand, is a recently proposed statistical model designed to address the challenges in modeling skewed lifetime and reliability data. By introducing a length-biased weighting mechanism to the classical Kpendidum distribution, this extended version provides enhanced flexibility in capturing the variability and asymmetry commonly observed in practical applications, particularly in environmental studies, reliability engineering, and biological processes. In this article, the definition of the length-biased weighted Kpendidum distribution is given with some of its properties. Also, the method of maximum likelihood is used to estimate the parameter of the proposed distribution. Moreover, five real life applications are presented to show that the constructed distribution is more efficient than the Kpendidum distribution and some other one-parameter length-biased weighted distributions. Finally the important practical use of the proposed one-parameter distribution over some other two-parameter distributions is illustrated in modeling some real life applications. This paper is organized as follows: in section 2, we give the definition of the length-biased weighted Kpendidum distribution. In section 3, we introduce some properties of this distribution including rth moment about the origin, central moment, mean, variance, standard deviation, coefficient of variation, index of dispersion, coefficient of skewness, and coefficient of kurtosis, moment generating function, characteristics function and stochastic ordering. In section 4, we use the maximum likelihood method to estimate the parameter of the proposed distribution. In section 5, five real life applications are introduced to compare the new model with other distributions. In the end, some conclusions are given in section 6.

2. The Length-Biased Weighted Kpendidum Distribution

The random variable X has the Kpendidum distribution (KD) if its probability density function for $x > 0$ takes the form, [21]

$$g(x;\theta) = \frac{\theta^4}{2(\theta^4+\theta+6)}\,(2x^3 + x^2 + 2\theta)e^{-\theta x}, \quad x > 0,\ \theta > 0 \quad \dots(1)$$

The length-biased weighted Kpendidum distribution (LBWKD) is a distribution in which its pdf takes the form: (shown in figure (1))

$$f(x;\theta) = \frac{x g(x;\theta)}{E[X]} = \frac{\theta^5}{2(\theta^4+3\theta+24)}\,(2x^4 + x^3 + 2\theta x)e^{-\theta x}, \quad x > 0, \theta > 0 \quad \dots(2)$$

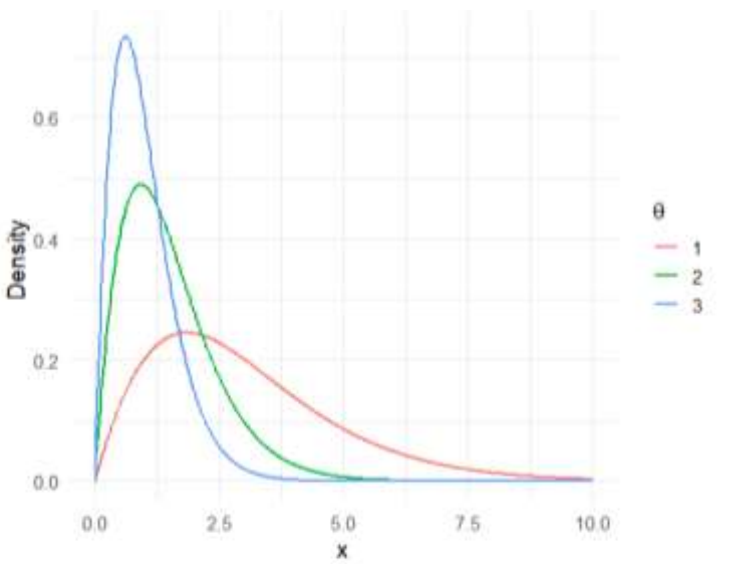

Figure 1: the pdf of LBWK distribution for different value of θ

Let g_1, g_2, g_3 be the pdf of $Gamma(2,\theta)$, $Gamma(4,\theta)$, $Gamma(5,\theta)$ distributions respectively. Then $f(x;\theta)=\sum_1^3 p_i g_i(x;\theta)$, where $p_1=\frac{2\theta^4}{2(\theta^4+3\theta+24)}$, $p_2=\frac{6\theta}{2(\theta^4+3\theta+24)}$ and $p_3=\frac{48}{2(\theta^4+3\theta+24)}$. That is f is a three component mixture distribution of $Gamma(2,\theta)$, $Gamma(4,\theta)$, $Gamma(5,\theta)$ with weights p_i, $i=1,2,3$ respectively. So, the corresponding cumulative density function of the constructed distribution is:

