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Mustafa Hasan Hadi
Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Iraq, pure.mustafa.hassan@uobabylon.edu.iq

Luay Abd Al haine Al swidi
Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Iraq, l.h.jabar64@gmail.com

ORIGINAL STUDY

Study of ∼-Topological Space by ∼-Binary Relation in Cluster System

Mustafa Hasan Hadi *, **Luay Abd Al haine Al swidi**

Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Iraq

ABSTRACT

In this paper, we introduced a new type of operators, which we called ∼-operator by cluster system, and studied its properties without conditions and with certain conditions. Through our study of this operator, we defined a new topology, which we called ∼-topological space, since it does not, in general, constitute a regular topological space. We also presented ∼-interior points, ∼-closure, ∼-exterior points and reviewed some theories and examples about this concept.

Keywords: ∼-operator, Ɛ-operator, Ⱨ-network, I-property, ƛ-additive property

1. Introduction

Operators play an important role in building new topologies that are more general than the topologies known on general topologies, as these operators are built based on specific binary relationships. Therefore, many scientists have resorted to studying operators and comparing them with the usual operators, as well as studying their properties and comparing them with the usual properties. Some researchers have provided the necessary conditions so that they can obtain equivalence between these properties.

In 1966, K. Kuratowski [1] provide a special kind of operator using ideal spaces and in 1947 G. Chopuet [2] defined operator by grill spaces. Irina Zvina [3] in 2006 a new type of operators is called $\preccurlyeq$ and $\approx$ it is considered a special type of ideal space. Yiezi. K and Luay. A [4] in 2020 use these ideas to build focal function and Ali. K and Luay. A [5] 2023 they built a *i*-subspace using it. Marie, Saad S. and Khalaf, Hatam [6] use Idea fuzzy spaces to define a new type of fuzzy socle T-ABSO submodules. Also, Sharqi, Moataz Sajid and khalil, Shuker [7] applied Idea soft to build soft permutation commutative Q-algebras and their applications.

In 1984, Milan. M [8, 9] introduced a new concept for classifying sets in topological spaces, which he called cluster system, which is a non-empty system defined in the form Ɛ ⊆ $2^{ꞩ}$ – {∅}, where $2^{ꞩ}$ is the power set. Through it, he defined an operator on the set Ⱨ, which he called Ɛ-operator and defined by Ɛ(Ⱨ) = {ꞩ ∈ Ꞩ : ∀Ɽ ∈ $Ŧ_{(ꞩ)}$ ∃ Ģ ∈ Ɛ ∋ Ģ ⊆ Ɽ ∩ Ⱨ}, studied some of its properties. In 2015 R. Thangamariappan and V. Renukadevi [10, 12, 13] they presented an idea Ⱨ-network in Ꞩ and defined by a cluster system Ɛ is a Ⱨ-network in Ꞩ iff Ɛ(Ꞩ) = Ꞩ. A cluster system Ɛ is satisfy the D-property iff ∀ Ģ ∈ Ɛ and Ⱨ, Ⱪ ⊆ Ꞩ such that Ģ ⊆ Ⱨ ∪ Ⱪ then there exists $Ģ_1$, $Ģ_2$ ∈ Ɛ ∋ $Ģ_1$ ⊆ Ⱨ or $Ģ_2$ ⊆ Ⱪ. Let Ǥ be any a nonempty collection of subsets of Ꞩ is satisfy: I) The stack-property iff ∀ Ⱨ ∈ Ǥ such that Ⱨ ⊆ Ⱪ, then Ⱪ ∈ Ǥ. II) The I-property iff ∀ Ⱨ, Ⱪ ∈ Ǥ then Ⱨ ∩ Ⱪ ∈ Ǥ [10]. III) The additive-property iff ∀ Ⱨ, Ⱪ ∈ Ǥ then Ⱨ ∪ Ⱪ ∈ Ǥ. IV) The ƛ-additive property iff for any index ⅄, $Ⱨ_{ƛ}$ ∈ Ǥ,∀ƛ ∈ ⅄ such that $\cup_{ƛ \in ⅄} Ⱨ_{ƛ}$ ∈ Ǥ.

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* Corresponding author.
E-mail addresses: pure.mustafa.hassan@uobabylon.edu.iq (M. H. Hadi), l.h.jabar64@gmail.com (L. A. A. haine Al swidi).

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2. ∼–Operator and properties

In this section we will review the properties that can be achieved in relation $\sim$. Some of these properties are achieved under certain conditions and others can be achieved without any conditions.

Definition 2.1: Let $\mathcal{E}$ be a cluster system on $\mathcal{S}$ and let $H, K \subseteq \mathcal{S}$, the relation $\sim$ on $\mathcal{S}$ defined by: $H \sim K$ if and only if $\forall\, G \in \mathcal{E} \ni G \nsubseteq H - K$. Negation of the sentence then, $H \not\sim K$ if and only if $\exists\, G \in \mathcal{E} \ni G \subseteq H - K$.

Example 2.2: *Let* $\mathcal{S} = \{h, \kappa, м\}$ *and* $\mathcal{E} = \{\{h\},\{\kappa\},\{м\},\{h, \kappa\},\{h, м\},\{\kappa, м\}, \mathcal{S}\}$. *If* $H = \{м\}$, $K = \{\kappa, м\}$, *then* $H \sim K$, *since* $H - K = \{м\} \cap \{h\} = \emptyset \nsupseteq G, \forall G \in \mathcal{E}$.

In this proposition we will introduce properties $\sim$-operator without condition.

Proposition 2.3: Let $\mathcal{E}$ be a cluster system on $\mathcal{S}$ and let $H, K, M \subseteq \mathcal{S}$, then the following properties are hold:

A) $H \sim \mathcal{S}, \forall\, H \subseteq \mathcal{S}$.
B) $H \sim H, \forall\, H \subseteq \mathcal{S}$.

