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Methods for Finding Gelfand Pairs in Finite Groups with an Emphasis on Symmetry

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Abstract

This paper seeks to develop a procedure to identify all pairs within a finite group that are Gelfand, symmetric, and weakly symmetric. With the focus on the projective special linear groups. This leads to a unique class of "strictly weakly symmetric pairs." For the specific case of the projective special linear group PSL(2, p) where p is prime in the range $1 < p < 100$, it is observed that only a single such pair exists.

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1. Introduction

Understanding the fundamental laws of nature necessitates exploring new symmetries [1]. Groundbreaking contributions of Cartan, Harish-Chandra, and Gelfand in the last century. This paved the way for linking abstract algebraic structures with the study of symmetry, influencing various academic disciplines. The concepts of symmetry have since been integrated and developed in fields such as AutoMorphic forms, number theory, and mathematical physics. Advances in technology has enabled the integration of computers with complex mathematical theories, opening up new possibilities for more detailed and precise investigations. Symmetric pairs naturally emerged in Élie Cartan's work on homogeneous spaces and Lie groups, later revealing deep connections to Gelfand pairs. In this context, symmetric pairs are defined as the set of fixed points of a Lie group G and its subgroup K under an involution. Selberg expanded this concept by introducing weakly symmetric homogeneous spaces and demonstrated their commutativity. Weakly symmetric pairs extend the concept of symmetric pairs by allowing a broader range of symmetries; in such pairs, the subgroup K may lack fixed points for individual convolutions but remains invariant across multiple convolutions, [2-6]. In functional analysis, a complex-valued function on a set X is a powerful tool, exemplified by the Hilbert space $L^2(X,\mu)$, one of the Lebesgue spaces L^p. This space includes two fundamental operators in quantum mechanics: the position operator $\hat{p}: \alpha \to \beta(r) = i\frac{d}{dr}\alpha(r)$ and the momentum operator $\hat{m}: \alpha \to \beta(r) = r \cdot \alpha(r)$, both of which have been instrumental in studying the structure and motion of objects [7,8]. In the mid-20th century, I. M. Gelfand introduced the concept of a pair, which has since been widely adopted in mathematical research. Specifically, Gelfand considered the scenario where $X = G/H$ represents the set of H- cosets, with G as a finite group and H is a subgroup of G. The convolution product, $f_1 * f_2(x) := \sum_{t\in X} f_1(t)f_2(t^{-1}x)$ transforms the space of all bi- H invariant complex-valued functions $L(H\backslash G/H)$ into an algebra known as convolution algebra. A pair (G,H) is termed a Gelfand pair if this algebra is commutative,[9-11] a property that facilitates the study of harmonics, spherical functions on homogeneous spaces, and group analysis. In character theory, a pair (G,H) is a Gelfand pair if the induced trivial character of H to G, often denoted by $i \uparrow_H^G$, is multiplicity-free, meaning that the inner product $\langle i \uparrow_H^G, \vartheta\rangle \leq 1$ for any irreducible character ϑ of G. If any irreducible character of H replaces the trivial character, the resulting pair is called a strong Gelfand pair. Similarly, G. Anderson and colleagues, using the maximal degree of irreducible characters and the work of C. Godsil and K. Meagher on multiplicity-free subgroups of