$$F(x;\theta)=\int_0^x f(t;\theta)dt = \\ =\frac{2\gamma(5,\theta x)+\theta\gamma(4,\theta x)+2\theta^4\gamma(2,\theta x)}{2(\theta^4+3\theta+24)}$$

where γ is the lower incomplete gamma function. The approximate cdf is plotted in figure (2).

On the other hand, the quantile function $Q(u)=F^{-1}(u)$ typically has no closed form because it is an inverse of a mixture of incomplete gamma functions. However, Numerical inversion via root finding could be found by $Q(u)=x>0$, where x is the unique solution of $F(x)-u=0$. However, figure (2) (pp plot) and figure (3) (QQ plot) show significant result. The Q–Q plot against the Normal distribution indicates a poor fit, with substantial deviations in the tails, underscoring the inadequacy of symmetric models for such data. By contrast, the PP-plot for LBWK distribution shows that the proposed distribution aligns closely with the empirical probabilities, particularly in the central range, confirming its suitability for modeling length-biased data.

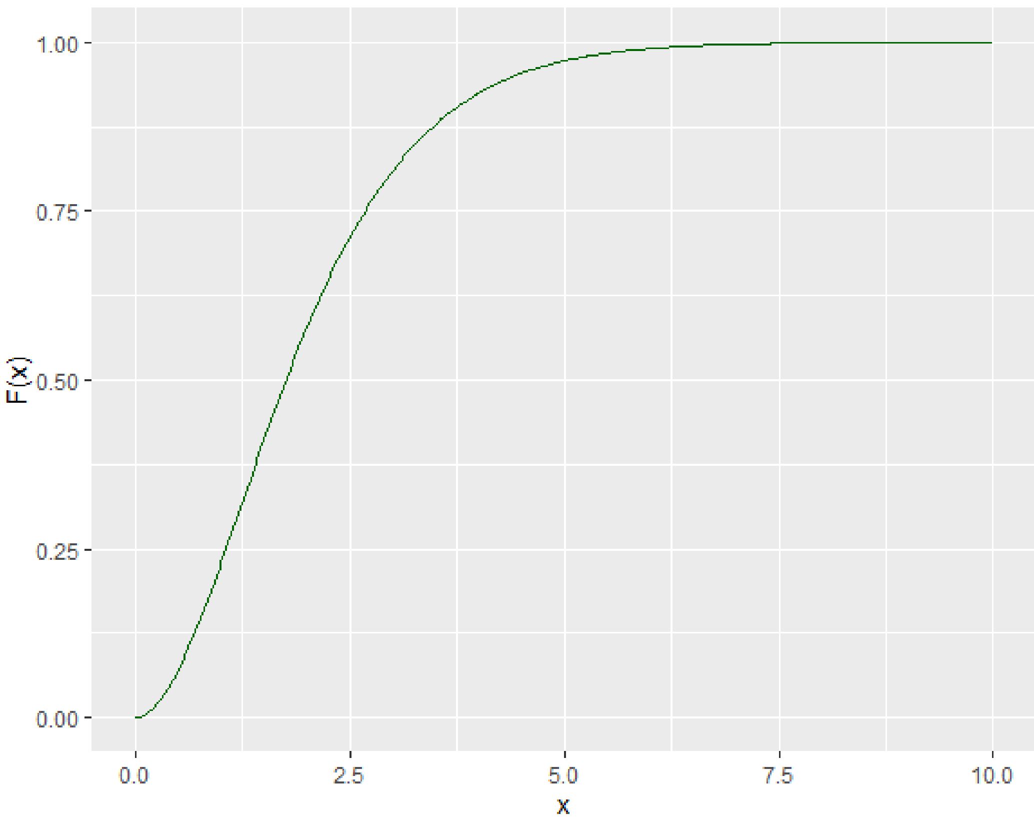

Figure 2: the cdf of LBWK distribution

3. Some Statistical Properties of the LBWKD

In this section, some of the main properties of the length-biased weighted Kpendidum distribution are presented. These properties are essential for the rest of the article.