Proof:

A) By Definition 2.1, $H \sim \mathcal{S}$ iff $H - \mathcal{S} = H \cap \emptyset = \emptyset \nsupseteq G, \forall\, G \in \mathcal{E}$.
B) If possible $H \not\sim H$, then $\exists G \in \mathcal{E} \ni G \subseteq H \cap (\mathcal{S} - H) = \emptyset$, then $G = \emptyset$, which is contradiction. So $H \sim H$.

Now, we will review in this proposition the properties of $\not\sim$-operator without conditions.

Proposition 2.4: Let $\mathcal{E}$ be a cluster system on $\mathcal{S}$ and let $H, K, M, Z \subseteq \mathcal{S}$, then the following properties are hold:

1) If $H \subseteq M$ and $H \not\sim K$, then $M \not\sim K$.
2) If $Z \subseteq K$ and $H \not\sim K$, then $H \not\sim Z$.
3) If $H_\lambda \not\sim K$, for any $\lambda \in \Lambda$, where Λ be any index, then $\cup_{\lambda \in \Lambda} H_\lambda \not\sim K$. (Where $\mathcal{E}$ is Λ-cluster system).
4) If $H \not\sim K_\lambda$, for any $\lambda \in \Lambda$, where Λ be any index, then $H \not\sim \cap_{\lambda \in \Lambda} K_\lambda$. (Where $\mathcal{E}$ is Λ-cluster system).
5) If $H_\lambda \not\sim K_\gamma$, for any $\lambda, \gamma \in \Lambda$, where Λ be any index, then $\cup_{\lambda \in \Lambda} H_\lambda \not\sim \cap_{\gamma \in \Lambda} K_\gamma$.
6) If $H - M \not\sim K \cup M$, then $H \not\sim K$ and $H \not\sim M$.
7) If $H \not\sim K$ and $H \not\sim M$, then $H \not\sim K \cap M$.
8) If $H \not\sim M$ and $(\mathcal{S} - M) \not\sim K$, then $H \cup (\mathcal{S} - K) \not\sim M$.

Proof:

1) Since $H \subseteq M$ and $H \not\sim K$, then $\exists G \in \mathcal{E} \ni G \subseteq H - K \subseteq M - K$, we get $G \subseteq M - K$, then $M \not\sim K$.
2) Since $Z \subseteq K$ and $H \not\sim K$, then $\exists G \in \mathcal{E} \ni G \subseteq H - K \subseteq H - Z$ so $G \subseteq H - Z$, we get $H \not\sim Z$.
3) Directly from part 1.
4) Directly from part 2.
5) Since $H_\lambda \subseteq \cup_{\lambda \in \Lambda} H_\lambda$ and $\cap_{\gamma \in \Lambda} K_\gamma \subseteq K_\gamma$ and $H_\lambda \not\sim K_\gamma$, then $\exists\, G_\lambda \subseteq H_\lambda - K_\gamma = H_\lambda \cap (\mathcal{S} - K_\gamma)$, for any $\lambda, \gamma \in \Lambda$, so $G_\lambda \subseteq \cup_{\lambda \in \Lambda} H_\lambda \cap (\mathcal{S} - \cap_{\gamma \in \Lambda} K_\gamma) = \cup_{\lambda \in \Lambda} H_\lambda - \cap_{\gamma \in \Lambda} K_\gamma$, then $\cup_{\lambda \in \Lambda} H_\lambda \not\sim \cap_{\gamma \in \Lambda} K_\gamma$.
6) Since $H \cap (\mathcal{S} - M) \not\sim K \cup M$, then $\exists G \in \mathcal{E} \ni G \subseteq H \cap (\mathcal{S} - M) \cap (\mathcal{S} - (K \cup M)) = H \cap (\mathcal{S} - M) \cap (\mathcal{S} - K) \cap (\mathcal{S} - M) = H \cap (\mathcal{S} - M) \cap (\mathcal{S} - K) \subseteq H \cap (\mathcal{S} - M)$, so $G \subseteq H \cap (\mathcal{S} - M)$, then $H \not\sim M$. And $G \subseteq H \cap (\mathcal{S} - K)$, then $H \not\sim K$.
7) Since $H \not\sim K$, then $\exists G_1 \in \mathcal{E} \ni G_1 \subseteq H - K$ and $H \not\sim M$, then $\exists G_2 \in \mathcal{E} \ni G_2 \subseteq H - M$, we get $G_1 \subseteq G_1 \cup G_2 \subseteq (H - K) \cup (H - M) = H - (K \cap M)$, then $H \not\sim K \cap M$.
8) Let $H \not\sim M$, then $\exists G_1 \in \mathcal{E} \ni G_1 \subseteq H - M$ and $(\mathcal{S} - M) \not\sim K$, then $\exists G_2 \in \mathcal{E} \ni G_2 \subseteq (\mathcal{S} - M) - K$, so $G_1 \subseteq G_1 \cup G_2 \subseteq (H - M) \cup ((\mathcal{S} - M) - K) = H \cup (\mathcal{S} - K) - M$, we get $H \cup (\mathcal{S} - K) \not\sim M$.

In this remark we will present examples of the inverse properties of $\not\sim$-operator without condition.

