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Oscillation Theorems for Solutions of Third Order Nonlinear Neutral Differential Equations with Periodic Coefficients

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Abstract

In this article, nonlinear third order neutral differential equations with several periodic coefficients and forced terms are studied, where certain necessary and sufficient conditions exist to ensure that only oscillatory solutions exist for this type of equations. All cases where no oscillatory solutions are possible are discussed, and appropriate conditions are found to eliminate them, The research showed the effect of periodic coefficients on the speed and presence of oscillation in solutions. Some illustrative examples are added to the results obtained.

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1. Introduction

Consider the equation neutral; with periodic coefficients (NEPC):

$$[\psi(\check{t}) - \mathrm{p}(\check{t}) * \psi(\tau(\check{t}))]''' + \sum_{l=1}^{n} Q_l(\check{t}) * f(\psi(\check{t} - \sigma_l)) = \sum_{v=1}^{k} \varrho_v(t) \quad \dots (1.1)$$

$\mathrm{p} \in C[[\check{t}_0, \infty); R^+]$, $\tau, Q_l, \varrho_v \in C[[\check{t}_0, \infty); R]$, $\tau(\check{t})$is Increasing function and $\lim_{\check{t}\to\infty} \tau(\check{t}) = \infty$, Q_l, ϱ_v are periodic functions of period σ, $\sigma_l = m_l\,\sigma$, $m_l \in N$, $l = 1,2,\cdots,n$.

$$\sigma_{\boldsymbol{min}} = \min\{\sigma_l,\ l = 1, 2, \cdots, n\} > 0,\ \ \sigma_{\boldsymbol{max}} = \max\{\sigma_l,\ l = 1,2,\cdots,n\},\ \ Q \not\equiv 0,\ \check{t} \geq 0$$

Let

$$\varphi(\check{t}) = \psi(\check{t}) - \mathrm{p}(\check{t})\,\psi(\tau(\check{t})) \quad \check{t}_0 \leq \check{t} \quad \dots (1.2)$$

This type of equation is important in many applications, such as the current connected to a nonlinear circuit in a transmission line without loss; many researchers have been interested in it. Abed et al. [1] Use two neutral integro-differential equations, some conditions are set to ensure that only oscillating solutions are obtained. Altun et al. [2] the estimates of solutions are found to show exponential decay at ∞ of a system neutral equation which is nonlinear and has periodic coefficients. Chatzarakis et al. In [3], the asymptotic behavior and fluctuating of equations(non linear) involving neutral terms have been studied. Fridman et al. [4] Studies the stability with quasi-periodic, continuous coefficients uses a method to find the maximum value of the small coefficient to ensure stability. Use a suitable Lyapunov function to find stability conditions in the form of inequalities. Hassan et al. [5] An asymptotic properties of nonlinear damped neutral equation were analyzed, supported by numerical examples. Matveeva I. [6] presents a study of nonlinear systems that comprising differential equations with recurring coefficients. Lyapunov functions were employed to determine the conditions for achieving exponential stability at zero and to estimate the exponential decay at infinity. Mohamad et al. [7] Explained the oscillation of a half-linear equation of 2nd order, and through the conditions obtained it turns out that the solutions are oscillatory. Mushtt et al. [8]Introduced Nonoscillatory Properties of 4th Order Neutral Differential Equation. Rama et al. [9] Discussed the oscillation solutions of neutral equations of the first order. Sufficient conditions have been obtained to contain all solutions of equations with deviating arguments and oscillatory coefficients. Sallam et al. [10] Adopt a nonlinear equation of order three without periodic coefficients and, through examples, show that the solution is an oscillatory or near zero. Sharba et al. In [11] new conditions were established to guarantee non-oscillatory solutions to the TOMDDE of order three, and they were illustrated with examples. Tollu. [12] Deals with nonlinear difference equations that include periodic coefficients. The results focus on proving the existence of periodic solutions and the conditions that satisfy them, along with an analysis of the solution behavior in terms of stability and periodicity. This paper investigates the oscillation of neutral, nonlinear, 3rd order equation that contains several periodic coefficients; sufficient conditions have been established to ensure that only oscillatory solutions exist for this equation. The importance of this study lies in the extent of the impact of periodic coefficients on the presence and speed of oscillation, as shown by the conditions that were found on the coefficients.

2. Results

We offer the results for the oscillating behavior of (1.1)

Theorem 1. Consider that $Q_l(\check{t})$ at least zero $, \sum_{v=1}^{k} \varrho_v \leq 0,\ 1 > p_0 \geq p(\check{t}) \geq \mathbf{0},\ \tau(\check{t}) \leq \check{t}$ and

$$n\hat{\sigma} > 1, \quad \hat{\sigma} = \min_{l} \int_0^{\sigma_l} Q_l(\check{t})\, d\check{t}, \quad l = 1,2,\dots,n. \qquad (2.1)$$

The solution ψ of (1.1) either resort to zero or oscillates or $|\psi(\check{t})| \to \infty$. As $\check{t} \to \infty$.

Proof. Let $\psi(\check{t})$ is not an oscillating solution to eq. (1.1)

Let $\psi(\check{t}) > 0,\ \psi\big(\tau(\check{t})\big) > 0,\ \psi(\check{t} - \sigma_l) > 0\ , \check{t} \geq \check{t}_0$. hence

$$\varphi'''(\check{t}) = -\sum_{l=1}^{n} Q_l(\check{t})\ f\big(\psi(\check{t} - \sigma_l)\big) + \sum_{v=1}^{k} \varrho_v(\check{t}) \leq 0. \qquad (2.2)$$

So φ'' is a non-increasing, so either $\varphi'(\check{t})$, $\varphi(\check{t})$ are monotones either $\varphi''(\check{t}) < 0$ or > 0, $\check{t} \geq \check{t}_1 \geq \check{t}_0$.

Case (1). $\varphi''(\check{t}) < 0,\ \varphi'''(\check{t}) \leq 0$ .Thus $\check{t}_1 \leq \check{t}_2$ s. t $\varphi'(\check{t}) < 0,\ \varphi(\check{t}) < 0$ and $\lim_{\check{t}\to\infty} \varphi(\check{t}) = -\infty$. from (1.2) we get $\psi(\check{t}) \leq p(\check{t}) * \psi\big(\tau(\check{t})\big) \leq \psi\big(\tau(\check{t})\big), \check{t} \geq \check{t}_2$.

That is $\psi(\check{t})$ is positive decreasing on $[\check{t}_2, \infty)$ so it is bounded on the other side from (1.2) we get $\varphi(\check{t}) \geq -p(\check{t})\psi\big(\tau(\check{t})\big) > -\psi\big(\tau(\check{t})\big)$ or $\psi\big(\tau(\check{t})\big) > -\varphi(\check{t}),\ \check{t} \geq \check{t}_2$. Hence, $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$. this means a contradiction.