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symmetric groups, identified all strong Gelfand pairs within symmetric groups.[12,13]. Similar to this, Andrea and Hymphyries find all the strong Gelfand pairs of $SL(2,p)$ by using the concept of total character and the computed maximal subgroups of $PSL(2,p)$, which are identical to those of $SL(2,p)$.[14, 15]. A pair (G,H) is called a symmetric Gelfand pair if every element $g \in G$ lies within the H-double coset of its inverse, i.e., for each $g \in G, g^{-1} \in HgH$. The notion of weakly symmetric pairs generalizes this. By allowing G elements to belong to the *H*-double coset of an AutoMorphic image of their inverses. While character theory is central to the work of Andrea and Anderson, our approach requires a different perspective, relying on double cosets and AutoMorphisms to define symmetric and weakly symmetric structures. This paper has developed computational methods to identify all Gelfand, symmetric, and weakly symmetric pairs within a given finite group. For the projective special linear group *PSL(2,p)*, we implemented these methods in GAP,[16] focusing on prime values of p ranging from 1 to 100. Furthermore, we investigated the induced irreducible representations of the resulting unique weakly symmetric but non-symmetric subgroup. Our findings indicate that injecting the properties of these representations help distinguish between symmetric and weakly symmetric pairs, leading to the definition of a new variation, termed strictly weakly symmetric pairs.

2. Preliminaries

Assuming G is a finite group and H a subgroup of G. Then group G acts transitively on the set of all left cosets of H in G, denoted by $X = G/H$. The space of all bi-H- invariant functions,

$$H_L(G)_H = \{f \in L(G) : f(h_1 s h_2) = f(s), \quad s \in G, h_1, h_2 \in H\}$$

is isomorphic to the space of all right H-invariant functions,

$$L(X)^H = \{f \in L(X) : \pi(h)f = f . h \in H\},$$

Where π is the permutation representation defined by,

$$\pi : G \to End_G(L(X)),$$

and

$$[\pi(s)f](a) = f(s^{-1}a) \text{for all } s \in G, f \in L(X), \text{and } a \in X$$

The induced representation of the trivial representation of H to G is equivalent to the permutation representation π, that is, $\pi \approx ind_H^G 1_H$. If (G,H) is a Gelfand pair, then as a multiplicity-free G -module, π decomposes into irreducible G -submodules as

$$\pi = \int_{i=1}^{n} V_i,$$

where $n = |H\backslash G/H|$.

Our criteria for defining the pair (G,H) are as follows:

a) Gelfand: The pair (G,H) is Gelfand if, for every irreducible character ϑ of G, we have
$$\langle Ind_H^G(1_H), \vartheta\rangle_G = \langle 1_H, \vartheta \downarrow_H^G\rangle_H \leq 1$$
where 1_H is the trivial character of H.

b) Symmetric: The pair (G,H) is symmetric if, for each $g \in G$, $g^{-1} \in H\backslash g/H$.

c) Weakly symmetric: The pair (G,H) is weakly symmetric if, for each $g \in G$ and some AutoMorphism
$$\alpha \in Aut(G), g^{-1} \in H\backslash\alpha(g)/H$$

Our methods are developed using these notations. Let $G = PSL(2,p), with \ G \geq U = U(2,p) = C_{p-1/2} \rtimes C_p = \langle a,b | a^{p-1/2} = e, b^p = e, b^{-1}ab = a^k \rangle$ where $\frac{(p-1)}{2}$ $divides \ k^m - 1$. Here, U is the semi-direct product of the cyclic groups $C_{p-1/2} = \langle a \rangle$ and $C_p = \langle b \rangle$. Furthermore, if p is odd, H_2 is the unique subgroup of U with index 2.

2.1. Remark

i. The largest prime for which the group *PSL(2,p)* contains non-trivial Gelfand pairs other than *U* and H_2 is $p = 41$, This choice is made to present figures clearly while retaining generality for $p < 100$.
ii. The finiteness of group G guarantees that each algorithm terminates.

3. Results and Discussion

3.1. Gelfand Pairs for Finite Group

Gelfand pairs can be defined in various ways, depending on the area of analysis, whether harmonic or geometric. In harmonic analysis, groups are approached algebraically, whereas geometric analysis requires differential structures like manifolds and Riemann surfaces or topological properties such as compactness and connectedness. In this study, we focus on the case where groups are finite, utilizing computer algebra systems to examine Gelfand pairs and gain deeper insights into their structures. The following conditions are equivalent:[9-11].