Remark 2.5:

1) The converse of (part 1) is not necessary true. For example, let $\mathcal{S} = \{h, \kappa, м\}$ and $\mathcal{E} = \{\{\kappa\}\}$. If $H = \{h\}$, $K = \{h, м\}$, $M = \{h, \kappa\}$, then $K \not\sim M$ and $H \subseteq M$, but $H \sim K$, because $H - K = \emptyset \nsupseteq \{\kappa\}$.
2) The converse of (part 2) is not necessary true. For example, let $\mathcal{S} = \{h, \kappa, z\}$ and $\mathcal{E} = \{\{\kappa\}\}$. If $H = \{\kappa\}$, $K = \{h, \kappa\}$, $Z = \{h\}$, then $H \not\sim Z$, and $Z \subseteq K$, but $H \sim K$, because $H - K = \emptyset \nsupseteq \{\kappa\}$.
3) The converse of (part 6) is not necessary true. For example, let $\mathcal{S} = \{h, \kappa, м\}$ and $\mathcal{E} = \{\{h\},\{\kappa\}\}$. If $H = \{h, \kappa\}$, $K = \{h\}$, $M = \{\kappa\}$, then $H \not\sim K$, because $\{h, \kappa\} \cap \{\kappa, м\} = \{\kappa\}$ and $H \not\sim M$, because $\{h, \kappa\} \cap \{h, м\} = \{h\}$, but $H - M \sim K \cup M$, because $\{h\} \cap \{м\} = \emptyset$.
4) The converse of (part 7) is not necessary true. For example, let $\mathcal{S} = \{h, \kappa, м\}$ and $\mathcal{E} = \{\{h\}\}$. If $H = \{h\}$, $K = \{h, \kappa\}$, $M = \{\kappa\}$, then $H \not\sim K \cap M$, because $\{h\} \cap \{h, м\} = \{h\} \supseteq \{h\}$ and $H \not\sim M$, because $\{h\} \cap \{h, м\} = \{h\} \supseteq \{h\}$, but $H \sim K$, because $\{h\} \cap \{м\} = \emptyset \nsupseteq \{h\}$.
5) The converse of (part 8) is not necessary true. For example, let $\mathcal{S} = \{h, \kappa, м\}$ and $\mathcal{E} = \{\{h\},\{\kappa, м\}\}$. If $H = \{h\}$, $K = \{h, м\}$, $M = \{\kappa, м\}$, then $H \cup (\mathcal{S} - K) \not\sim M$, because $\{h, \kappa\} \cap \{h\} = \{h\}$ and $H \not\sim M$, because $\{h\} \cap \{h\} = \{h\}$, but $(\mathcal{S} - M) \sim K$, because $\{h\} \cap \{\kappa\} = \emptyset \nsupseteq \{h\}$.

In this proposition we will introduce the properties of the $\not\sim$-operator using stack-property.

Proposition 2.6: Let $\mathcal{E}$ be acluster system on $\mathcal{S}$ with stack-property and let $H, K, M, \subseteq \mathcal{S}$, then the following properties are hold:

1) Ḩ ≃ ∅ iff Ḩ ∈ Ɛ.
2) If Ḩ ≃ Ķ, ∀ Ķ ∈ Ꞩ, then Ḩ ∈ Ɛ.

Proof:

1) ⇒) Let Ḩ ≃ ∅, then ∃Ģ ∈ Ɛ ∋ Ķ ⊆ Ḩ − ∅ = Ḩ ∩ Ꞩ = Ḩ, since Ɛ with stack-property, then Ḩ ∈ Ɛ.
⇐) Let Ḩ ∈ Ɛ, since Ḩ ⊆ Ḩ, so Ḩ − ∅ = Ḩ ∩ Ꞩ = Ḩ ∈ Ɛ, since Ɛ with stack-property, so Ḩ ≃ ∅.
2) Since Ḩ ≃ Ķ, then ∃Ģ ∈ Ɛ ∋ Ģ ⊆ Ḩ − Ķ ⊆ Ḩ, since Ɛ with stack-property, there for Ḩ ∈ Ɛ.

Remark 2.7: Condition the stack-property in above proposition is indispensable. For example, let Ꞩ = {h, κ, м} and Ɛ = {{h}, Ꞩ}. If Ḩ = {h, κ}, Ķ = {κ} then:

1) Ḩ ≃ ∅ because, {h, κ} ∩ Ꞩ = {h, κ} ⊇ {h}, but Ḩ ∉ Ɛ.
2) Ḩ ≃ Ķ, because {h, κ} ∩ {h, м} = {h} ⊇ {h}, but Ḩ ∉ Ɛ. (Where Ɛ dose not satisfy the stack-property in this example).

Now, we will present the ≃-operator properties that can be achieved using I-property.

Proposition 2.8: Let Ɛ be a cluster system with I-property and Ḩ, Ķ, M̧ ⊆ Ꞩ, then the following properties are hold:

1) If Ḩ ≃ Ķ and Ḩ ≃ M̧, then Ḩ ≃ Ķ ∪ M̧.
2) If Ḩ ≃ M̧ and (Ꞩ − M̧) ≃ Ķ, then Ḩ ≃ Ķ.

Proof:

1) Since Ḩ ≃ Ķ, then ∃Ģ$_1$ ∈ Ɛ ∋ Ģ$_1$ ⊆ Ḩ − Ķ, and Ḩ ≃ M̧, then ∃Ģ$_2$ ∈ Ɛ ∋ Ģ$_2$ ⊆ Ḩ − M̧, since Ɛ with I-property, then Ģ$_1$ ∩ Ģ$_2$ ⊆ (Ḩ − Ķ) ∩ (Ḩ − M̧) = Ḩ ∩ ((Ꞩ − Ķ) ∩ (Ꞩ − M̧)) = Ḩ ∩ (Ꞩ − (Ķ ∪ M̧)) = Ḩ − (Ķ ∪ M̧), so Ḩ ≃ Ķ ∪ M̧.
2) It is clear.

Remark 2.9: Condition the I-property in above proposition is indispensable. For example: Let Ꞩ = {h, κ, м} and Ɛ = {{h, κ},{κ, м}}. If Ḩ = Ꞩ, Ķ = {h}, M̧ = {м}, then Ḩ ≃ Ķ, because Ꞩ ∩ {κ, м} = {κ, м} and Ḩ ≃ M̧, because Ꞩ ∩ {h, κ} = {h, κ}, but Ḩ ∼ Ķ ∪ M̧, because Ꞩ ∩ {κ} = {κ}.

The next proposition we will show the properties of ≃-operator using D-property.

Proposition 2.10: Let Ɛ be a cluster system on Ꞩ with D-property and Ḩ, Ķ, M̧ ⊆ Ꞩ, then the following properties are hold:

1) If Ḩ ≃ Ķ, then Ḩ ≃ M̧ and M̧ ≃ Ķ.
2) If Ḩ ∪ M̧ ≃ Ķ ∩ (Ꞩ − M̧), then Ḩ ≃ Ķ and M̧ ≃ Ķ.