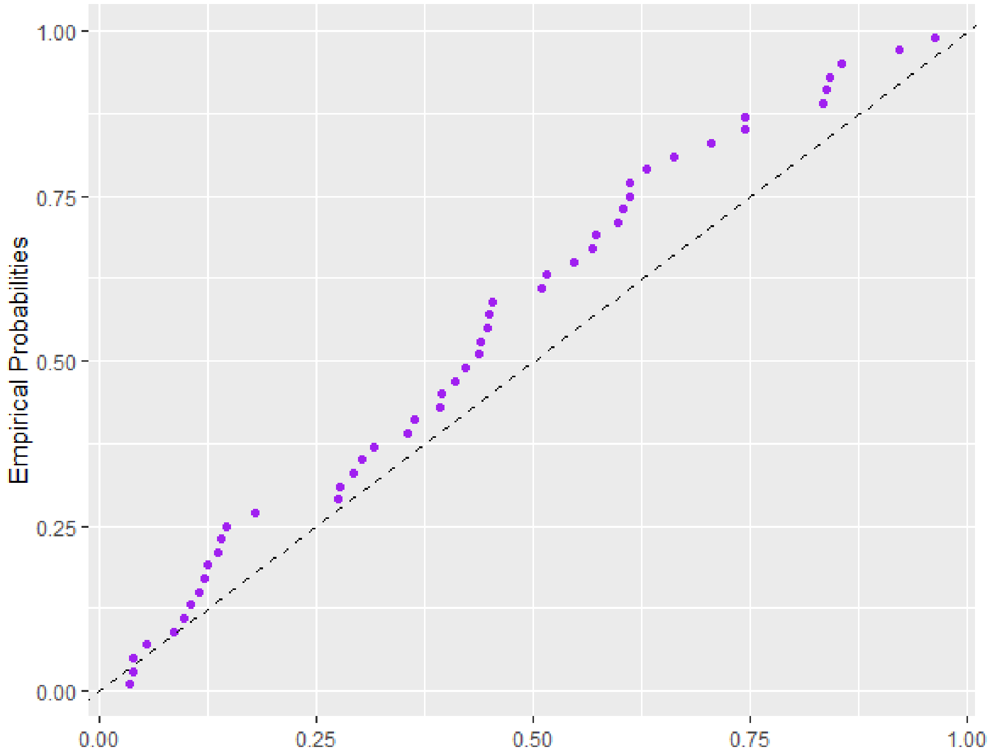

Figure 3: the PP plot of LBWK distribution

3.1. Moments and Related Measures

The r[th] moment about the origin is determined as follows:

$$\mu_r' = E[X^r] = \int_0^\infty x^r f(x;\theta)dx$$

$$= \frac{\theta^5}{2(\theta^4+3\theta+24)}\left[\frac{2\Gamma(5+r)}{\theta^{5+r}} + \frac{\Gamma(4+r)}{\theta^{4+r}} + \frac{2\Gamma(2+r)}{\theta^{r+1}}\right]$$

Therefore, the mean of X, μ is given by:

$$\mu = \mu_1' = E[X] = \frac{2(\theta^4+6\theta+60)}{\theta(\theta^4+3\theta+24)}$$