Case (2) $\varphi''(\check{t}) > 0,\ \check{t} \geq \check{t}_1$, So $\check{t}_2 \geq \check{t}_1$, $\varphi'(\check{t}) < 0$ or $\varphi'^{(\check{t})} > 0$, $\breve{\boldsymbol{t}} \geq \check{t}_2$.

Case 2.1 Let $\varphi'(\check{t}) < 0, \check{t} \geq \check{t}_2, \check{t}_3 \geq \check{t}_2$ either $\varphi(\check{t})$**less or more** 0 for $\check{t} \geq \check{t}_3$, so there are two subcases to consider:

Case 2.1.1 $\varphi(\check{t}) < 0$, there is $\check{t}_4 \geq \check{t}_3$ and b > 0, $\varphi(\check{t}) \leq -b, \check{t}_4 \leq \check{t}$ .From equation (1.2)

$$\varphi(\check{t}) \geq -p(\check{t}) * \psi\big(\tau(\check{t})\big) \geq -\psi(\tau(\check{t})), \text{ that is}$$

$$-\psi(\check{t}) \leq \varphi\big(\tau^{-1}(\check{t})\big),$$

$$-\psi(\check{t} - \sigma_l) \leq \varphi(\tau^{-1}(\check{t} - \sigma_l)),\ \check{t} \geq \check{t}_4, \qquad (2.3)$$

Substitute (2 .3) in (1.1) yields

$$\begin{aligned}\varphi'''(\check{t}) &= -\sum_{l=1}^{n} Q_l(\check{t})\, \boldsymbol{f}\big(\psi(\check{t} - \sigma_l)\big) + \sum_{v=1}^{k} \varrho_v(\check{t}) \\ &\leq -\beta \sum_{l=1}^{n} Q_l(\check{t})\, \psi(\check{t} - \sigma_l) + \sum_{v=1}^{k} \varrho_v(\check{t})\end{aligned}$$

$$\varphi'''(\check{t}) \leq -\beta b \sum_{l=1}^{n} Q_l(\check{t}) + \sum_{v=1}^{k} \varrho_v(\check{t}). \qquad (2.4)$$

Integrating(2.4) from $\check{t}$ to $\check{t} + \breve{\sigma}$

$$\begin{aligned}\varphi'(\check{t} + \sigma) - \varphi'(\check{t}) &\leq -\beta b \int_{\check{t}}^{\check{t}+\sigma} \sum_{l=1}^{n} Q_1(s)\, ds + \int_{\check{t}}^{\check{t}+\sigma} \sum_{v=1}^{k} \varrho_v(s)\ ds \\ &\leq -\beta b \sum_{l=1}^{n} \int_{\check{t}}^{\check{t}+\sigma} Q_l(s)\, ds + \sum_{v=1}^{k} \int_{\check{t}}^{\check{t}+\sigma} \varrho_v(s)\, ds\end{aligned}$$

$$\varphi'(\check{t}) \geq (\beta b - k)\breve{\sigma}, \quad \breve{\sigma} = \min_{l,v} \left(\int_0^{\sigma} Q_l(s) - \varrho_v(\check{t}) \right) d\check{t},\ l = 1,2,..,n,$$

$v = 1,2,\dots,k$. Integrating on $[\check{t}\ ,\ \check{t} + \breve{\sigma}]$ Yields

$$\varphi'(\check{t}+\breve{\sigma}) - \varphi'(\check{t}) \geq (n\beta b - k)\breve{\sigma}^2, \qquad \varphi'(\check{t}) \leq -(n\beta b - k)\breve{\sigma}^2 . \tag{2.5}$$

By integration

$$\varphi(\check{t}) - \varphi(\check{t}_4) \leq -(\beta nb - k)\breve{\sigma}^2(\check{t} - \check{t}_4), \tag{2.6}$$

As $\check{t} \to \infty$, one can conclude that (2.6) implies that $\lim_{\check{t} \to \infty} \varphi(\check{t}) = -\infty$, from (1.2) one can deduce that $\varphi(\check{t}) \geq -p(\check{t}) * \psi\left(\tau(\check{t})\right) \geq -\psi\left(\tau(\check{t})\right)$, this implies $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$. on the other side since $\varphi(\check{t}) < 0$. It yields $\psi(\check{t}) \leq p(\check{t})\psi\left(\tau(\check{t})\right) \leq \psi\left(\tau(\check{t})\right)$ so ψ is positive decreasing, so it is bounded a contradiction. Then $\varphi(\check{t}) \geq L$.

Case2.1.2.If $\varphi(\check{t}) > 0$, $\lim_{\check{t}\to\infty} \varphi(\check{t}) = L \geq 0$ then $\varphi(\check{t}) \geq L$. from (1.2) it yields $\varphi(\check{t}) \leq \psi(\check{t})$ and $\varphi(\check{t} - \sigma_l) \leq \psi(\check{t} - \sigma_l)$ then it follows from (1.1)

$$\varphi^{(3)}(\check{t}) = -\sum_{l=1}^{n} Q_l\, f\left(\psi(\check{t} - \sigma_l)\right) + \sum_{v=1}^{k} \varrho_v \leq -\beta \sum_{l=1}^{n} Q_l\, \psi\,(\check{t} - \sigma_l) + \sum_{v=1}^{k} \varrho_v$$
$$\leq -\text{ß} \sum_{l=1}^{n} Q_l(\check{t}) * \varphi(\check{t} - \sigma_l) + \sum_{v=1}^{k} \varrho_v(\check{t}) \leq -\beta L \sum_{l=1}^{n} Q_l(\check{t}) + \sum_{v=1}^{k} \varrho_v .$$

Integrating from t to $t + \sigma$

$$\varphi''(\check{t} + \sigma) - \varphi''(\check{t}) \leq -\beta L \int_{\check{t}}^{\check{t}+\sigma} \sum_{l=1}^{n} Q_l(s)\, ds + \int_{\check{t}}^{\check{t}+\sigma} \sum_{v=1}^{k} \varrho_v(s)\, ds$$
$$\varphi''(\check{t}) \geq \beta L \int_{\check{t}}^{\check{t}+\sigma} \sum_{l=1}^{n} Q_l(s)\, ds - \int_{\check{t}}^{\check{t}+\sigma} \sum_{v=1}^{k} \varrho_v(s)\, ds$$
$$\geq \beta L\, n\hat{\sigma} - k\hat{\sigma} = (\beta Ln - k)\hat{\sigma}. \tag{2.7}$$

Now, integrating equation (2.7)

$$\varphi'(\check{t} + \hat{\sigma}) - \varphi'(\check{t}) \geq (\beta Ln - k)\hat{\sigma}^2$$
$$\varphi'(\check{t}) \leq -(\beta Ln - k)\breve{\sigma}^2. \tag{2.8}$$

Integrating (2.8) on $[\,\breve{t}_4 , \check{t}\,]$

$$\varphi(\check{t}) - \varphi(\check{t}_4) \leq -(\beta Ln - k)\breve{\sigma}^2(\check{t} - \check{t}_4). \tag{2.9}$$

This implies that $\lim_{\check{t}\to\infty} \psi(\check{t}) = -\infty$, a contradiction.