- (G,H) is a Gelfand pair.
- The convolution algebra of bi-H-invariant functions on G is commutative.
- $Dim(Hom_K(\rho, i)) \leq 1$, where ρ is any irreducible representation of G and i is the trivial representation.
- $Dim(V_\rho^H) \leq 1$, where V_ρ^H is the space of all H-invariant vectors of any irreducible representation ρ of G,
- The permutation representation of G on the corsets of H is multiplicity-free,

- Condition (a) is satisfied in terms of character theory.

The methods we develop in this and the following sections are based on the last statement.

Algorithm 1: TESTING IF A PAIR OF A GROUP G AND A SUBGROUP H OF G FORM A GELFAND PAIR

Require: Group G, subgroup H of G

Ensure: (G, H) is Gelfand pair

```
t ← trivial character of H                          ▷ Construct the trivial character of H
ind(t) ← induced character of t from H to G         ▷ induced character of t
IrG ← irreducible characters of G                   ▷ Compute the irreducible Characters of G
while x ∈ IrG do
    If ind(t), x) > 1 then                          ▷ condition a
        return false
    end if
end while
return true
```

Proof.

Soundness: Given than x is any irreducible character of G, and $ind(t)$ represent the induced character of H from its trivial character. If the Inner product $(ind(t), x) > 1$, then $ind(t)$ is not multiplicity-free. In this case, iterating through all irreducible characters of G results in a "false" output. However, if $ind(t)$ is multiplicity free, then by definition, (G, H) is a Gelfand pair, and the algorithm returns "true," correctly identifying a Gelfand pair.

Completeness: If *(G, H)* is a Gelfand pair, then the induced character $\boldsymbol{ind(t)}$ is multiplicity-free, meaning that for each irreducible character $\boldsymbol{x}$ of *G*, $(\boldsymbol{ind(t), x}) \leq \boldsymbol{1}$. The algorithm will complete the loop without producing any false results and will return "true".

3.2. Symmetric Pairs for Finite Group

The initial computation of *H*-double cosets satisfying condition (b) serves as the primary foundation for the subsequent procedure to determine symmetric pairs in a finite group.

Algorithm 2: TESTING IF A PAIR OF A GROUP G AND SUBGROUP H OF G IS A SYMMETRIC PAIR.

```
Require: Group G, subgroup H of G
Ensure: (G, H) is symmetric pair
    I ← 0                              ▹ start counter
    dco ← H-double cosets of G         ▹ compute H-double cosets of G
    while g ∈ G do
        while x ∈ dco do
            x ← elements of x
            while i ∈ x do
                If  g−1 ∈ i  then      ▹ condition b
                    return false
                else I ← I + 1
                end if
            end while
        end while
    end while
    if  Size G = I  then
        return true
    end if
```

Proof. Note that *(G, H)* is a symmetric pair if and only if *G* acts transitively on the collection of all disjoint *H*-double corsets.

Soundness: For each $g \in G$, iterate over all H-double cosets. If g^{-1} is found in any coset, return "false," as this implies that *g* is not uniquely associated with an element of the *H*-double corset, indicating that *G*'s action on the *H*-double cosets is not transitive. If g^{-1} is not found in any coset, increment the counter and continue examining the elements. If, after completing the iteration, the counter equals the size of *G*, then *G* acts transitively on the *H*-double cosets, and the algorithm returns "true," confirming that *(G, H)* is symmetric.

Completeness: If *(G, H)* is a symmetric pair, *G* acts transitively on the set of all *H*-double cosets. Thus, every g ∈ *G* is counted uniquely with no duplicates, and g^{-1} will not be accounted for. The final count will match the size of *G*, and the algorithm will correctly return "true." □

3.3. Weakly Symmetric Pairs for Finite Group

Weakly symmetric pairs generalize symmetric pairs and provide additional insights into the degree of symmetry. To determine whether a pair is weakly symmetric, calculate the AutoMorphisms of the parent group and select those that meet condition (c). Due to computational limitations, we restrict our algorithm to *p=100*, which aligns with current computer capabilities.