Hence the variance σ^2, the coefficient of variation CV and the index of dispersion γ are given as:

$$\sigma^2 = \mu_2' - \mu^2 = \frac{2(\theta^8+15\theta^5+192\theta^4+18\theta^2+360\theta+1440)}{\theta^2(\theta^4+3\theta+24)^2}$$

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{2(\theta^8+15\theta^5+192\theta^4+18\theta^2+360\theta+1440)}}{2(\theta^4+6\theta+60)}$$

and

$$\gamma = \frac{\sigma^2}{\mu} = \frac{\theta^8+15\theta^5+192\theta^4+18\theta^2+360\theta+1440}{\theta(\theta^4+3\theta+24)(\theta^4+6\theta+60)}$$

Moreover, the central moment of the LBWKD is:

$$\mu_r = \frac{\theta^5}{2(\theta^4+3\theta+24)}\sum_{i=0}^{r}\binom{r}{i}\left[\frac{2\Gamma(5+i)}{\theta^{5+i}} + \frac{\Gamma(4+i)}{\theta^{4+i}} + \frac{2\Gamma(2+i)}{\theta^{i+1}}\right]\left[\frac{2(\theta^4+6\theta+60)}{\theta(\theta^4+3\theta+24)}\right]^{r-i}$$

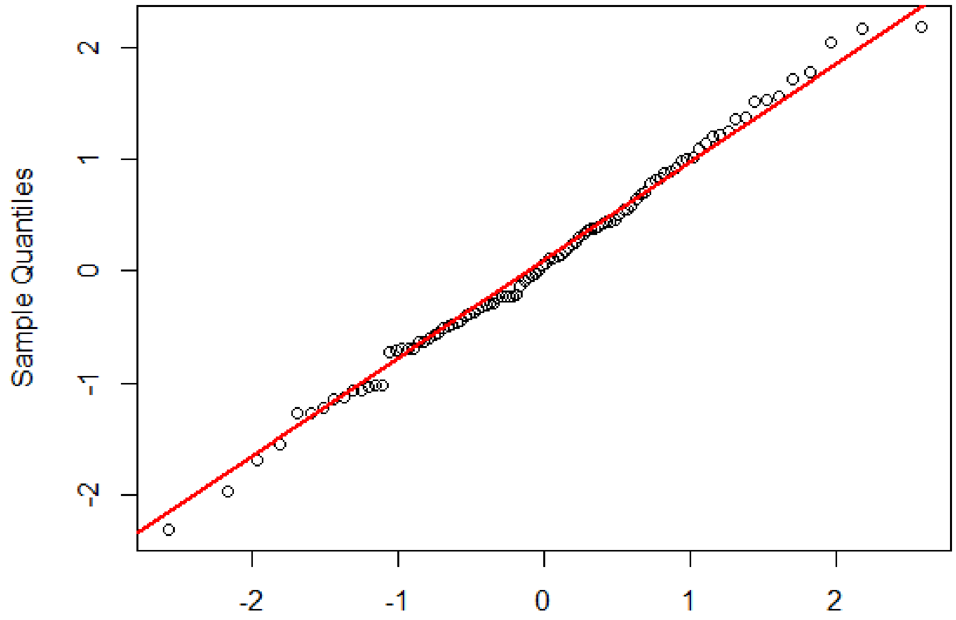

Figure 4: the QQ plot of LBWK distribution

Thus the coefficient of skewness is

$$S = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{A}{B}$$

Where

$$A = 12\theta^{14} - 20\theta^{12} + 252\theta^{11} + 3096\theta^{10} - 360\theta^9 - 3276\theta^8 + 25488\theta^7 + 125928\theta^6 - 42444\theta^5 - 193320\theta^4 + 463536\theta^3 + 1360800\theta^2 - 881280\theta - 2764800$$

And

$$xB = \sqrt{8(\theta^8 + 15\theta^5 + 192\theta^4 + 18\theta^2 + 360\theta + 1440)^3}$$