Case 2.2 $\varphi'(\check{t}) > 0, \check{t} \geq \check{t}_3$, in this case $\varphi(\check{t}) > 0$, $\lim_{\check{t}\to\infty} \varphi(\check{t}) = -\infty$, but $\psi(\check{t}) > \varphi(\check{t})$ then $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$.

Theorem 2. Suppose that $Q_l(\check{t}) \leq 0, \sum_{\upsilon=1}^{k} \varrho_\upsilon(\check{t}) \geq 0, 1 < p_0 \leq p(\check{t}) \leq k,\ \tau(\check{t}) \geq \check{t}$ and

$$n\breve{\sigma} > 1,\ \ \breve{\sigma} = \min_{l} \int_{0}^{\sigma_l} |Q_l(\check{t})|\, d\check{t}\, , l = 1, \ldots, n. \tag{2.10}$$

Hence, $(\check{t})$) either oscillates or approaches to zero or $|\psi(\check{t})| \to \infty$ as $\check{t} \to \infty$.

Proof. Suppose that the solution to the equation (1.1) non-oscillatory , let $\psi(\check{t}) > 0,\ \psi\left(\tau(\check{t})\right) > 0,\ \psi(\check{t} - \sigma_l) > 0, l = 1, \ldots, n, \check{t} \geq \check{t}_0$ then

$$\varphi'''(\check{t}) = \sum_{l=1}^{n} |Q_l| * f(\psi(\check{t} - \sigma_l)) + \sum_{1}^{m} \varrho_\upsilon \geq 0, \tag{2.11}$$

Hence φ'' is nondecreasing function, so either $\varphi''(\check{t}) > 0$ or < 0 , $\breve{t} \geq \check{t}_1 \geq \check{t}_0$.

Case (1). If $\varphi''(\check{t}) > 0, \check{t} \geq \check{t}_1$, it follows $\varphi'(\check{t}) > 0, \varphi(\check{t}) > 0$ and $\lim_{\check{t}\to\infty} \varphi(\check{t}) = \infty$. from (1.2) we get $\varphi(\check{t}) \leq \psi(\check{t})$ which implies $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$. on the other side, one can obtain $\psi(\check{t}) \geq p(\check{t})\, \psi\left(\tau(\check{t})\right) > \psi\left(\tau(\check{t})\right)$, thus $\psi(\check{t})$ is decreasing and positive so it is bounded, leading to a contradiction, so, case 1 cannot happen.

Case (2) $\varphi'(\check{t}) < 0$, $\check{t} \geq \check{t}_1$ so either $\varphi'(\check{t}) > 0$ or $\varphi'(\check{t}) < 0$, $\check{t} \geq \breve{t}_2 \geq \check{t}_1$, thus there are two possible subcases to consider:

Case 2.1 let $\varphi'^{(\check{t})} > 0$, $\check{t} \geq \check{t}_2$. there is $\check{t}_3 \geq \check{t}_2$, either $\varphi(\check{t}) < 0$ or > 0 for$\check{t} \geq \check{t}_3$, thus there are two possible subcases to consider:

Case 2.1.1 $\varphi(\check{t}) < 0, \check{t} \geq \check{t}_3$ let $\lim_{t\to\infty} \varphi(\check{t}) = L \leq 0$, we suppose that $L = 0$, otherwise $\varphi(\check{t}) \leq L < 0$, then (2.3) leads to

$\varphi(\check{t}) \geq -p(\check{t})\psi\big(\tau(\check{t})\big) \geq -k\psi\big(\tau(\check{t})\big)$, that is $-\psi(\check{t}) \leq \frac{1}{k}\varphi\big(\tau^{-1}(\check{t})\big)$,

$$\varphi(\check{t} - \sigma_l) \geq -\frac{1}{k}\psi(\tau^{-1}(\check{t} - \sigma_l)) , \quad l = 1,2, \dots , n. \qquad (2.12)$$

Hence, eq. (1.1) reduced to

$$\varphi'''(\check{t}) = \sum_{l=1}^{n} |Q_l(\check{t})|\, f(\psi(\check{t} - \sigma_l)) + \sum_{v=1}^{m} \varrho_v(\check{t}) \geq \beta \sum_{l=1}^{n} |Q_l(\check{t})| * \psi(\check{t} - \sigma_l) + \sum_{v=1}^{m} \varrho_v(\check{t})$$

$$\geq -\frac{\beta}{k}\sum_{l=1}^{n} |Q_l(\check{t})| \varphi\big(\tau^{-1}(\check{t} - \sigma_l)\big) + \sum_{v=1}^{m} \varrho_v(\check{t}),$$

$$\varphi'''(\check{t}) \geq -\frac{L\beta}{k}\sum_{l=1}^{n} |Q_l(\check{t})| + \sum_{v=1}^{m} \varrho_v(\check{t}) , \qquad (2.13)$$

Integrating (2.13) over $[\ t, \check{t} + \sigma_l]$

$$\varphi''(\check{t} + \sigma_l) - \varphi''(\check{t}) \geq \int_{\check{t}}^{t+\sigma} \Big(-\frac{L\beta}{k}\sum_{i=1}^{n} |Q_l(s)| + \sum_{e=1}^{m} \varrho_v(\check{t})\Big)\, ds,$$

$$\varphi''(\check{t}) \leq \Big(\frac{L\beta n}{k} - m\Big)\breve{\sigma}, \text{ where } \breve{\sigma} = \min_{l}\Bigg(\int_{0}^{\sigma_l} (|Q_l(\check{t})| + \varrho_v(\check{t})\Bigg)\, d\check{t} \,, l = 1,2, \dots , n, v = 1,2, \cdots , m. \qquad (2.14)$$

Integrating (2.14) yields

$$\varphi'(\check{t} + \hat{\sigma}) - \varphi'(\check{t}) \leq \frac{Ln\hat{\sigma}^2}{k},$$
$$\varphi'(\check{t}) \geq -\Big(\frac{L\beta n}{k} - m\Big)\breve{\sigma}^2 .$$

Integrating yield

$\varphi(\,\breve{t}\,) - \varphi(\,\check{t}_3\,) \geq -\Big(\frac{L\beta n}{k} - m\Big)\breve{\sigma}^2(\check{t}\_\check{t}_3\,), \check{t} \geq \check{t}_3.$ (2.15)