Proof:

1) Let Ḩ ≃ Ķ, then ∃Ģ ∈ Ɛ ∋ Ģ ⊆ Ḩ − Ķ, but Ḩ ∩ (Ꞩ − Ķ) = Ḩ ∩ (Ꞩ − Ķ) ∩ Ꞩ = (Ḩ ∩ (Ꞩ − Ķ) ∩ M̧ ∪ (Ꞩ− M̧)) = (Ḩ ∩ (Ꞩ − Ķ) ∩ M̧) ∪ (Ḩ ∩ (Ꞩ − Ķ) ∩ (Ꞩ − M̧)), then Ģ ⊆ Ḩ ∩ (Ꞩ − Ķ) ∩ M̧ ∪ (Ḩ ∩ (Ꞩ − Ķ) ∩ (Ꞩ − M̧)), by D-property then there exists Ģ$_1$, Ģ$_2$ ∈ Ɛ ∋ Ģ$_1$ ⊆ Ḩ ∩ (Ꞩ − Ķ) ∩ M̧ ⊆ M̧ ∩ (Ꞩ − Ķ), then M̧ ≃ Ķ and Ģ$_2$ ⊆ Ḩ ∩ (Ꞩ − Ķ) ∩ (Ꞩ − M̧) ⊆ Ḩ ∩ (Ꞩ − M̧), then Ḩ ≃ M̧.
2) Since Ḩ ∪ M̧ ≃ Ķ ∩ (Ꞩ − M̧), then ∃Ģ ∈ Ɛ ∋ Ģ ⊆ (Ḩ ∪ M̧) ∩ Ꞩ − (Ķ ∩ (Ꞩ − M̧)) = (Ḩ ∪ M̧) ∩ ((Ꞩ − Ķ) ∪ M̧) = (Ḩ ∩ (Ꞩ − Ķ)) ∪ (M̧ ∩ (Ꞩ − Ķ)) ∪ ((Ḩ ∪ M̧) ∩ M̧), by D-property, then ∃Ģ$_1$, Ģ$_2$ ∈ Ɛ ∋ Ģ$_1$ ⊆ Ḩ ∩ (Ꞩ − Ķ), so Ḩ ≃ Ķ. And Ģ$_2$ ⊆ M̧ ∩ (Ꞩ − Ķ), so M̧ ≃ Ķ.

Remark 2.11: Condition the D-property in above proposition is indispensable. For example:

1) Let Ꞩ = {h, κ, м} and Ɛ = {{h, κ}}. If Ḩ = {h, κ}, Ķ = {м}, M̧ = {h}, then Ḩ ≃ Ķ, because {h, κ} ∩ {h, κ} = {h, κ}, but Ḩ ∼ M̧, because {h, κ} ∩ {κ, м} = {κ} ⊉ {h, κ}, and M̧ ∼ Ķ, because {h} ∩ {h, κ} = {h} ⊇ {h, κ}. (Where Ɛ dose not the D-property).
2) Let Ꞩ = {h, κ, м} and Ɛ = {{h, κ}}. If Ḩ = ∅, Ķ = {м}, M̧ = Ꞩ, then Ḩ ∪ M̧ ≃ Ķ ∩ (Ꞩ − M̧), because Ꞩ ∩ Ꞩ = Ꞩ ⊇ {h, κ}, and M̧ ≃ Ķ, because Ꞩ ∩ {h, κ} = {h, κ}, but Ḩ ∼ Ķ, because ∅ ∩ {h, κ} = ∅ ⊇ {h, κ}. (Where Ɛ dose not satisfy the D-property).

3. ∼-Topological space and its applications

In this section we will introduce the definition of ∼-topological space using the ∼-operator and review some of its properties and related examples.

Definition 3.1: Let Ɛ be a cluster system on Ꞩ is a family Ŧ$_\sim$ of subsets of Ɛ, that satisfies the conditions:

1) ∅, Ꞩ ∈ Ŧ$_\sim$.
2) For any Ḩ, Ķ ∈ Ŧ$_\sim$, there exists a proper subset M̧ ∈ Ŧ$_\sim$ such that Ḩ ∩ Ķ ∼ M̧.
3) For any index ⅄, Ḩ$_ƛ$ ∈ Ŧ$_\sim$, ∀ƛ ∈ ⅄, there exists a proper subset M̧ ∈ Ŧ$_\sim$ such that ∪$_{ƛ∈⅄}$Ḩ$_ƛ$ ∼ M̧.
4) Ɛ is a Ɥ-network in Ꞩ. Then Ꞩ$_Ɛ^{Ŧ_\sim}$ is called ∼-topological space.

Example 3.2: *Let* Ꞩ = {h, κ, м, ʐ}, Ŧ$_\sim$ = {Ꞩ,∅,{{h, κ},{κ, ʐ},{h, κ, ʐ}} *and* Ɛ = {{h, κ},{κ, ʐ}}.

1) ∅, Ꞩ ∈ Ŧ$_\sim$.
2) {h, κ} ∼ {h, κ},{κ, ʐ} ∼ {κ, ʐ}, {h, κ, ʐ} ∼ {h, κ, ʐ} *and* {h, κ} ∩ {κ, ʐ} = {κ} ∼ {h, κ}, *because* {κ} ∩ {м, ʐ} = ∅.

3) {ɦ, ĸ} ∪ {ĸ, ʐ} = {ɦ, ĸ, ʐ} ∼ {ɦ, ĸ, ʐ}.
4) Ɛ(Ꞩ) = Ꞩ. *Then* $Ꞩ_{Ɛ}^{Ŧ\sim}$ *is ∼-topological space.*

Example 3.3: *From the example above if* Ɛ = {{ɦ, ĸ}}, *then* $Ꞩ_{Ɛ}^{Ŧ\sim}$ *is not ∼-topological space, because* Ɛ(Ꞩ) ≠ Ꞩ.