The coefficient of kurtosis is $K = \frac{\mu_4}{{\mu_2}^2} = \frac{A}{B}$, where

$$A = -72\theta^{19} - 1728\theta^{16} - 22896\theta^{15} - 10152\theta^{13} - 279648\theta^{12} - 1881792\theta^{11} - 4968\theta^{10} - 584496\theta^9 - 110833392\theta^8 - 54652320\theta^7 + 2327616\theta^6 + 126748800\theta^5 - 657791280\theta^4 - 516093120\theta^3 + 945561600\theta^2 + 604661760 + 1428295680$$

And

$$B = 8\theta^{16} + 240\theta^{13} + 3072\theta^{12} + 2088\theta^{10} + 51840\theta^9 + 317952\theta^8 + 4320\theta^7 + 141696\theta^6 + 1451520\theta^5 + 4426272\theta^4 + 103680\theta^3 + 1451520\theta^2 + 8294400\theta + 16588800$$

3.2. The Moment Generating Function and Characteristics Function

The moment generating function of the LBWKD is determined as:

$$M(t) = \int_0^\infty e^{tx} f(x;\theta)dx = \frac{\theta^5}{2(\theta^4 + 3\theta + 24)}\left[\frac{2\Gamma(5)}{(\theta - t)^5} + \frac{\Gamma(4)}{(\theta - t)^4} + \frac{2\theta\Gamma(2)}{(\theta - t)^2}\right]$$

And the characteristic function of the LBWKD is determined as:

$$Q(t) = E[e^{itX}] = \frac{\theta^5}{2(\theta^4 + 3\theta + 24)}\left[\frac{2\Gamma(5)}{(\theta - it)^5} + \frac{\Gamma(4)}{(\theta - it)^4} + \frac{2\theta\Gamma(2)}{(\theta - it)^2}\right]$$

3.3. Stochastic Ordering

Let X and Y be random variables of LBWK with the following pdf's:

$$f_X(x;\theta_1) = \frac{{\theta_1}^5}{2\left({\theta_1}^4 + 3\theta_1 + 24\right)}(2x^4 + x^3 + 2\theta_1 x)e^{-\theta_1 x}, x, \theta_1 > 0$$

$$f_Y(x;\theta_2) = \frac{{\theta_2}^5}{2\left({\theta_2}^4 + 3\theta_2 + 24\right)}(2x^4 + x^3 + 2\theta_2 x)e^{-\theta_2 x}, \quad x, \theta_2 > 0$$

Then

$$\frac{d}{dx}log\left(\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)}\right) = -\frac{(\theta_1 - \theta_2)A}{(2x^3 + x^2 + 2\theta_1)(2x^3 + x^2 + 2\theta_2)}$$

where

$$A = 4x^6 + 4x^5 + x^4 + 4\theta_2 x^3 + 4\theta_1 x^3 + 2\theta_2 x^2 + 2\theta_1 x^2 + 12x^2 + 4x + 4\theta_1\theta_2$$

Therefore, for $\theta_1 \geq \theta_2$, $\frac{d}{dx} log\left(\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)}\right) \leq 0$. This means that LBWKD is ordered with respect to the likelihood ratio ordering. Moreover, by using [20], the LBWKD is ordered with respect to the stochastic order, hazard order and mean residual life order.

4. Maximum Likelihood Estimation of the LBWKD

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from LBWKD distribution, then the likelihood function takes the form:

$$L(\theta) = \left[\frac{\theta^5}{2(\theta^4 + 3\theta + 24)}\right]^n e^{-\theta\sum_{i=1}^n x_i} \prod_{i=1}^{n} (2x_i^4 + x_i^3 + 2\theta x_i)$$

Therefore

$$logL(\theta) = 5n\log\theta - nlog2 - nlog(\theta^4 + 3\theta + 24) - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} log\,(2x_i^4 + x_i^3 + 2\theta x_i)$$

To estimate θ , we must solve the nonlinear algebraic equation:

$$\frac{\partial logL(\theta)}{\partial\theta} = \frac{5n}{\theta} - \frac{n(4\theta^3 + 3)}{\theta^4 + 3\theta + 24} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2x_i}{2x_i^4 + x_i^3 + 2\theta x_i} = 0$$

To do this, one can use any computer package/software say R, SAS, Ox, Matlab, Maple, Mathematica.