Letting $\check{t} \to \infty$, one can conclude that (2.15) implies $\lim_{\check{t}\to\infty} \varphi(\check{t}) = \infty$, a contradiction. Hence $L = 0$, let $\limsup_{\check{t}\to\infty} \psi(\check{t}) = l \geq 0$, from (1.2) we get

$p_0\psi\big(\tau(\check{t})\big) \leq p(\check{t})\psi\big(\tau(\check{t})\big) = \psi(\check{t}) - \varphi(\check{t})$, Then $\Psi\big(\tau(\check{t})\big) \leq \frac{\psi(\check{t})-\varphi(\check{t})}{p_0}$, so

$$\Psi(\check{t}) \leq \frac{\psi(\tau^{-1}(\check{t})) - \varphi\,(\tau^{-1}(\check{t}))}{p_0} . \qquad (2.16)$$

Letting $\check{t} \to \infty$, we obtain $L \leq \frac{l}{p_0}$, which is possible only when$L = 0$.

Case 2.1.2. $\varphi(\check{t})$, $\varphi'(\check{t}) > 0$, $\check{t} \geq \check{t}_3$, there exist $b > 0$, such that $\varphi(\check{t}) \geq b$, from (1.2) we obtain $\varphi(\check{t}) \leq \psi(\check{t})$, $\check{t} \geq \check{t}_4 \geq \check{t}_3$, then eq. (2.12) reduce to

$$\varphi'''(\check{t}) \geq \sum_{l=1}^{n} |Q_l(\check{t})|\, \varphi(\check{t} - \sigma_l) + \sum_{v=1}^{m} \varrho_v(\check{t}) \geq b\sum_{l=1}^{n} |Q_l(\check{t})| + \sum_{v=1}^{m} \varrho_v(\check{t}). \qquad (2.17)$$

Integrating (2.17) from $\check{t}$ to $\check{t} + \sigma$ to get

$\varphi''(\check{t} + \sigma_l) - \varphi''(\check{t}) \geq \int_{\check{t}}^{\check{t}+\sigma_l}(b\sum_{l=1}^{n}|Q_l(s)|)++\sum_{v=1}^{m} \varrho_v(\check{t}))$ ds

$$\varphi''(\check{t}) \leq -(bn + m)\breve{\sigma}. \qquad (2.18)$$

Integrating (2.18) between $\check{t}$ and $\check{t} + \breve{\sigma}$

$$\varphi'(\check{t} + \breve{\sigma}) - \varphi'(\check{t}) \leq -(bn + m)\breve{\sigma}^2,$$
$$\varphi'(\check{t}) \geq (bn + m)\breve{\sigma}^2 .$$

Integrating from $\check{t}_4$ to $\check{t}$ to get

$$\varphi(\check{t}) - \varphi(\check{t}_4) \geq (bn + m)\breve{\sigma}^2(\check{t} - \check{t}_4)$$

As $\check{t} \to \infty$ one can get $\lim_{\check{t}\to\infty} \varphi(\check{t}) = \infty$. since $\varphi(\check{t}) \leq \psi(\check{t})$, $\check{t} \geq \check{t}_4$ it can be easily conclude that $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$.

Case 2 .2 *If* $\varphi'(\check{t}) < 0$, $\check{t} \geq \check{t}_3$, then $\varphi(\check{t}) < 0$ and $\lim_{\check{t}\to\infty}\varphi(\check{t}) = -\infty$, which entails that $\lim_{t\to\infty} \psi(\check{t}) = \infty$.

Theorem 3. Assume that $Q_l(\check{t}) \geq 0, \sum_{v=1}^{k} \varrho_v \leq 0, \quad 1 < p_0 \leq p(\check{t}) \leq k, \tau(\check{t}) \leq \check{t}$ and

$$\prod_{l=0}^{\infty} p(\tau^{-l}(\check{t})) < \infty, \ \check{t} \geq \check{t}_0. \tag{2.19}$$

Then, each solution to (1.1) is either oscillatory or close to zero as $\check{t} \to \infty$.

proof. Let φ be a non-oscillating solution to eq. (1.1), and let $\varphi(\check{t})$, $\varphi(\tau(\check{t}))$, $\varphi(\check{t} - \sigma_l) > 0$ then

$$\varphi'''(\check{t}) = -\sum_{l=1}^{n} Q_l(\check{t}) f(\psi(\check{t} - \sigma_l)) + \sum_{v=1}^{m} \varrho_v(\check{t}) \leq 0, \tag{2.20}$$

Hence $\varphi''(\check{t})$ is non-increasing function, so either $\varphi''^{(\check{t})}$ greater or less 0 for$\check{t} \geq \check{t}_1 \geq \check{t}_0$.

Case (1). If $\varphi''(\check{t}) < 0, \check{t} \geq \check{t}_1$, hence $\varphi'(\check{t}) < 0$, $\varphi(\check{t}) < 0$ and $\lim_{\check{t}\to\infty} \varphi(\check{t}) = -\infty$, then there is $\check{t}_2 \geq \check{t}_1$ and δ greater than 0where $\varphi(\check{t}) \leq -\delta, \check{t} \geq \check{t}_2$. from (1.2)

$$\psi(\check{t}_2) = \varphi(\check{t}_2) + p(\check{t}_2)\psi(\tau(\check{t}_2)) \leq -\delta + p(\check{t}_2)\psi(\tau(\check{t}_2)$$

$$\psi(\tau^{-1}(\check{t}_2)) \leq -\delta + p(\tau^{-1}(\check{t}_2))\varphi(\check{t}_2) \leq -\delta + p(\tau^{-1}(\check{t}_2))[-\delta + p(\check{t}_2)\varphi(\tau(\check{t}_2))]$$

$$\psi(\tau^{-1}(\check{t}_2)) \leq -\delta(1 + p(\tau^{-1}(\check{t}_2)) + p(\tau^{-1}(\check{t}_2))p(\check{t}_2)\psi(\tau(\check{t}_2))$$

$$\psi(\tau^{-2}(\check{t}_2)) \leq -\delta[1 + p(\tau^{-2}(\check{t}_2)) + p(\tau^{-2}(\check{t}_2))] + p(\tau^{-2}(\check{t}_2))p(\tau^{-1}(\check{t}_2))p(\check{t}_2)\psi(\tau(\check{t}_2))$$

$$\psi(\tau^{-m}(\check{t}_2)) \leq -\delta(1 + \sum_{l=0}^{m-1} \prod_{v=l}^{m-1} p(\tau^{v-m}(\check{t}_2)) + \psi(\tau(\check{t}_2)) \prod_{l=0}^{m} p(\tau^{-i}(\check{t}_2)) \tag{2.21}$$

As $m \to \infty$ by using (2.11) it follows, $\lim_{m\to\infty} \psi(\tau^{-m}(\check{t}_2)) = -\infty$, this is regarded as a contradiction.