This remark will give us the most important properties for the new ∼-topology.

Remark 3.4:

1) The members of $Ŧ_{\sim}$ is called ∼-open set and the complement of ∼-open set is called ∼-closed set.
2) The intersection of two ∼-open set is not necessary ∼-open set. From Example 3.2, then {ɦ, ĸ} ∩ {ĸ, ʐ} = {ĸ} ∉ $Ŧ_{\sim}$.
3) The union of two ∼-open set is not necessary ∼-open set. For example, let Ꞩ = {ɦ, ĸ, ӎ, ʐ}, $Ŧ_{\sim}$ = {Ꞩ,∅,{ĸ},{ɦ, ĸ},{ĸ, ʐ}} and Ɛ = {{ĸ}}, then {ɦ, ĸ} ∪ {ĸ, ʐ} = {ɦ, ĸ, ʐ} ∉ $Ŧ_{\sim}$.
4) In general, the ∼-topological space may be not a topological space, ideal topological space and grill topological space.

Now, we will employ the new definition to build concept ∼-interior.

Definition 3.5: Let $Ꞩ_{Ɛ}^{Ŧ\sim}$ be a ∼-topological space. Apoint ꞩ ∈ Ꜧ is called ∼-interior of Ꜧ ⊆ Ꞩ iff there exists ∼-open set Ꞧ such that ꞩ ∈ Ꞧ ⊆ Ꜧ and the set of all ∼-interior points of Ꜧ is denoted by $\sim_{int}$ (Ꜧ). And Ꜧ is called ∼-neighbourhood of ꞩ, the set of all ∼-neighbourhood of ꞩ, is called the neighbourhood system of points and denoted by ∼ −Ŋ(ꞩ).

Proposition 3.6: Let $Ꞩ_{Ɛ}^{Ŧ\sim}$ be a ∼-topological space. And Ꜧ ⊆ Ꞩ. If Ꜧ is ∼-open set then Ꜧ is a ∼-neighbourhood of each of its points.

Proof: Let Ꜧ is ∼-open set and ꞩ ∈ Ꜧ, then ʐ ∈ Ꜧ ⊆ Ꜧ, so Ꜧ is ∼-neighbourhood of each of its points.

Remark 3.7:

1) The converse of (Proposition 3.6) is not necessary true.
2) The ∼-interior set is not necessary ∼-open set.
3) The ∼ −Ŋ(ꞩ) system satisfies of all property of system nbd in general topology except that intersection ∼ −Ŋ(ꞩ) may be not ∼ −Ŋ(ꞩ).

Example 3.8: *Let* Ꞩ = {ɦ, ĸ, ӎ, ʐ}, *where* Ꞩ *refers of the number internet companies in Iraq, where*(ɦ) *refers to the number of Hrins users,* (ĸ) *refers to the number of Alwatani users,* (ӎ) *refers to the number of Zain users and* (ʐ) *refers to the number of Asiacell users,* $Ŧ_{\sim}$ = {Ꞩ,∅,{ĸ, ӎ},{ӎ, ʐ}} *and* Ɛ = {{ӎ}}. *If* Ꜧ = {ĸ, ӎ, ʐ} *then,* $\sim_{int}$ (Ꜧ) = {ĸ, ӎ, ʐ} *and it is neighbourhood for each of its points, but it is not ∼-open set.*

The following proposition will present the properties of $\sim_{int}$ for any set and we note that there are properties that achieve equivalence in the general topology and do not achieve equivalence in the new definition of topology.

Proposition 3.9: Let $Ꞩ_{Ɛ}^{Ŧ\sim}$ be a ∼-topological space and Ꜧ, Ꝃ ⊆ Ꞩ, then:

1) $\sim_{int}$ (Ꜧ) = ∪{Ꞧ ∈ $Ŧ_{\sim}$: Ꞧ ⊆ Ꜧ}.
2) If Ꜧ ⊆ Ꝃ, then $\sim_{int}$ (Ꜧ) ⊆ $\sim_{int}$ (Ꝃ).
3) $\sim_{int}$ (Ꜧ) ⊆ Ꜧ.
4) $\sim_{int}$ ($\sim_{int}$ (Ꜧ)) ⊆ $\sim_{int}$ (Ꜧ).
5) If Ꜧ ∈ $Ŧ_{\sim}$, then $\sim_{int}$ (Ꜧ) = Ꜧ.
6) $\sim_{int}$ (Ꞩ) = Ꞩ and $\sim_{int}$ (∅) = ∅.
7) $\sim_{int}$ (Ꜧ ∩ Ꝃ) ⊆ $\sim_{int}$ (Ꜧ)∩ $\sim_{int}$ (Ꝃ).
8) $\sim_{int}$ (Ꜧ)∪ $\sim_{int}$ (Ꝃ) ⊆ $\sim_{int}$ (Ꜧ ∪ Ꝃ).

Proof of (3): Let ꞩ ∈ $\sim_{int}$ (Ꜧ), then there exists a ∼-open set Ꞧ ⊆ Ꜧ such that ꞩ ∈ Ꞧ ⊆ Ꜧ, we get ꞩ ∈ Ꜧ, then $\sim_{int}$ (Ꜧ) ⊆ Ꜧ.

In this note we will show the equivalence between some relations using conditions ƛ-additive property and I-property and also give the opposite examples.