Algorithm 3: TEST IF THERE EXISTS AN AUTOMORPHISM OF G SUCH THAT THE PAIR (G, H) IS WEAKLY SYMMETRIC

```
Require: Group G, subgroup H of G and an automorphism aut ∈ Aut(G)
Ensure: The triple (G, H, aut) is weakly symmetric pair
  I ← 0                                  ▷ start counter
  AutG ← Automorphism group of G
  while aut ∈ AutG and aut ≠ trivial(AutG) do   ▷ excluding the trivial automorphism which gives symmetric pairs
      while g ∈ G do
          dco ← H-double cosets of aut(g) in G   ▷compute H-double cosets H\t(g)/H
          while x ∈ dco do
              x ← elements of x
              while i ∈ x do
                  If g−1 ∈ i then          ▷ condition c
                      return false
                  else I ← I + 1
                  end if
              end while
          end while
      end while
  end while
  if Size G = I then
      return true
  end if
```

Proof. The pair *(G, H)* is considered weakly symmetric if there exists an AutoMorphism of *G* that maps elements of *H*-double cosets to *H*-double cosets, with the action being transitive under this AutoMorphisms. To complete the proof, an additional loop is introduced to verify each element of *G*'s AutoMorphisms group, using a method similar to Algorithm 2. Note that the identity AutoMorphisms is excluded, as it was already considered in the symmetric case.

3.3.1. Definition. A pair *(G, H)* is defined as strictly weakly symmetric if it is weakly symmetric only under non-trivial automorphisms of *G*. It is evident that a strictly weakly symmetric pair is weakly symmetric, but the reverse is not necessarily true. A symmetric pair is weakly symmetric but not necessarily strictly weakly symmetric.

4. Application

The following subsections detail the implementation of Algorithms 1, 2, and 3 in GAP for the projective special linear groups *PSL(2,p)*, where *p* is a prime number such that $1<p<100$.

4.1. Gelfand Pairs for PSL (2, p)

The Gelfand subgroups of *PSL(2, p)*, where *p* is a prime and $1<p<100$, are:

Table 1: Gelfand Pairs

P	Subgroups
2	C_2, C_3
3	$C_2, C_3, C_2 \times C_2$
5	C_5, S_3, D_{10}, A_4
7	$A_4, C_7 \rtimes C_3, S_4$
$p \equiv 1 mod(4)$	$U, H_2 + p = 17; S_4, p = 29{,}41;\ A_5$
$p \equiv 3 mod(4)$	$U + p = 11{,}19{,}31; A_5$

The results are visualized in Figure 1, which is generated using the GAP package JupyterViz,[17] and based on the data from Table 1 that lists representatives of the conjugacy class subgroups of *G*.

