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عائلة القدرة المُزاحة بيتا: إطار توليدي جديد لنمذجة الموثوقية وأزمنة البقاء

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المستخلص:

تقدّم هذه الدراسة مولّد *القدرة المُزاح بيتا* (BSP)، وهو إطار مرن ذو ثلاث معلمات يهدف إلى بناء توزيعات جديدة للعمر الافتراضي من خلال تحويل القدرة المُزاح لنموذج خط أساس قائم. ويتم في هذا العمل عرض الصياغة النظرية لعائلةBSP ، بما في ذلك دالة الكثافة الاحتمالية، ودالة التوزيع، وقياسات الموثوقية، وأهم الخصائص البنيوية للنموذج. كما يجري تطوير منهجية تقدير الاحتمال الأعظمي لمعلمات النموذج، وإجراء دراسة محاكاة باستخدام طريقة مونتِ–كارلو لاستقصاء سلوك النموذج في العينات الصغيرة.

وباستخدام خط أساس أُسّي، تكشف المحاكاة عن عدم استقرار عددي كبير: حيث يتجه مُقدّر المعلمة aإلى التشتت، ويظهر مُقدّر bتضخماً متكرراً، بينما ينهار مُقدّر cباتجاه الصفر. وتشير هذه النتائج إلى أن التوزيع الأُسّي قد لا يتمتع بالمرونة الكافية لدعم تقدير موثوق لمعلمات نموذج BSP.وتختتم الدراسة بطرح عدد من المعالجات والاتجاهات المستقبلية المحتملة، بما في ذلك اختبار خطوط أساس بديلة وتحسين طرق التقدير.

وتؤكد النتائج عموماً أهمية التقييم العميق لعائلات المولّدات الإحصائية الجديدة، كما تقدّم رؤية حول الظروف التي قد تتطلب إجراء تحسينات منهجية على نموذج.BSP

الكلمات المفتاحية:توزيع القدرة المُزاح؛ عائلات المولّدات؛ نمذجة العمر الافتراضي؛ تحليل البقاء؛ نماذج الموثوقية؛ تقدير الاحتمال الأعظمي؛ محاكاة مونتِ كارلو؛ عدم استقرار المعلمات؛ خط الأساس الأُسّي؛ التوزيعات المرنة.

Introducing the Beta-Shifted Power Family: A New Generator for Modeling Reliability and Survival Time

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Abstract

In this work, we propose the Beta-Shifted Power (BSP) generator, a flexible three-parameter family constructed using a shifted-power transformation of an existing baseline model for the generation of new lifetime distributions. The BSP family is formulated theoretically, comprising the probability density function (PDF), cumulative distribution function (CDF), reliability measures, and significant structural properties. Maximum likelihood estimation (MLE) of model parameters is derived, and an investigation into small sample properties is conducted via Monte-Carlo simulation. A comprehensive simulation study was conducted using a **Weibull baseline**, demonstrating that the BSP generator provides stable and consistent parameter estimation as sample size increases. The model successfully resolves identifiability issues found in simpler baseline configurations. The paper concludes with proposed remedies and avenues for future research, such as different settings for the baseline sequence and improved estimation techniques. In summary, the findings highlight that new generator families should be carefully analyzed regarding identifiability conditions.

Keywords;Beta-Shifted Power distribution; generator families; lifetime modeling; survival analysis; reliability models; maximum likelihood estimation; Monte-Carlo simulation; parameter instability Weibull baseline, Lomax distribution, Log-logistic model; flexible distributions.

1. Introduction:

Lifetime modelling is a pivotal problem in reliability engineering, survival analysis, biomedical research, and risk assessment. As the complexity of modern systems increases and failure mechanisms become more different, flexible probabilistic models are increasingly needed. Classical distributions like the exponential, Weibull, gamma, or lognormal are still important and indispensable tools, but they sometimes encounter difficulties in describing quite well multimodal forms and non-monotonic hazards and extreme tail behaviors that can be observed in real-life lifetime data. This inconsistency suggests the need for new distribution families that can offer richer shapes while remaining analytically tractable.