Remark 3.10:

1) If $Ꞩ_{Ɛ}^{Ŧ\sim}$ is satisfy the ƛ-additive property, then: a) $\sim_{int}$ ($\sim_{int}$ (Ꜧ)) = $\sim_{int}$ (Ꜧ). b) $\sim_{int}$ (Ꜧ) is a ∼-open set. ∀Ꜧ ⊆ Ꞩ.
2) If $Ꞩ_{Ɛ}^{Ŧ\sim}$ is satisfy the I-property, then $\sim_{int}$ (Ꜧ ∩ Ꝃ) = $\sim_{int}$ (Ꜧ)∩ $\sim_{int}$ (Ꝃ), ∀Ꜧ, Ꝃ ⊆ Ꞩ.
3) The converse of (part 2) is not necessary true, by (Example 3.8), if Ꜧ = {ĸ} and Ꝃ = {ӎ, ʐ}, then $\sim_{int}$ (Ꜧ) = ∅ and $\sim_{int}$ (Ꝃ) = {ӎ, ʐ}. Then $\sim_{int}$ (Ꜧ) ⊆ $\sim_{int}$ (Ꝃ), but Ꜧ ⊈ Ꝃ.
4) The converse of (part 3) is not necessary true, by (Example 3.8), if Ꜧ = {ӎ}, then $\sim_{int}$ (Ꜧ) = ∅, but Ꜧ ⊈ $\sim_{int}$ (Ꜧ).
5) In general, the opposite direction is not true of (part 5), by (Example 3.8), if Ꜧ = {ĸ, ӎ, ʐ}, then $\sim_{int}$ (Ꜧ) = {ĸ, ӎ, ʐ}, but Ꜧ ∉ $Ŧ_{\sim}$.
6) In general, the equality is not achieved of (part 7), by (Example 3.8), if Ꜧ = {ĸ, ӎ} and Ꝃ = {ӎ, ʐ}, then $\sim_{int}$ (Ꜧ) = {ĸ, ӎ} and $\sim_{int}$ (Ꝃ) = {ӎ, ʐ}, but $\sim_{int}$ (Ꜧ ∩ Ꝃ) = ∅, so $\sim_{int}$ (Ꜧ) ∩ $\sim_{int}$(Ꝃ) ⊈ $\sim_{int}$(Ꜧ ∩ Ꝃ).
7) The converse of (part 8) is not necessary true, by (Example 3.8), if Ꜧ = {ӎ} and Ꝃ = {ʐ}, then $\sim_{int}$ (Ꜧ) = ∅ and $\sim_{int}$ (Ꝃ) = ∅, but $\sim_{int}$ (Ꜧ ∪ Ꝃ) = $\sim_{int}$ ({ӎ, ʐ}) = {ӎ, ʐ}, so $\sim_{int}$ (Ꜧ ∪ Ꝃ) ⊈ $\sim_{int}$ (Ꜧ)∪ $\sim_{int}$ (Ꝃ).

Proposition 3.11: Let $Ꞩ_{Ɛ}^{Ŧ_\sim}$ be a $\sim$-topological space with ƛ-additive property and $Ⱨ \subseteq Ꞩ$, $Ⱨ$ is $\sim$-open set iff $Ⱨ$ is a neighbourhood of each of its points.

Proof: $\Rightarrow$) From direct of (Proposition 3.6).
$\Leftarrow$) Let $Ⱨ$ is a neighbourhood of each of its points. Let $ꞩ \in Ⱨ$, then there exists $Ꞧ_ꞩ$ is $\sim$-open set such that $ꞩ \in Ꞧ_ꞩ \subseteq Ⱨ$, then $Ⱨ = \cup_{ꞩ \in Ⱨ}\{ꞩ\} \subseteq \cup_{ꞩ \in Ⱨ}Ꞧ_ꞩ \subseteq \cup\ Ⱨ = Ⱨ$, we get $Ⱨ = \cup_{ꞩ \in Ⱨ}Ꞧ_ꞩ$, but $Ŧ_\sim$ with ƛ-additive property, then $\cup_{ꞩ \in Ⱨ}Ꞧ_ꞩ$ is a $\sim$-open set, therefore $Ⱨ$ is $\sim$-open set.

Now, we will review the definition of $\sim_{c\ell}$ for the new $\sim$-operator and also provide examples.

Definition 3.12: Let $Ꞩ_{Ɛ}^{Ŧ_\sim}$ be a $\sim$-topological space and let $Ⱨ \subseteq Ꞩ$, the $\sim$-closure of $Ⱨ$ is the intersection of all $\sim$-closed sets contains of $Ⱨ$ and is denoted by $\sim_{c\ell}(Ⱨ)$, i.e. $\sim_{c\ell}(Ⱨ) = \cap\{F̧ : F̧ \text{ is } \sim\text{-closed}, Ⱨ \subseteq F̧\}$.

Remark 3.13: The $\sim$-closure set is not necessary $\sim$-closed set.

Example 3.14: *Let $Ꞩ = \{h, ĸ, м, z\}$, where $Ꞩ$ refers to a squadron of aircraft, (h) refers to fixed-wing aircraft, $(ĸ)$ refers to helicopters, $(м)$ refers to light aircraft, and (z) refers to drones, $Ŧ_\sim = \{\emptyset, Ꞩ, \{ĸ\}, \{ĸ, м\}, \{ĸ, z\}\}$ and $Ɛ = \{\{ĸ\}\}$ then $\sim$-closed sets are: $Ꞩ, \emptyset, \{h, м, z\}, \{h, z\}, \{h, м\}$. If $Ⱨ = \{h\}$, then $\sim_{c\ell}(Ⱨ) = \{h\}$, but $\{h\}$ is not $\sim$-closed set.*

The proposition shows the most important properties of $\sim_{c\ell}$ and we notice that some properties were proven in one direction because they do not achieve the union between the new topological sets.