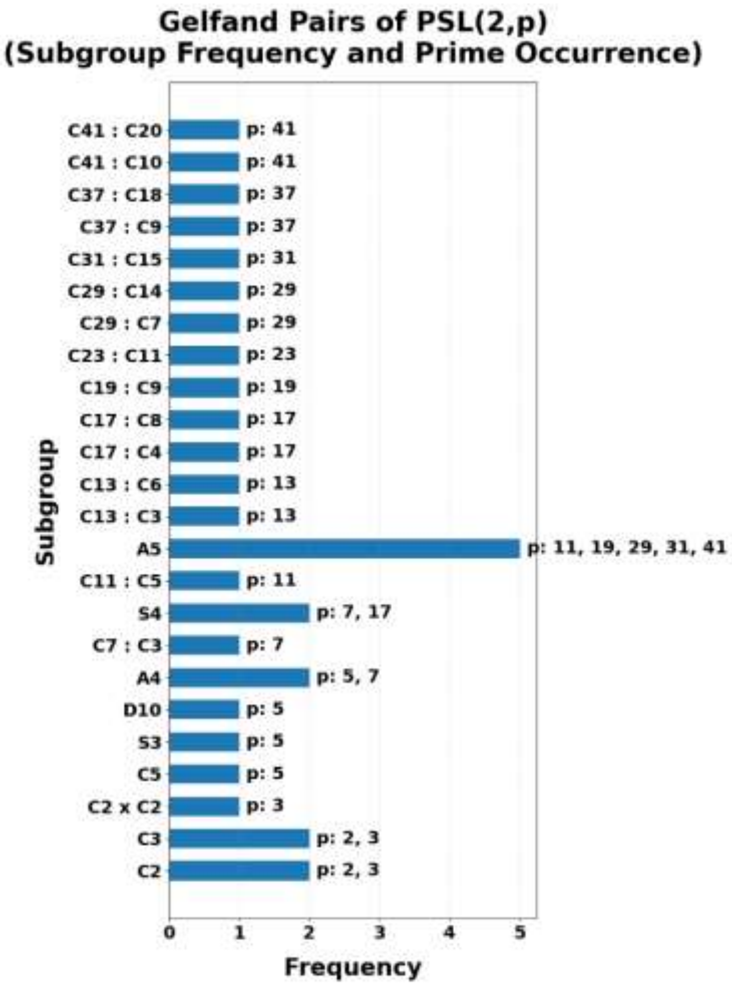

Figure 1. Multiplicity-free subgroups of $PSL(2, p)$.

4.2. Symmetric Pairs for PSL(2, p)

The symmetric pairs of *PSL(2, p)*, where p is a prime number with $1<p<100$, include all Gelfand pairs except C_2 and $C_2 \times C_2$, which appear at $p = 3$. A comparison between the symmetric pairs and Gelfand pairs is illustrated in Figure 2.