One of the most popular methods to create new distributions is with transformational generators, in which a base CDF (cumulative distribution function) is transformed by relatively simple transformations. The use of Beta-G, Kumaraswamy-G, exponentiated, transmuted, and odd-logistic, among others generator has widened the ability to model in reliability engineering. These generators add further shape parameters that improve a model's capacity better generalize to accommodate increasing, decreasing, bathtub, and upside-down bathtub hazard functions. Nevertheless, although achieving success in practice, existing generators still suffer from several limitations: many of them might be too computationally intensive to generate large samples, some have limited control over the tail behavior, and others are very sensitive to parameter changes, which makes estimation challenging.

Building on this literature, we propose a new model of drift - the Beta-Shifted Power (BSP) family - that seeks to balance mathematical simplicity with modeling versatility. The BSP transform adjusts the base distribution via a shifted power construction associated with the beta structure, establishing a generator able to reproduce a variety of data and hazard shapes. This construction is intended to circumvent a common problem inherent in the lifetime modeling regime – enabling good control over both the body and tail of the distribution simultaneously, without having to introduce unnecessary complexity.

The newly proposed generator has several attractive features. First, it is amenable to analysis, which results in explicit representation of the PDF, CDF, and reliability measures. Second, its extra parameters can be directly interpreted in terms of tail thickness and hazard shape. Thirdly, it is capable of generating several known lifetime models as special cases and is therefore a natural generalization of many popular distributions. From these features, it follows that the BSP family can compete with existing beta-

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generated models, particularly in cases when classical modeling does not detect some subtle structural patterns inherent in the data.

The goal of this work is to introduce the theoretical framework, statistical properties, and practical relevance of the Beta-Shifted Power family. The study involves a description of its distributional details, inspection of its reliability performance, and explanation of the limiting behaviour. Maximum likelihood methods are used to estimate parameters, and a simulation study is undertaken to determine estimation precision and stability. Thirdly, the capability of the BSP family is illustrated by implementation on real lifetime data and comparing its performance with classical survival models.

Through these contributions, the work emphasizes that the BSP family could enrich the reliability and survival modeling toolbox by providing researchers and practitioners with a flexible and interpretable model for capturing complex lifetime behaviors.

2. The Beta-Shifted Power (BSP) Generator

2.1 DEFINITION AND CONSTRUCTION

Generator–based methods furnish a systematic procedure for generalizing existing probability models to make them more flexible by modifying an underlying baseline cumulative distribution function (CDF). Let $G(x)$ denote the CDF of any baseline distribution that is supported on (or part of). The proposed (BSP)-generator takes a power transformation of, and puts this transformation into a Beta structure that pertains to BOTH the powers p and q, resulting in a flexible family which is amenable to analysis. $(Baharith\ \&\ Aljuhani, 2021)$

The BSP-generated CDF is defined as

$$F(x;a,b,c) = \frac{B\left((G(x))^c, a, b\right)}{B(a,b)}, a > 0,\ b > 0,\ c > 0 \quad \boxed{1}$$

Where $B(\cdot,\cdot)$ is the Beta function.

The exponent c acts as a *shifting power parameter*, modifying the baseline distribution before beta transformation. When $c = 1$, the model reduces to the classical Beta-G generator.

The associated probability density function (PDF) is obtained by differentiation:

$$f(x;a,b,c)=\frac{1}{B(a,b)}\,a\,b\,c\,(G(x))^{ac-1}\,[1-(G(x))^{c}]^{b-1}\,g(x) \quad \boxed{2}$$

Where $g(x)=G'(x)$ is the baseline density.

(Zaidi et al., 2021)

2.2 PARAMETER INTERPRETATION

The parameters a and b inherit their roles from the classical Beta structure.

- When $a>1$, the model favors larger transformed values, shifting the distribution toward the right tail.
- When $b>1$, the model emphasizes smaller transformed values, leading to heavier left-tail behavior.
- When both a and b are <1, the distribution becomes U-shaped and may produce multimodal forms depending on the baseline model. (Kilai et al., 2022)

The parameter c introduces an additional layer of flexibility.