Proposition 3.15: Let $Ꞩ_{Ɛ}^{Ŧ_\sim}$ be a $\sim$-topological space and let $Ⱨ, Ⱪ \subseteq Ꞩ$, then:

1) If $Ⱨ$ is $\sim$-closed set, then $\sim_{c\ell}(Ⱨ) = Ⱨ$.
2) $Ⱨ \subseteq \sim_{c\ell}(Ⱨ)$.
3) $\sim_{c\ell}(Ⱨ) \subseteq \sim_{c\ell}(\sim_{c\ell}(Ⱨ))$.
4) $\sim_{c\ell}(Ꞩ) = Ꞩ$ and $\sim_{c\ell}(\emptyset) = \emptyset$.
5) If $Ⱨ \subseteq Ⱪ$, then $\sim_{c\ell}(Ⱨ) \subseteq \sim_{c\ell}(Ⱪ)$.
6) $\sim_{c\ell}(Ⱨ \cap Ⱪ) \subseteq \sim_{c\ell}(Ⱨ) \cap \sim_{c\ell}(Ⱪ)$.
7) $\sim_{c\ell}(Ⱨ) \cup \sim_{c\ell}(Ⱪ) \subseteq \sim_{c\ell}(Ⱨ \cup Ⱪ)$.

Proof of (5): Let $ꞩ \in \sim_{c\ell}(Ⱨ)$, then $ꞩ \in F̧$ is a $\sim$-closed set and $Ⱨ \subseteq F̧$. If possible $ꞩ \in \sim_{c\ell}(Ⱪ)$, then $ꞩ \notin Ȩ$, for any $Ȩ$ is $\sim$-closed set, $Ⱪ \subseteq Ȩ$ but $Ⱨ \subseteq Ⱪ$, then $ꞩ \notin Ȩ$, for any $Ȩ$ is $\sim$-closed set, so $Ⱨ \subseteq Ⱪ \subseteq Ȩ$, then $ꞩ \notin \sim_{c\ell}(Ⱨ)$, this is a contradiction to the assumption.

Remark 3.16:

1) If $Ꞩ_{Ɛ}^{Ŧ_\sim}$ is satisfy the ƛ-additive property, then:
   a) $\sim_{c\ell}(\sim_{c\ell}(Ⱨ)) = \sim_{c\ell}(Ⱨ)$
   b) $\sim_{c\ell}(Ⱨ)$ is $\sim$-closed set, $\forall Ⱨ \subseteq Ꞩ$.
2) If $Ꞩ_{Ɛ}^{Ŧ_\sim}$ is satisfy the additive-property then, $\sim_{c\ell}(Ⱨ) \cup \sim_{c\ell}(Ⱪ) = \sim_{c\ell}(Ⱨ \cup Ⱪ), \forall Ⱨ, Ⱪ \subseteq Ꞩ$.
3) The converse of (part 2) is not necessary true, by (Example 3.14), if $Ⱨ = \{ĸ\}$, then $\sim_{c\ell}(Ⱨ) = Ꞩ$, so $\sim_{c\ell}(Ⱨ) \nsubseteq Ⱨ$.
4) In general, the opposite direction is not true of (part 5), by (Example 3.14), if $Ⱨ = \{h, z\}$ and $Ⱪ = \{ĸ\}$, then $\sim_{c\ell}(Ⱨ) = \{h, z\}$ and $\sim_{c\ell}(Ⱪ) = Ꞩ$, so $\sim_{c\ell}(Ⱨ) \subseteq \sim_{c\ell}(Ⱪ)$, but $Ⱨ \nsubseteq Ⱪ$.
5) The converse of (part 6) is not necessary true, by (Example 3.14), if $Ⱨ = \{h\}$ and $Ⱪ = \{z\}$, then $\sim_{c\ell}(Ⱨ) = \{h\}$ and $\sim_{c\ell}(Ⱪ) = \{h, z\}$ and $\sim_{c\ell}(Ⱨ \cap Ⱪ) = \emptyset$, we get $\sim_{c\ell}(Ⱨ) \cap \sim_{c\ell}(Ⱪ) \nsubseteq \sim_{c\ell}(Ⱨ \cap Ⱪ)$.
6) In general, the equality is not achieved of (part 7), see example the following.

Example 3.17: *Let $Ꞩ = \{h, ĸ, м, z\}$, $Ŧ_\sim = \{\emptyset, Ꞩ, \{h, ĸ\}, \{ĸ, z\}, \{h, ĸ, z\}\}$ and $Ɛ = \{\{h, ĸ\}, \{ĸ, z\}, \{ĸ\}, \{z\}\}$ the $\sim$-closed sets are: $Ꞩ, \emptyset, \{м, z\}, \{h, м\}, \{м\}$. If $Ⱨ = \{м, z\}$ and $Ⱪ = \{h, м\}$, then $\sim_{c\ell}(Ⱨ) = \{м, z\}$, $\sim_{c\ell}(Ⱪ) = \{h, м\}$ and $\sim_{c\ell}(Ⱨ) \cup \sim_{c\ell}(Ⱪ) = \{h, м, z\}$, but $\sim_{c\ell}(Ⱨ \cup Ⱪ) = \sim_{c\ell}(\{h, м, z\}) = Ꞩ$, then $\sim_{c\ell}(Ⱨ \cup Ⱪ) \nsubseteq \sim_{c\ell}(Ⱨ) \cup \sim_{c\ell}(Ⱪ)$.*

Now, we will review the definition of $\sim_{ext}$ using the $\sim$-operator and we will study its properties.

Definition 3.18: Let $Ꞩ_{Ɛ}^{Ŧ_\sim}$ be a $\sim$-topological space. A $\sim$-exterior of $Ⱨ \subseteq Ꞩ$ denoted by $\sim_{ext}(Ⱨ)$ and we define the $\sim$-exterior of $Ⱨ$ by $\sim_{ext}(Ⱨ) = \sim_{int}(Ꞩ - Ⱨ)$. In the other word $ꞩ \in \sim_{ext}(Ⱨ)$ iff there exists a $\sim$-open set $Ꞧ$ such that $ꞩ \in Ꞧ \subseteq Ꞩ - Ⱨ$ or $ꞩ \in Ꞧ$ and $Ꞧ \cap Ⱨ = \emptyset$.

Proposition 3.19: Let $Ꞩ_{Ɛ}^{Ŧ_\sim}$ be a $\sim$-topological space, and $Ⱨ \subseteq Ꞩ$, then $\sim_{ext}(Ⱨ) = \cup\{Ꞧ : Ꞧ \text{ is a } \sim\text{-open set}, Ꞧ \subseteq Ꞩ - Ⱨ\}$.