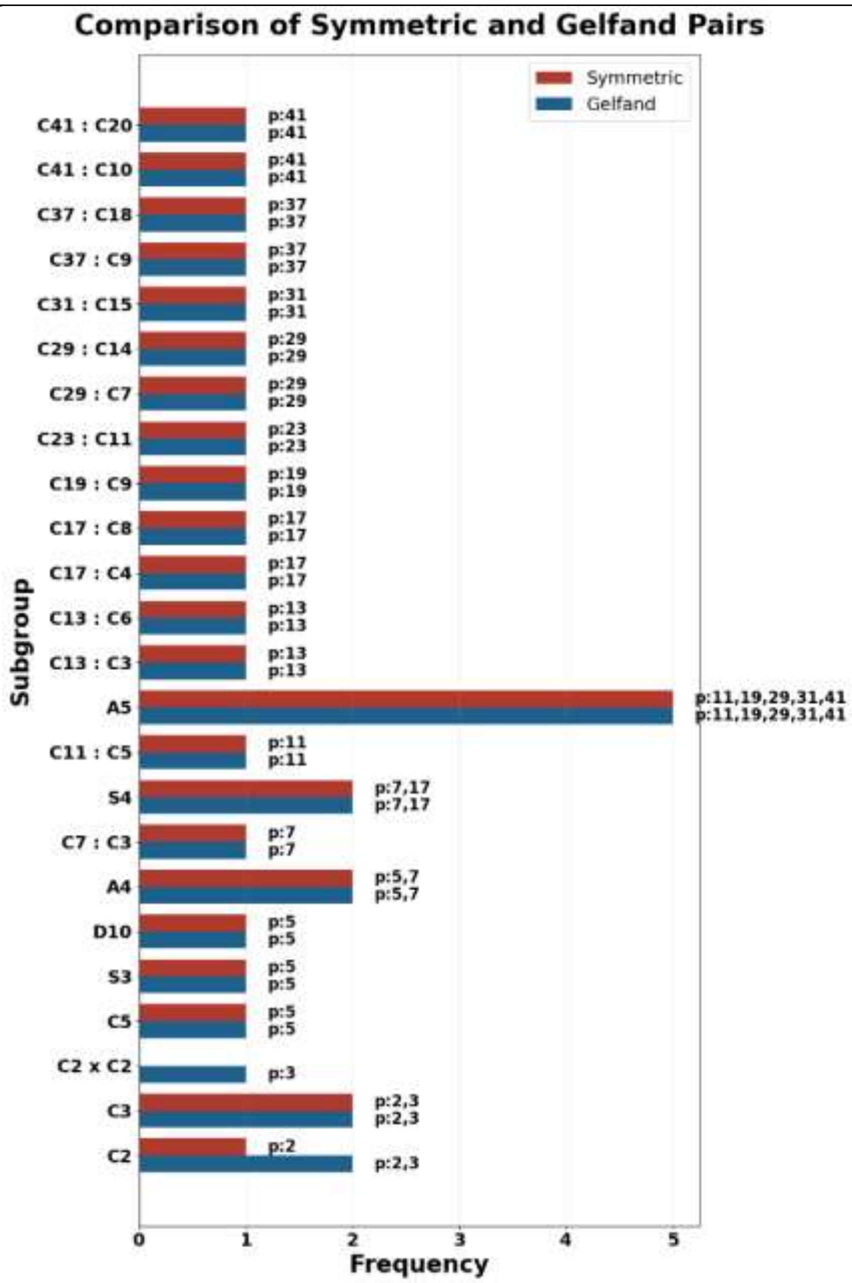

Figure 2. Gelfand and symmetric pairs of PSL(2, p), where p<100

4.3

. Weakly symmetric of PSL(2, p)

For the group $G = PSL(2, p)$ with $11 < p < 100$, all symmetric pairs are also weakly symmetric. Table 2 shows the cases for $p \leq 11$, including the unique strictly weakly symmetric pair $(PSL(2,3), C_2 \times C_2)$.

Table 2: Weakly Symmetric Pairs

P	Subgroups
2	C_2, C_3
3	$C_3, C_2 \times C_2$
5	C_5, S_3, D_{10}, A_4
7	S_4, U_7
11	A_5, U_{11}

Figure 3 presents a comparison between symmetric and weakly symmetric pairs.

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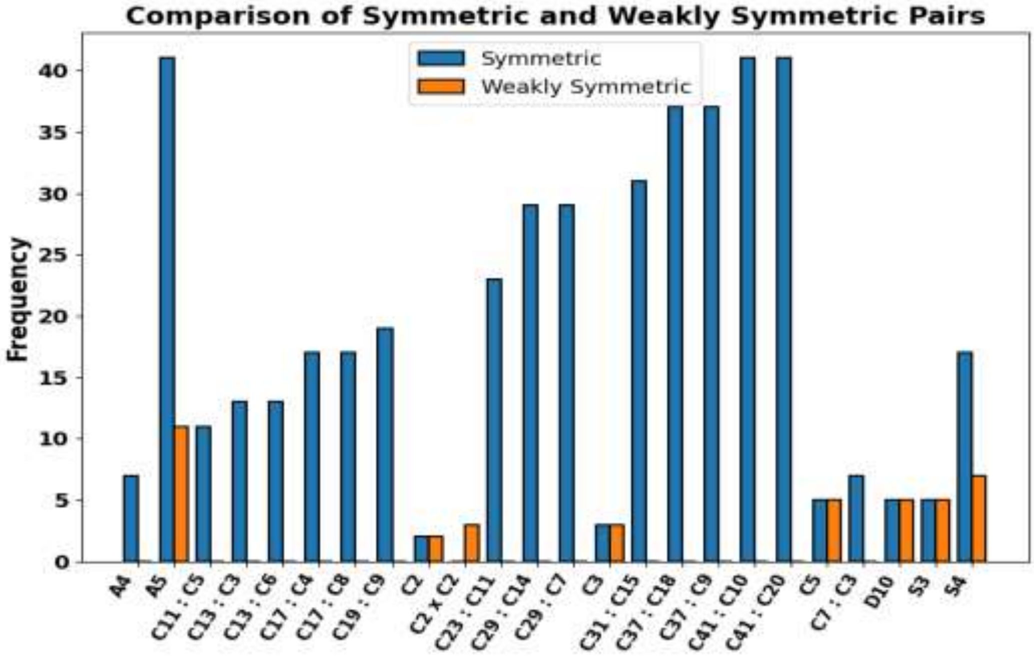