5. Some Real-Life Applications

In this section, five real life applications are presented to illustrate the flexibility of the one-parameter length-biased weighted Kpendidum distribution as compared with the Kpendidum distribution (KD) and the following known one-parameter length-biased weighted distributions.

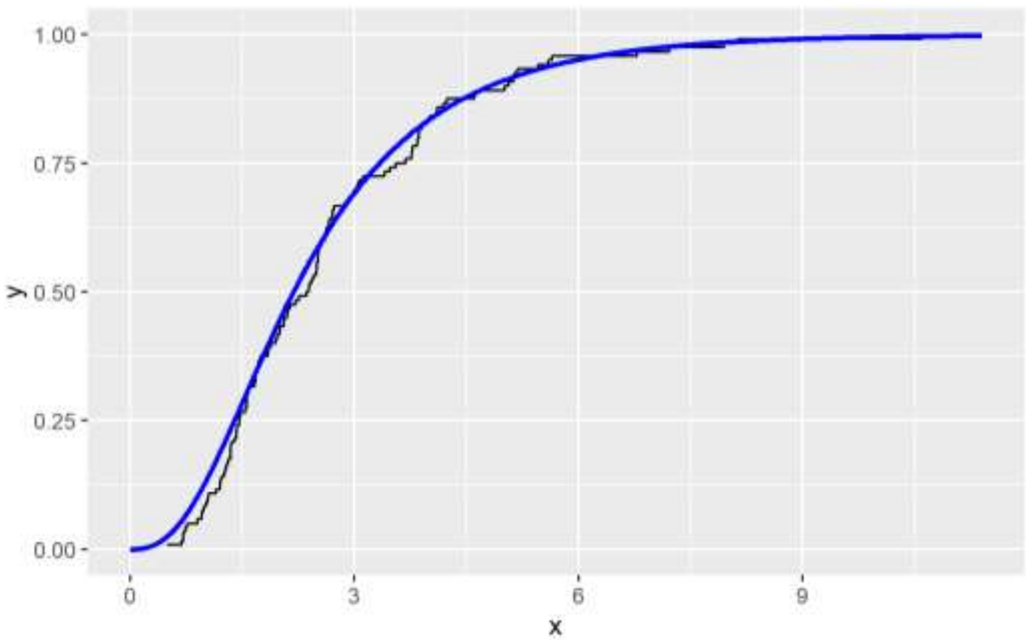

Figure 5: the fitted cdf of LBWK distribution from set data

The length-biased weighted exponential distribution (LBWED), [14]:

$$f(x;\theta) = \theta^2\, xe^{-\theta x}, x > 0,\ \ \theta > 0$$

The length-biased weighted Lindley distribution (LBWLD), [14]:

$$f(x;\theta) = \frac{\theta^3\, x(1+x)}{2+\theta}\, xe^{-\theta x}, x > 0,\ \ \theta > 0$$

The length-biased weighted Aliamujia distribution (LBWAD), [23]:

$$f(x;\theta) = 4\theta^3\, x^2 e^{-2\theta x},\qquad x > 0,\ \ \theta > 0$$

The length-biased weighted Aradhana distribution (LBWARD), [19]:

$$f(x;\theta) = \frac{\theta^4}{\theta^2 + 4\theta + 6} x(1+x)^2 e^{-\theta x},\qquad x > 0,\ \ \theta > 0$$

The length-biased weighted Shanker distribution (LBWSD), [24]:

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(x^2+\theta x)e^{-\theta x}, \qquad x>0,\ \theta>0$$

Moreover, it is shown that the proposed one-parameter distribution is also more appropriate than the two-parameter distributions. For this purpose, the maximum likelihood estimation is used for each model to estimate the value of its parameter (θ) Also some of the well-known information criterion metrics are employed for each model. Figure (5) demonstrates that the LBWK distribution provides a strong distributional fit across the full support. On the other hand, The histogram–PDF comparison (Figure (6)) highlights that the LBWK distribution adequately captures both the central tendency and the skewness of the data. Together, they confirm that the LBWK distribution offers superior flexibility and accuracy compared to existing one- and two-parameter models in handling skewed, length-biased, and reliability-type data.