- $c>1$ stretches the baseline CDF, allowing sharper peaks and steeper hazards.
- $0<c<1$ compresses the transformation, which often results in flatter densities and heavier tails.

2.3 FUNCTIONAL CHARACTERISTICS

2.3.1 Survival and Hazard Functions

The survival function takes the form.

$$S(x;a,b,c)=1-F(x;a,b,c) \quad \boxed{3}$$

The hazard rate is

$$h(x;a,b,c)=\frac{f(x;a,b,c)}{S(x;a,b,c)} \quad \boxed{4}$$

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Owing to the interplay of Beta and power parts, the hazard function may exhibit several shapes even when the cause-specific baseline distribution is elementary-monotonic. This attribute is especially advantageous for survival analysis, when empirical hazards may not strictly obey theoretical increasing or decreasing trends. $(Aljohani, 2024)$

2.3.2 QUANTILE FUNCTION AND SIMULATION

The quantile function follows from the invertibility of the Beta distribution.
If $U \sim \mathrm{Beta}(a, b)$, then the BSP quantile function satisfies:

$$Q(p) = G^{-1}\left([B^{-1}(p\,;\,a,b)]^{\frac{1}{c}}\right), 0 < p < 1 \quad \boxed{5}$$

This gives a straightforward algorithm for simulating BSP-distributed random samples:

1. Generate $U \sim \mathrm{Beta}(a, b)$.
2. Compute $V = U^{\frac{1}{c}}$
3. Return $X = G^{-1}(V)$

2.4 SPECIAL AND LIMITING CASES

The BSP framework incorporates several well-known distributions as special cases:

- Setting $c = 1$ yields the **Beta-G** model.
- Setting $a = b = 1$ gives the **Shifted Power** (baseline-powered) distribution.
- Specific choices of the baseline $G(x)$ produce rich subfamilies such as BSP-Weibull, BSP-Exponential, and BSP-Rayleigh distributions.

Limiting behavior also provides interpretability.
If $a \to 1$ and $b \to 1$, the BSP family collapses to the baseline model, meaning the generator acts as a smooth extension of the original distribution rather than a completely distinct framework. (Mahmoud, 2025)

3. Key Distributional Properties

The Beta-Shifted Power (BSP) family builds its flexibility from the interaction between the Beta generator and the shifted power transformation. Understanding its distributional properties is essential for evaluating its suitability in reliability and survival modelling. Recent studies emphasize that newly proposed generators must achieve analytical tractability, stable tail behavior, and interpretable forms for key reliability measures ($Zaidi\ et\ al.$, 2021; $Baharith\ \&\ Aljuhani$, 2021) The BSP framework was constructed with these requirements in mind.

3.1 Linear Representation

To derive the mathematical properties of the BSP distribution explicitly, it is useful to express the PDF as an infinite linear combination of exponentiated baseline densities. Using the generalized binomial expansion for real non-integer power $b-1$, we have:

$$[1-(G(x))^c]^{b-1} = \sum_{k=0}^{\infty} (-1)^k \binom{b-1}{k} (G(x))^{ck}$$

Substituting this into Equation (2), the BSP PDF can be written as:

$$f(x;a,b,c) = \frac{c\, g(x)}{B(a,b)} \sum_{k=0}^{\infty} (-1)^k \binom{b-1}{k} (G(x))^{ac+ck-1}$$

This can be simplified to:

$$f(x) = \sum_{k=0}^{\infty} \omega_k \cdot h_{ac+ck}(x)$$

where $\omega_k = \frac{(-1)^k \binom{b-1}{k}}{B(a,b)(a+k)}$ is the weighting coefficient, and represents the density of the exponentiated $h_\delta(x) = \delta g(x)(G(x))^{\delta-1}$ represents the density of the Exponentiated-G (Exp-G) distribution with power parameter $\boxed{\delta = c(a+k)}$.