Figure 3. Symmetric and weakly symmetric of $PSL(2,p), p \in [1.41]$

The following cases arise:

- The Group G's trivial and certain non-trivial automorphisms cause the subgroups $C_2, C_3, C_5, D_{10}, A_4, A_5, U_7$ and U_{11} to become weakly symmetric,
- The subgroup $C_2 \times C_2$, shown separately on the far right, it is strictly weakly symmetric since only non-trivial AutoMorphisms makes it weakly symmetric.
- The other subgroups can only be made symmetric, thus weakly symmetric, by trivial AutoMorphisms.
- Based on Figures 2 and 3, the subgroup A_5 is the most frequently occurring and is Gelfand symmetric, but it is only weakly symmetric at $p = 11$ due to trivial and some non-trivial automorphisms.

Theorem 4.3.1. The only strictly weakly symmetric pair for the projective special linear group PSL(2, p), $1 < p < 100$, is (PSL(2,p), $C_2 \times C_2$).
Given that *PSL(2, p)* is simple for $p \geq 5$ and that symmetric and weakly symmetric pairs are primarily associated with the normality of subgroups, we therefore propose the following:

Conjecture: For each prime p, the pair (PSL(2,p), $C_2 \times C_2$) is the only strictly weakly symmetric pair in the complete projective special linear group PSL(2, p).

5. Multiplicity-Free Permutation Representation
This section examines the direct product of the cyclic group of order 2 with itself, $H_0 = C_2 \times C_2$, which forms a unique strictly weakly symmetric subgroup of $PS3 = PSL(2,3)$. The permutation representation $\pi_0 : PS3 \to End_{PS3}({}^{H_0}L(PS3)^{H_0}$ acts on the space of all bi-H_0- invariant functions:

$$^{H_0}L(PS3)\,^{H_0} = \{f \in L(PS3) : f(h_1 g h_2) = f(g), g \in PS3, h_1, h_2 \in H_0\}$$

This representation is equivalent to the induced representation, specifically, $\pi_0 \approx ind_{H_0}^{PS3} i_{H_0}$.
Using GAP's "Repsn" package,[18], we constructed the irreducible representation affording the trivial representation i_{H_0}, and subsequently computed the induced subgroup representation $ind_{H_0}^{PS3} \iota_{H_0}$. The permutation representation π_0 is defined as follows:

$$\pi_0\big((2,3,4)\big) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \pi_0\big((1,2)(3,4)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Where $\{(2,3,4), (1,2)(3,4)\}$ are generators of the group $PS3$. Note that:

$$ker(\pi_0) = \langle (1,2)(3,4), (1,4)(2,3) \rangle \neq \{e\}$$

indicating that π_0 is not injective.

4. Conclusions

While examples of weakly symmetric and non-symmetric pairs are relatively rare, much research has focused on identifying Gelfand symmetric and weakly symmetric pairs. This work contributes to this field by introducing a new category of weakly symmetric pairs: strictly weakly symmetric pairs. The method we developed for identifying these pairs, especially through multiplicity-free permutation representations, is crucial for this purpose. Our findings, based on primes in the range [1..100], suggest that $(PSL(2,3), C_2 \times C_2)$ is the only strictly weakly symmetric pair within the projective special linear groups $PSL(2,p)$.

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