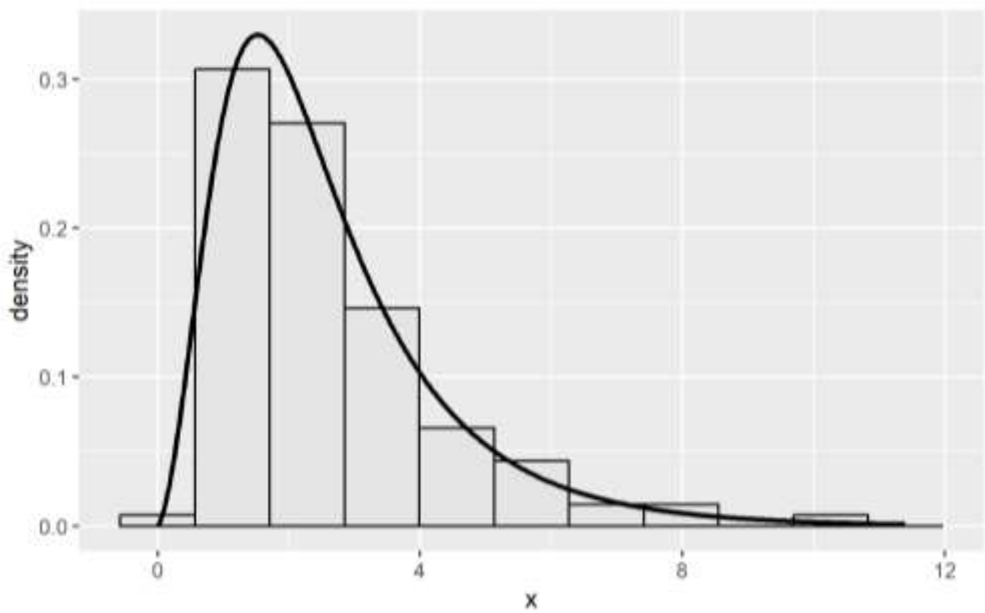

Figure 6: the fitted histogram of the pdf of LBWK distribution

5.1. Data Set (1)

The third data is the data that was reported by Murthy et al. in [25]. In 2022, this data set is analyzed by Hosseini et al. in [26] to investigate the flexibility of the following one-parameter weighted-Lindley distribution (WLD):

$$f(x;\theta)=\frac{2\theta^2(1+x)}{(1+\theta)\left[1+\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^2}\,e^{-\theta x}, x>0,\ \theta>0$$

as compared with Weibull, Rayleigh, Gamma, Lindley, exponential, log normal and half-logistic exponential distributions. Here, we will show that LBWED is the best fitting as compared with WLD, KD and the above one-parameter length-biased weighted distributions.

5.2. Data Set (2)

The fourth data is the data that was reported by Birnbaum and Saunders in [27]. In 2022, this data set is analyzed by Andure and Ade in [28] to investigate the flexibility of the following two-parameter length-biased weighted quasi-Lindley distribution (LBWQLD):

$$h(x;\theta,\alpha)=\frac{\theta^4}{2(\theta^2+3\alpha)}(\theta+\alpha x)x^2e^{-\theta x}, x>0,\ \theta>0, \alpha>0$$

Here, we will show that LBWKD is the best fitting as compared with LBWQLD, KD and the above one-parameter length-biased weighted distributions.