Case (2). If $\varphi''(\check{t}) > 0, \check{t} \geq \check{t}_1$, then $\exists\ \check{t}_2 \geq \check{t}_1$ for which either $\varphi' < 0$ or > 0, $\check{t} \geq \check{t}_2$.

Case 2.1 $\varphi'(\check{t}) < 0, \check{t} \geq \check{t}_2$, there is $\check{t}_3 \geq \check{t}_2$ s. t either $\varphi(\check{t}) > 0$ or $\varphi(\check{t}) < 0$ for$\check{t} \geq \check{t}_3$ thus there are two possible subcases to consider:

Case 2.1.1 $\varphi(\check{t}) > 0, \check{t} \geq \check{t}_3$, then (1.2) leads to $\psi(\check{t}) \geq p(\check{t})\psi(\tau(\check{t})) > \psi(\tau(\check{t}))$, $\check{t} \geq \check{t}_2$, which means that $\psi(t)$ is increasing. Let $\lim_{t\to\infty} \varphi(\check{t}) = L \geq 0$.

1. If $L > 0$, hence $\varphi(\check{t}) \geq L > 0$, from (1.2) it follows $\psi(\check{t}) \geq \varphi(\check{t})$, then eq. (1.1) reduce to

$$\varphi'''(\check{t}) = -\sum_{l=1}^{n} Q_l(\check{t}) f(\psi(\check{t} - \sigma_l)) + \sum_{v=1}^{k} \varrho_v$$

$$\leq -\lambda \sum_{l=1}^{n} Q_l(\check{t})\, \psi(\check{t} - \sigma_l), \tag{2.22}$$

$$\varphi'''(\check{t}) \leq -L\lambda \sum_{l=1}^{n} Q_l \qquad \check{t} \geq \check{t}_3 \geq \check{t}_2\,. \tag{2.23}$$

Integrating (2.23) from $\check{t}$ to $\check{t} + \sigma$ to get

$$\varphi''(\check{t}) \geq L\lambda \sum_{l=1}^{n} \int_0^{\sigma} Q_l(s)\ ds \ \geq \ Ln\lambda\hat{\sigma}. \tag{2.24}$$

Integrating (2.24)

$$\varphi'(\check{t} + \hat{\sigma}) - \varphi'(\check{t}) \geq L\lambda n\hat{\sigma} \int_{\check{t}}^{\check{t}+\hat{\sigma}} ds$$

$$\varphi'(t) \leq -L\lambda n\hat{\sigma}^2. \tag{2.25 }$$

Integrating (2.25)

$$\varphi(\check{t}) - \varphi(\check{t}_4) \leq -L\lambda n\hat{\sigma}^2(\check{t} - \check{t}_4).$$

This implies that $\lim_{t\to\infty} \varphi(\check{t}) = -\infty$, a contradiction.

2. If $L = 0$ that is $\lim_{\check{t}\to\infty} \varphi(\check{t}) = 0$, by (1 . 2) yield

$$\psi(\tau(\check{t})) \leq \frac{\psi(\check{t}) - \varphi(\check{t})}{p_0}, \check{t} \geq \check{t}_3. \tag{2.26}$$

Let $\lim_{\check{t}\to\infty} \psi(\check{t}) = L \in (0, \infty)$, by letting $\check{t} \to \infty$ in (2. 26), one can conclude that $L \leq$, Which is possible only when $L = 0$, and that is impossible since $\psi(\check{t})$ is increasing.

Case 2.1.2If $\varphi(\check{t}) < 0, \check{t} \geq \check{t}_3$ then $\check{t}_4 \geq \check{t}_2$, $k > 0$ such that

$$\varphi(\check{t}) \leq -k, \ \check{t} \geq \check{t}_4 \text{ , from (1.2)}$$

$$\varphi(\check{t}) \geq -p\,(\check{t})\,\psi(\tau(\check{t})) \geq -k\psi(\tau(\check{t})), \text{ That is } -\psi(\check{t}) \leq \frac{1}{k}\ (\ \varphi(\tau^{-1}(\check{t})))$$

$$\_\,\psi(\check{t} - \sigma_l) \leq \frac{\varphi(\tau^{-1}(\check{t} - \sigma_l))}{k}, \ \ \check{t} \geq \check{t}_2, \qquad l = 1,2, \dots, n.$$

Then eq. (1.1) will be

$$\varphi'''(\check{t}) = \sum_{l=1}^{n} -Q_i(\check{t}) * f(\psi(\check{t}-\sigma_l)) \le -\lambda \sum_{l=1}^{n} Q_l\ \psi(\check{t}-\sigma_l) \le \lambda \sum_{l=1}^{n} Q_l(\check{t})\frac{\varphi\left(\tau^{-1}(\check{t}-\sigma_l)\right)}{k}$$

$$\varphi'''(\check{t}) \le -\frac{\lambda}{k}\sum_{l=1}^{n} Q_l(\check{t})\,, \check{t} \ge \check{t}_2. \qquad (2.27)$$

Integrating (2.27) from $\check{t}$ to $\check{t}+\sigma$

$$\varphi''(\check{t}) \ge \frac{\lambda}{k}\sum_{l=1}^{n}\int_0^{\sigma} Q_l(s)ds.$$

$$\varphi''(\check{t}) \ge \frac{n\lambda\hat{\sigma}}{k}.$$

Integrating the last inequality yields

$$\varphi'(\check{t}+\hat{\sigma}) - \varphi'(\check{t}) \ge \frac{n\lambda\hat{\sigma}^2}{k},$$

$$\varphi'(\check{t}) \le -\frac{n\lambda\hat{\sigma}^2}{k}. \qquad (2.28)$$

Integrating (2.28) from $\check{t}_4$ to $\check{t}$

$$\varphi(\check{t}) - \varphi(\check{t}_4) \le -\frac{n\lambda\hat{\sigma}^2}{k}\,(\check{t}-\check{t}_4)$$

As $\check{t} \to \infty$ one can get that $\lim_{\check{t}\to\infty} \varphi(\check{t}) = -\infty$, a contradiction.

Case(2. 2) $\varphi'(\check{t}) > 0$, $\check{t}_3 \le t$, in that case $\varphi(\check{t}) > 0$, limit $\varphi(\check{t}) = \infty\ as \check{t} \to \infty$, but $\psi(\check{t}) > \varphi(\check{t})$, hence $\lim_{t\to\infty} \psi(\check{t}) = \infty$.