Example 3.20: $Ꞩ = \{h, ĸ, м, z\}$, *where $Ꞩ$ represents some of the car manufacturers in the world, (h) represents cars manufactured by ford, $(ĸ\ )$ represents cars manufactured by Nissan, $(м)$ represents cars manufactured by Hyundai, and finally (z) represents cars manufactured by BMW, $Ŧ_\sim = \{Ꞩ, \emptyset, \{h, ĸ\}, \{h, м\}, \{h, ĸ, z\}\}$ and $Ɛ = \{\{h\}\}$. If $Ⱨ = \{h, м, z\}$, then $\sim_{ext}(Ⱨ) = \sim_{int}(Ⱨ - Ꞩ) = \emptyset$.*

Proposition 3.21: Let $Ꞩ_{Ɛ}^{Ŧ_\sim}$ be a $\sim$-topological space, and $Ⱨ, Ⱪ \subseteq Ꞩ$, then:

1) $\sim_{ext}(\emptyset) = Ꞩ$ and $\sim_{ext}(Ꞩ) = \emptyset$.
2) $\sim_{ext}(Ⱨ) \subseteq Ꞩ - Ⱨ$.
3) $\sim_{ext}(Ꞩ - \sim_{ext}(Ⱨ)) \subseteq \sim_{ext}(Ⱨ)$.
4) If $Ⱨ \subseteq Ⱪ$, then $\sim_{ext}(Ⱪ) \subseteq \sim_{ext}(Ⱨ)$.
5) $\sim_{ext}(Ⱨ \cup Ⱪ) \subseteq \sim_{ext}(Ⱨ) \cap \sim_{ext}(Ⱪ)$.

6) $\sim_{ext}(Ḩ) \cup \sim_{ext}(Ķ) \subseteq \sim_{ext}(Ḩ \cap Ķ)$.
7) $\sim_{int}(Ḩ) \subseteq \sim_{ext}(\sim_{ext}(Ḩ))$.

Proof of (7): Since $\sim_{int}(Ḩ) \subseteq Ḩ$, we get $Ꞩ - Ḩ \subseteq Ꞩ - \sim_{int}(Ḩ)$, so $\sim_{int}(Ꞩ - Ḩ) \subseteq \sim_{int}(Ꞩ - \sim_{int}(Ḩ)) = \sim_{ext}(\sim_{int}(Ḩ)) = \sim_{ext}(\sim_{ext}(Ꞩ - Ḩ))$, so $\sim_{int}(Ḩ) \subseteq \sim_{ext}(\sim_{ext}(Ḩ))$.

Remark 3.22:

1) If $Ꞩ^{T\sim}_{\mathcal{E}}$ is satisfy the λ-additive property then, $\sim_{ext}(Ꞩ - \sim_{ext}(Ḩ)) = \sim_{ext}(Ḩ)$, $\forall Ḩ \subseteq Ꞩ$.
2) If $Ꞩ^{T\sim}_{\mathcal{E}}$ is satisfy the I-property then, $\sim_{ext}(Ḩ \cup Ķ) = \sim_{ext}(Ḩ) \cap \sim_{ext}(Ķ)$, $\forall Ḩ, Ķ \subseteq Ꞩ$.
3) In general, the equality is not achieved of (part 2), see (Example 3.20), then $Ꞩ - Ḩ \nsubseteq \sim_{ext}(Ḩ)$.
4) The converse of (part 4) is not necessary true, see (Example 3.20), if $Ḩ = \{м\}$ and $Ķ = \{ḩ, κ\}$, then $\sim_{ext}(Ḩ) = \{ḩ, κ, ȥ\}$ and $\sim_{ext}(Ķ) = \emptyset$, then $\sim_{ext}(Ķ) \subseteq \sim_{ext}(Ḩ)$, but $Ḩ \nsubseteq Ķ$.
5) In general, the equality is not achieved of (part 5), see (Example 3.20), if $Ḩ = \{м, ȥ\}$ and $Ķ = \{κ, ȥ\}$, then $\sim_{ext}(Ḩ) = \{ḩ, κ\}$ and $\sim_{ext}(Ķ) = \{ḩ, м\}$, but $\sim_{ext}(Ḩ \cup Ķ) = \emptyset$, we get $\sim_{ext}(Ḩ) \cap \sim_{ext}(Ķ) \nsubseteq \sim_{ext}(Ḩ \cup Ķ)$.
6) In general, the opposite direction is not true of (part 6), see (Example 3.20), if $Ḩ = \{ḩ\}$ and $Ķ = \{κ\}$, then $\sim_{ext}(Ḩ) = \emptyset$ and $\sim_{ext}(Ķ) = \{ḩ, м\}$, but $\sim_{ext}(Ḩ \cap Ķ) = Ꞩ$, therefore $\sim_{ext}(Ḩ \cap Ķ) \nsubseteq \sim_{ext}(Ḩ) \cup \sim_{ext}(Ķ)$.
7) The converse of (part 7) is not necessary true, see (Example 3.20), if $Ḩ = \{ḩ\}$, then $\sim_{ext}(\sim_{ext}(Ḩ)) = Ꞩ$, but $\sim_{int}(Ḩ) = \emptyset$, therefore $\sim_{ext}(\sim_{ext}(Ḩ)) \nsubseteq \sim_{ext}(Ḩ)$.

4. Conclusion

In general, ∼-topological space does not constitute a topology because the process of creating this operator was done through the binary ∼-operation. Also, the intersection and union process with respect to ∼-open set is not closed process this can be achieved provided that I-property and λ–additive property are respectively. Also, we showed that ∼-interior and ∼-closure do not constitute ∼-open and ∼-closed respectively. Also, the basic concepts in topology can be generalized using the ideas in this paper and the definition of dual soft set can be developed in [11]. The results can also be generalized in [12].

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Conflict of interest

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