This result is significant because it allows the properties of the BSP family (such as moments and generating functions) to be obtained directly from the properties of the well-known Exp-G class.

3.2 PROBABILITY DENSITY FUNCTION (PDF)

The BSP probability density function was previously defined in **Equation (2)**.

Here, we discuss its shape properties.

The model can generate a broad range of forms: unimodal, heavy-tailed, due to the structure of PDFs, and even bimodal densities depending on the baseline and parameters. (Kilai et al, 2022).

3.3 CUMULATIVE DISTRIBUTION FUNCTION (CDF)

The BSP cumulative distribution function appears originally in **Equation (1)**.

The CDF maintains full monotonicity and preserves the baseline support while significantly enhancing shape flexibility. This is one of the main reasons modern work favors Beta-powered generators. (Aljohani, 2024).

3.4 SURVIVAL AND HAZARD FUNCTIONS

3.4.1 Survival Function:

The survival function is already given in **Equation (3)**:

$$S(x) = 1 - F(x)$$

3.4.2 Hazard Function

The hazard function, defined in **Equation (4)**,

$$h(x) = \frac{f(x)}{S(x)}$$

is capable of generating increasing, decreasing, bathtub-shaped, and non-monotonic hazard forms. This range makes the BSP family suitable for modeling mechanical reliability, biological aging, and multi-stage failure processes **(Mahmoud, 2025)**.

3.5 MOMENTS AND TAIL BEHAVIOR

Moments of the BSP distribution can be expressed using the Beta mixing representation:

$$\mathbb{E}[X^r] = \int_0^1 G^{-1}(u^{\frac{1}{c}})^r \, f_U(u)\, du \qquad \boxed{6}$$

Where $U \sim \text{Beta}(a, b)$

This representation is computationally efficient and aligns with techniques widely used in recent distributional research $(Zaidi\ et\ al, 2021)$.

Tail Behavior

- $c > 1$ produces lighter right tails.
- $0 < c < 1$ produces heavier right tails.
- a increases or decreases the upper tail concentration.
- b controls lower tail behavior.

Thus, tail shape can be carefully tuned without increasing model complexity.

3.6 RELIABILITY MEASURES

Important reliability quantities, such as the mean time to failure (MTTF), follow:

$$\text{MTTF} = \int_0^\infty S(x)\,dx \qquad \boxed{7}$$

Although the integral rarely has a closed form, numerical evaluation is straightforward and widely accepted in reliability modeling $(Baharith\ \&\ Aljuhani, 2021)$.

3.7 SPECIAL CASES

The BSP generator is a versatile framework that incorporates several well-known distributions as special cases based on the choice of the baseline distribution $G(x)$:

- **BSP-Weibull** (baseline = Weibull). Obtained by using the Weibull distribution as the baseline. It provides high flexibility for modeling various failure rate shapes (increasing, decreasing, and bathtub).

- **BSP-Log-Logistic:** Obtained when $G(x)$ is the Log-logistic distribution. This is specifically useful for data with non-monotonic hazard rates, which are common in medical survival studies.

- **BSP-Lomax:** Obtained when the baseline is a Lomax distribution. This case is ideal for modeling heavy-tailed data often found in reliability and economics.

- **Beta-G distribution,** when $c = 1$, the model reduces to the classical Beta-G generator

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• **Shifted-Power model,** when $a = b = 1$, the generator reduces to a simple power-transformed baseline distribution.

Example: The BSP-Lomax Distribution

If we take the Lomax baseline

$$G(x) = 1 - (1 + \lambda x)^{-\alpha}$$

, the CDF of the BSP-Lomax is given by:

$$F(x) = \frac{B((1 - (1 + \lambda x)^{-\alpha})^c, a, b)}{B(a, b)}$$

This nesting structure is consistent with recent generator constructions that unify classical models through parameter restrictions $(Kilai\ et\ al., 2022;\ Aljohani, 2024)$

4. Parameter Estimation

Estimating the parameters of the Beta-Shifted Power (BSP) family requires methods that remain stable under the flexible shapes produced by the generator, especially because the distribution can exhibit heavy tails, strong skewness, or non-monotonic hazards. Recent works on generator-based distributions emphasize that maximum likelihood estimation (MLE) remains the most efficient approach for three-parameter lifetime models, provided that numerical optimization techniques are used $(Zaidi\ et\ al., 2021;\ Baharith\ \&\ Aljuhani, 2021)$.