Theorem 4. Suppose that $Q_l(\check{t}) \le 0, \sum_{v=1}^{k} \varrho_v \ge 0, 0 < p(\check{t}) \le p_0 < 1,\ \tau(\check{t}) \le \check{t}$ and

$$\limsup_{\check{t}\to\infty} \sum_{l=0}^{m} p\left(\tau^{-l}(\check{t})\right) = \infty, \check{t} \ge \check{t}_0. \qquad (2.29)$$

Every solution $\psi(\check{t})$ of eq. (1.1) either $\lim_{\check{t}\to\infty}|\psi(\check{t})| = \infty$ or oscillates .

Proof. Let $\psi(\check{t})$ be a solution that is not oscillatory of eq. (1.1), let $\psi(\check{t})$, $\psi\left(\tau(\check{t})\right) > 0$, $\psi(\check{t}-\sigma_l) > 0, \check{t} \ge \check{t}_0, l = 1, \dots, n.$ then eq. (1.1) leads to

$$\varphi'''(\check{t}) = \sum_{l=1}^{n} |Q_l(\check{t})| * \mathrm{f}(\psi(\check{t}-\sigma_l)) + \sum_{v=1}^{k} \varrho_v \ \ge 0.$$

Hence $\varphi'''(\check{t}) \ge \lambda \sum_{l=1}^{n} |Q_l\,|\psi(\check{t}-\sigma_l)$ and $\varphi''(\check{t})$ are not decreasing, so either
$\varphi''(\check{t}) > 0$ or < 0 For $\check{t} \ge \check{t}_1 \ge \check{t}_0$.

Case 1. If $\varphi''(\check{t}) > 0$, then $\varphi'(\check{t}) > 0, \varphi(\check{t}) > 0$ and $\lim_{\check{t}\to\infty} \varphi(\check{t}) = \infty$ then these is $\check{t}_2 \ge \check{t}$ and $\gamma > 0$,

$\varphi \ge \gamma\,, \check{t} \ge \check{t}_2$, from (1, 2), we get

$$p(\check{t}_2)\psi\left(\tau(\check{t}_2)\right) = \psi(\check{t}_2) - \varphi(\check{t}_2),$$

$$\psi\left(\tau(\check{t}_2)\right) = \frac{1}{p(\check{t}_2)}\psi(\check{t}_2) - \varphi(\check{t}_2),$$

$$= \frac{1}{p(\check{t}_2)}\psi(\check{t}_2) - \frac{1}{p(\check{t}_2)}\varphi(\check{t}_2) \le \frac{1}{p(\check{t}_2)}\psi(\check{t}_2) - \gamma. \qquad (2.30)$$

Then

$$\psi(\check{t}_2) \le \frac{1}{p(\tau^{-1}(\check{t}_2))}\psi(\tau^{-1}(\check{t}_2)) - \gamma.$$

Hence using (2.29)

$$\psi\big(\tau(\check{t}_2)\big) \leq [\frac{1}{p(\tau^{-1}(\check{t}_2))}\psi(\tau^{-1}(\check{t}_2)) - \gamma]\frac{1}{p(\check{t}_2)} - \gamma,$$

$$\leq \frac{1}{p(\tau^{-1}(\check{t}_2))}\psi(\tau^{-1}(\check{t}_2))\frac{1}{p(\check{t}_2)} - 2\gamma.$$

Again

$$\psi\left(\tau\big(\tau(\check{t}_2)\big)\right) \leq \frac{1}{p(\tau(\check{t}_2))p(\check{t}_2)}\psi(\check{t}_2) - 2\gamma,$$

$$\leq \frac{1}{p(\check{t}_2)p(\tau(\check{t}_2))}[\frac{1}{p(\tau^{-1}(\check{t}_{2}))}\psi(\tau^{-1}(\check{t}_2)) - \gamma] - 2\gamma,$$

$$\psi\big(\tau^2(\check{t}_2)\big) \leq \frac{1}{p(\check{t}_2)p(\tau(\check{t}_2))p(\tau^{-1}(\check{t}_2))}\psi(\tau^{-1}(\check{t}_2)) - 3\gamma.$$

By repeating the above procedure m time we obtain

$$\psi\big(\tau^m(\check{t}_2)\big) \leq \prod_{l=0}^{m}\frac{1}{p(\tau^{l-1}(\check{t}_2))}\psi(\tau^{-1}(\check{t}_2)) - (m+1)\gamma. \tag{2.31}$$

Taking into account condition (2.30), by letting $m \to \infty$, one can conclude that (2.31) implies that $\limsup_{\check{t}\to\infty}\psi(\check{t}) < \infty$. On the other side, $\lim_{\check{t}\to\infty} \varphi(\check{t}) = \infty$ implies $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$, which gives a contradiction.

Case 2 If $\varphi''(\check{t}) < 0, \check{t} \geq \check{t}_1$, then these is$\check{t}_2 \geq t_1$, either $\varphi'(\check{t})$exceed or $< 0, \check{t} \geq \check{t}_2$.

Case 2.1 If $\varphi'(\check{t}) > 0$, $\check{t} \geq \check{t}_2$then either $\varphi(\check{t}) > 0$ or $\varphi(\check{t}) < 0$ for$\check{t} \geq \check{t}_3 \geq \check{t}_2$.

Case 2.1.1$If\ \varphi(\check{t}) > 0,\ \check{t} \geq \check{t}_3$; thus,$\exists\ b > 0$ s. t $\varphi(\check{t}) \geq b$, from (1, 2) it follows,$\psi(\check{t}) \geq \varphi(\check{t})$ Then $\psi(\check{t} - \sigma_l) \geq \varphi(\check{t} - \sigma_l), l = 1, \ldots \ldots n$

Therefore, eq. (1.1) reduces to

$$\varphi'''(\check{t}) \geq \lambda \sum_{l=1}^{n} |Q_l(\check{t})|\, \psi(\check{t} - \sigma_l) \geq \lambda \sum_{l=1}^{n} |Q_l(\check{t}_1)|\, \psi(\check{t} - \sigma_l),$$

$$\geq \lambda b \sum_{l=1}^{n} |Q_l(\check{t})|. \tag{2.32}$$

Integrating (2.32) from $\check{t}$ to $\check{t} - \sigma_l$ yields

$$\varphi''(\check{t} + \sigma_l) - \varphi''(\check{t}) \geq \lambda b \int_{\check{t}}^{\check{t}+\sigma_l} \sum_{v=1}^{n} |Q_v(s)|\, ds,$$

$$\varphi''(\check{t}) \leq -\lambda b \int_{t}^{t+\sigma_l} \sum_{v=1}^{n} |Q_v(s)|\, ds \leq -n\lambda b\check{\sigma}.$$

Integrating from $\check{t}_3$ to $\check{t}$ thus, as $\check{t} \to \infty$ it follows that $\lim \varphi'(\check{t}) = -\infty$ a contradiction.