5.3. Data Set (3)

The fifth data is the data that was reported by Xu et al. in [29]. This data set is studied by Ganaie and Rajagopalan, [30] to indicate the flexibility of the two-parameter length-biased weighted new quasi Lindley distribution (LBWNQLD):

$$h(x;\theta,\alpha)=\frac{\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)xe^{-\theta x}, x>0,\ \theta>0,\alpha>0$$

Here, we will show that LBWKD is the best fitting as compared with LBWNQLD, KD and the above length-biased weighted distributions. From tables (1)-(3), it is obvious that the constructed distribution (LBWKD) provides a better fit for the above five real data sets as compared with the Kpendidum distribution and the other existing well-known weighted distributions since it has lowest value of (-2lnL), (AIC), (BIC), (AICC), (HQIC) and (CAIC).

Table 1. Estimation of parameter and measures of goodness of fit for data set (1)

Model	Θ	-2lnL(θ)	AIC	BIC	AICC	HQIC	CAIC
LBWKD	1.89643	199.643	201.643	203.786	201.706	202.486	201.708
WLD	0.99000	201.990	203.990	206.133	204.053	204.833	204.056
KD	1.36794	202.358	204.358	206.501	204.422	205.201	204.424
LBWSD	1.23278	204.800	206.800	208.943	206.863	207.643	206.866
LBWLD	1.25387	204.908	206.908	209.052	206.972	207.751	206.974
LBWARD	1.55587	205.610	207.610	209.753	207.673	208.452	207.675
LBWED	0.95911	205.747	207.747	209.890	207.810	208.590	207.813
LBWAD	1.43866	214.304	216.304	218.447	216.368	217.147	216.370

Table 2. Estimation of parameter and measures of goodness of fit for data set (2)

Model	θ (α)	-2lnL(θ)	AIC	BIC	AICC	HQIC	CAIC
LBWKD	0.07229	936.248	938.248	940.863	938.287	939.306	938.288
LBWQLD	0.05801 (74.01668)	945.778	949.778	955.008	949.898	951.895	949.900
KD	0.0578	946.150	948.150	950.766	948.190	949.209	948.191
LBWARD	0.05739	946.659	948.659	950.893	948.717	949.545	948.718
LBWAD	0.04345	962.847	964.847	966.281	964.976	965.315	964.985
LBWSD	0.04344	962.882	964.882	966.316	965.349	965.349	965.020
LBWLD	0.04315	963.643	965.643	967.077	965.772	966.111	965.781
LBWED	0.02897	992.967	994.967	996.401	995.096	995.434	995.105

Table 3. Estimation of parameter and measures of goodness of fit for data set (3)

Model	θ (α)	-2lnL(θ)	AIC	BIC	AICC	HQIC	CAIC
LBWKD	0.77842	179.042	181.042	182.731	181.142	181.653	181.147
KD	0.61307	184.252	186.252	187.941	186.352	186.862	186.357
LBWARD	0.58411	185.959	187.959	189.648	188.059	188.570	188.064
LBWAD	0.47981	189.006	191.006	192.694	191.106	191.616	191.111
LBWSD	0.46155	190.681	192.681	194.115	192.810	193.148	192.819
LBWNQLD	0.47980 4.32705×103	189.006	193.006	196.384	193.314	194.227	193.330
LBWLD	0.45041	192.182	194.182	195.871	194.282	194.793	194.288
LBWED	0.31987	201.026	203.026	204.715	203.126	203.637	203.131

6. Conclusions

In this paper, a new form of the one-parameter length-biased weighted distribution was presented. This model is the length-biased weighted distribution associated with the Kpendidum distribution. It has been shown that this model is a convex mixture of three components of Gamma distributions. Subsequently, the flexibility of this model is illustrated by studying five real life applications which gives a better fit than the Kpendidum distribution and some of the well-known one-parameter length-biased weighted distributions. Also, the important useful of this one-parameter distribution in comparison with some two-parameter distributions in modeling some real data sets is illustrated.

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