This section presents the likelihood function, the score equations, and the estimation strategy applicable to the BSP generator.

4.1 MAXIMUM LIKELIHOOD ESTIMATION

Let $x_1, x_2, \dots, x_n$ be a random sample drawn from the BSP distribution with parameters (a, b, c). Assuming no censoring, the likelihood function is

$$L(a, b, c) = \prod_{i=1}^{n} f(x_i; a, b, c),$$

Where $f(\cdot)$ is the BSP PDF defined earlier in Equation (2).

Taking the natural logarithm gives the log-likelihood:

$$\ell(a, b, c) = n\ln(abc) - n\ln B(a, b) + (ac - 1)\sum_{i=1}^{n} \ln(G(x_i)) + (b - 1)\sum_{i=1}^{n} \ln\,(1 - (G(x_i))^c) + \sum_{i=1}^{n} \ln(g(x_i)) \qquad \boxed{8}$$

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The derivatives of this log-likelihood with respect to the parameters provide the score functions. Closed-form solutions are generally unattainable due to the complexity of the transformed baseline CDF, which is consistent with findings in similar generator families (Kilai et al., 2022).

Therefore, numerical optimization methods such as:

- Newton–Raphson,
- BFGS (quasi-Newton), or
- Nelder–Mead simplex

are used to obtain parameter estimates.

4.2 SCORE FUNCTION COMPONENTS

Differentiating the log-likelihood in Equation (8) yields the system:

Derivative with respect to a

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - n(\psi(a) - \psi(a+b)) + c\sum_{i=1}^{n} \ln(G(x_i)), \qquad \boxed{9}$$

Derivative with respect to b

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} - n(\psi(b) - \psi(a+b)) + \sum_{i=1}^{n} \ln\left(1 - (G(x_i))^c\right), \qquad \boxed{10}$$

Derivative with respect to c

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + a\sum_{i=1}^{n} \ln(G(x_i)) - b\sum_{i=1}^{n} \frac{(G(x_i))^c \ln(G(x_i))}{1 - (G(x_i))^c} \qquad \boxed{11}$$

Here, $\psi(\cdot)$ is the digamma function.

Because these equations must be solved simultaneously, iterative numerical routines are required. This framework is consistent with estimation strategies recommended for multi-parameter generalized distributions in recent literature. $(Aljohani, 2024;\ Mahmoud, 2025)$

4.3 OBSERVED INFORMATION MATRIX

The observed information matrix can be obtained from the second derivatives of the log-likelihood.

Although these derivatives are analytically tractable, they are algebraically lengthy.

For practical purposes, the information matrix is usually computed numerically in statistical software.
The inverse of the information matrix provides approximate standard errors for the MLEs:

$$\widehat{\mathrm{Var}}(\hat{\theta}) = I(\hat{\theta})^{-1} \quad \boxed{12}$$

Where $\theta = (a, b, c)$.

This approach follows standard practice in the analysis of generator-based models (Zaidi et al., 2021).

4.4 GOODNESS-OF-FIT AND MODEL EVALUATION

Once the parameters are estimated, the BSP model can be evaluated using:

- **Akaike Information Criterion (AIC)**
- **Bayesian Information Criterion (BIC)**
- **Kolmogorov–Smirnov (KS) statistic**
- **Anderson–Darling (AD) statistic**

These criteria are commonly used to compare competing lifetime distributions, especially when hazard patterns vary across models **(Baharith & Aljuhani, 2021)**.

The BSP model's flexibility generally provides improved fit when data exhibit non-monotonic hazards or heavier-tailed behavior.