Case 2.1.2 Let $\varphi(\check{t}) < 0, \check{t} \geq \check{t}_{3,}$ then by (1.2) $\psi\ < p(\check{t})\psi(\tau(\check{t}_1)) < \psi(\tau(\check{t}))$
That is $\psi(\check{t}_1)$is increasing function.

Let $\lim_{\check{t}\to\infty} \varphi(\check{t}) = L \leq 0$:

1- If $L < 0, \quad \varphi(\check{t}) \leq L < 0,$ from (1.2) we get

$\varphi(\check{t}) \geq -p(\check{t})\psi\big(\tau(\check{t})\big) > -\psi\big(\tau(\check{t})\big),$

$\psi(\check{t}) \geq -\varphi(\tau^{-1}(\check{t})),$

$\psi(\check{t}-\sigma_l) \geq -\varphi(\tau^{-1}(\check{t}-\sigma_l))$

Substituting in eq. (1.1) lead to

$$\varphi'''(\check{t}) \geq \lambda \sum_{l=1}^{n} |Q_l(\check{t})|\, \psi(\check{t}-\sigma_l) \geq \lambda \sum_{l=1}^{n} |Q_l(\check{t})| \varphi(\tau^{-1}(\check{t}-\sigma_l)) \geq -L\lambda \sum_{l=1}^{n} |Q_l(\check{t})|, \check{t} \geq \check{t}_3, \quad l = 1, 2 \ldots n. \qquad (2.33)$$

Integrating (2.33) from $\check{t}$ to $\check{t}+\sigma_l$

$$\varphi''(\check{t}+\sigma_l) - \varphi''(\check{t}) \geq -L\lambda \int_{\check{t}}^{\check{t}+\sigma_l} \sum_{v=1}^{n} |Q_v(s)|\, ds,$$

$$\varphi''(\check{t}) \leq L\lambda \sum_{l=1}^{n} \int_{\check{t}}^{\check{t}+\sigma_l} |Q_l(s)|\, ds \leq n\lambda L\breve{\sigma}. \qquad (2.34)$$

Integrating (2.34) from $\check{t}$ to $\check{t}+\breve{\sigma}$

$$\varphi'(\check{t}+\breve{\sigma}) - \varphi'(\check{t}) \leq nL\lambda\, \breve{\sigma}^2,$$
$$\varphi'(\check{t}) \geq -nL\lambda\, \breve{\sigma}^2.$$

Using integration

$$\varphi(\check{t}) - \varphi(\check{t}_3) \geq -nL\lambda\, \breve{\sigma}^2(\check{t}-\check{t}_3).$$

Hence, $\lim_{t\to\infty} \varphi(\check{t}) = \infty$ a contradiction.

2- If $L = 0$, that is $\lim_{\check{t}\to\infty} \varphi(\check{t}) = 0$, from (1.2)

$$\psi(\tau(\check{t})) = \frac{\psi(\check{t}) - \varphi(\check{t})}{p(\check{t})} \geq \frac{\psi(\check{t}) - \varphi(\check{t})}{p_0} \qquad (2.35)$$

Let $\lim_{t\to\infty} \psi(\check{t}) = L \geq 0$.

Letting $\check{t} \to \infty$, inequality, (2.35) lead to $L \geq \frac{l}{p_o}$ then $L(p_0 - 1) \geq 0$.

This is possible only when $L = 0$, however, this is impossible since $\psi(\check{t})$ is increasing and positive.

Case 2.2 $\varphi'(\check{t}) < 0, \check{t} \geq \check{t}_3$. This intends that $\varphi(\check{t}) < 0$ and $\lim_{t\to\infty} \varphi(\check{t}) = -\infty$, hence $\lim_{\check{t}\to\infty} \psi(\check{t}) = \infty$.

3. Ȩxamples’

This part provides examples to exhibit the results obtained
Example 1. Look at the equation with periodic coefficients

$$\left[\varphi(t) - \left(1 - \frac{1}{4}\sin^2 2t - \cos^2 2t\right)\varphi(t-\pi)\right]''' + 118\sin^2 2t\, \varphi\left(t - \frac{\pi}{4}\right) + 44\cos^2 t\, \varphi\left(t - \frac{3\pi}{4}\right) = 0 \qquad (3.1)$$

Note that $p(t) = 1 - \frac{1}{4}\sin^2 2t - \cos^2 2t$, then, $0 \leq p(t) < \frac{4}{5}$, $Q_1(t) = 118\sin^2 2t$, $Q_2(t) = 44\cos^2 2t$.
Hence $Q_1, Q_2 > 0$, $n = 2$, $\varrho_l(t) \equiv 0$, $\tau(t) = t - \pi$, $\sigma_1 = \frac{\pi}{4}, \sigma_2 = \frac{3\pi}{4}$.

$$\int_0^{\frac{\pi}{4}} Q_1(t)dt = 118\int_0^{\frac{\pi}{4}} \sin^2 2t dt = \frac{59\pi}{4}, \quad \int_0^{\frac{\pi}{4}} Q_2(t)dt = 44\int_0^{\frac{\pi}{4}} \cos^2 2t\, dt = \frac{11\pi}{2}$$
$$\hat{\sigma} = \min\{\frac{59\pi}{4}, \frac{11\pi}{2}\}, \quad n\hat{\sigma} = 11\pi.$$

So all conditions of theorem1 hold, hence, according to theorem1, the solutions of Eq. (3.1) are oscillates, see in Figure 1, $\varphi(t) = \sin 2t$, is

such an oscillatory solution.

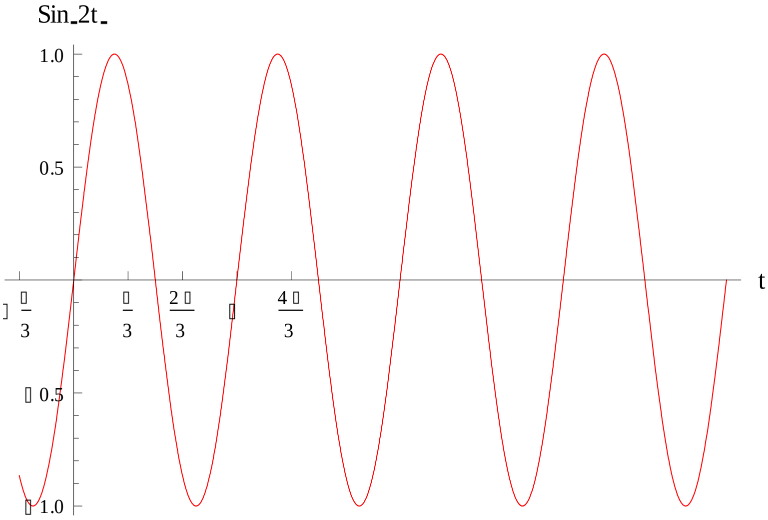

Figure1: The oscillatory solution $\varphi(t)$=sin 2t

Example 2. Consider the equation with periodic coefficients

$$[\varphi\,(t)-(2+\sin t+\sin^2\mathrm{t}\,)*\varphi\,(t-2\pi)]'''-6\cos^2\mathrm{t}\,\varphi\left(\mathrm{t}-\frac{\pi}{2}\right)-21\sin^2 t\,\varphi\left(t-\frac{3\pi}{2}\right)=6\cos^2 t+21\sin^2 t. \tag{3.2}$$

Hence $p\,(t)=\sin^2 t+\sin t+\ 2$then $1<\frac{7}{4}\le p(t)\le 4$
$Q_1(t)=6\cos^2 t\,,\ Q_2(t)=21\,\sin^2 t,$. Hence $Q_1,Q_2\le 0,\ n=2,\ \varrho_1\,(t)=6\cos^2 t\,,\ \varrho_2\,(t)=21\sin^2 t,\ \varrho_1\,,\varrho_2\ge 0,\ \tau(t)=t-2\pi,\ \sigma_1=\frac{\pi}{2},\sigma_2=\frac{3\pi}{2}$.

$$\int_0^{\frac{\pi}{2}}Q_1(t)dt=6\int_0^{\frac{\pi}{2}}\cos^2 t dt=\frac{3\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}}Q_2(t)dt=21\int_0^{\frac{\pi}{2}}\sin^2 t dt=\frac{21\pi}{4}$$

$$\hat{\sigma}=\min\{\frac{3\pi}{2},\frac{21\pi}{4}\},\ \ n\hat{\sigma}=3\pi\,.$$

So all conditions at theorem2 hold, thus according to theorem2, all solutions of the equation (3.2) are oscillatory, in figure 2, the solution $\varphi(t)=\sin t-1$, is such an oscillatory solution.

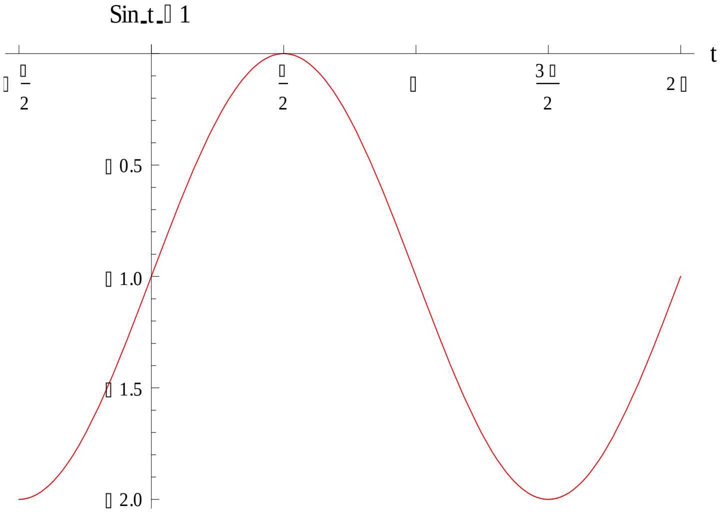

Figure2: The oscillatory solution $\varphi(t)$=sin t-1

Example 3. Look at the equation with periodic coefficients

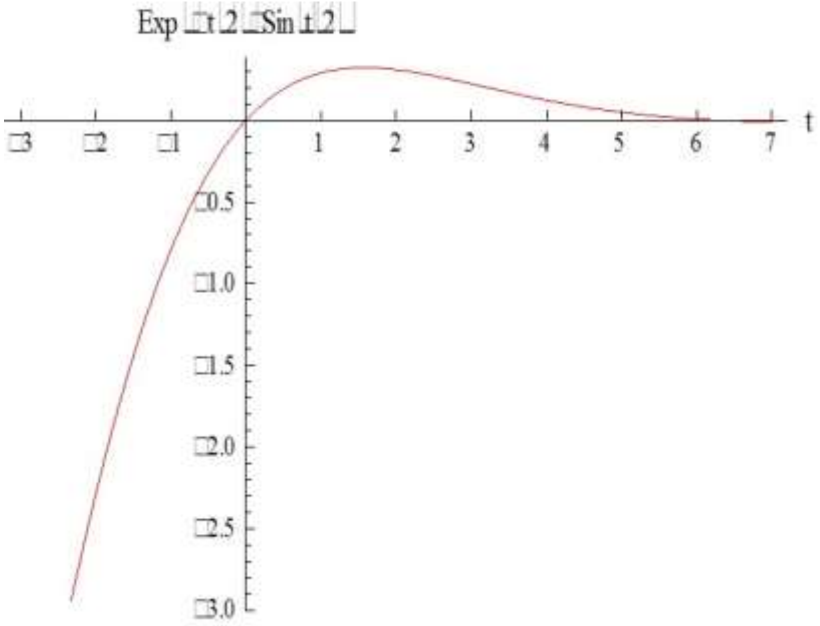

Figure3: The oscillatory solution $\varphi(t)$ = $e^{-t/2}$ sin(t/2)

$$[\varphi(t) - e^{-\pi}\varphi(t-2\pi)]''' + \frac{1}{2}e^{-\frac{\pi}{2}}\varphi(t-\pi) + \frac{1}{2}e^{-\pi}\varphi(t-2\pi) = 0 \qquad (3.3)$$

Note that $p(t) = e^{-\pi}$, then, $0 \le p(t) < \frac{1}{8}$ $Q_1 = \frac{1}{2}e^{-\frac{\pi}{2}}$, $Q_2 = \frac{1}{2}e^{-\pi}$. Hence $Q_1, Q_2 > 0$, $n = 2$, $\varrho_l(t) \equiv 0, l = 1,2$, $\tau(t) = t - 2\pi$, $\sigma_1 = \pi, \sigma_2 = 2\pi$.

$$\prod_{l=0}^{\infty} p(\tau^{-l}(\check{t})) < \infty,$$

So all conditions of theorem 3 hold, hence, according to theorem 3, the solutions of Eq. (3.3) are oscillates, see in Figure 3, $\varphi(t) = e^{-\frac{t}{2}}\sin\frac{t}{2}$, is such an oscillatory solution and converge to zero.

4. Conclusions

This study has derived appropriate criteria to ensure the oscillation for each solution for neutral equations with periodic coefficients. These criteria have demonstrated the extent to which periodic coefficients affect the oscillation property, and we believe the resulting conditions are straightforward and easy to use, as demonstrated by the cases mentioned above and the examples presented.

Conflicts of Interest

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