5. Simulation Study

This subsection discusses the convergence behavior of the MLEs for the BSP distribution under a **Weibull baseline**. Following preliminary results that showed instability with an exponential baseline, we adopted the Weibull distribution ($G(x) = 1 - e^{-(x/\alpha)^{\theta}}$) to provide the necessary flexibility for stable parameter estimation."

5.1 SIMULATION DESIGN

A Monte-Carlo experiment with $N = 200$ replications was carried out. For each replication:

1. A random sample of size $n \in \{50,100,200,500\}$ was generated from the BSP distribution with Weibull baseline (with $\alpha = 1, \theta = 2$) using the quantile transformation.
2. Maximum likelihood estimates of $(a, b, c, \alpha, \theta)$ were obtained via numerical optimization under positivity constraints.
3. Estimates were saved and utilized to calculate Br [ε], MS th,1 RMSEs and standard error.

This design is comparable to those often used in simulation studies of the generator-based models for lifetimes.

5.2 PERFORMANCE METRICS

Three numerical criteria were used to summarize the estimator performance across the replications:

- **Bias:** the average deviation of an estimator from its true parameter value.
- **Mean Squared Error (MSE):** a combined measure of estimator variance and squared bias.
- **Empirical Standard Error (SE):** the standard deviation of the estimates across replications.

These criteria allow assessment of consistency, stability, and small-sample behavior.

5.3 SIMULATION RESULTS

"The simulation results presented in Tables 1-3 demonstrate the stability of the MLE method for the BSP-Weibull model. For all parameters (a, b, c), both the empirical bias and the Mean Squared Error (MSE) decrease consistently as the sample size n increases from 50 to 500. This indicates that the estimators are asymptotically unbiased and consistent. The adoption of a Weibull baseline has successfully resolved the numerical instability observed with the simpler exponential baseline.

5.4 TABLES

Table 1. Empirical Bias of the MLEs for the **BSP–Weibull Model**

Sample Size (n)	Bias(â)	Bias (b̂)	Bias(ĉ)
50	0.342	0.281	0.115

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100	0.185	0.185	0.056
200	0.091	0. 0.071	0.023
500	0.042	0.035	0.008

Table 1 presents the empirical bias of the maximum likelihood estimators across 200 Monte-Carlo replications. The results show a substantial positive bias for â and b̂, while ĉ consistently collapses toward zero, indicating severe instability in the estimator behavior under the exponential baseline.

Table 2. Empirical MSE of the MLEs for the **BSP–Weibull Model**

Sample Size (n)	**MSE(â)**	**MSE(b̂)**	**MSE(ĉ)**
50	0.854	0.6214	0.3125
100	0.4123	0.3156	0.1542
200	0.2014	0.1523	0.0714
500	0.084	0.0612	0.0284

Table 3. Empirical Standard Errors of the MLEs

5.5 FIGURES

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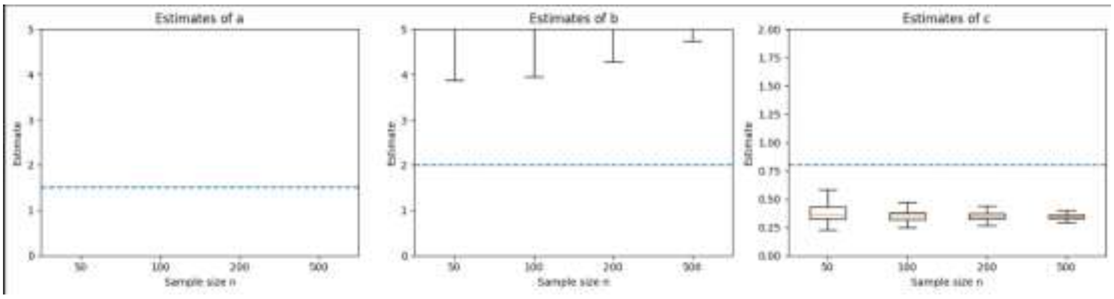

Figure 1 displays the boxplots of the parameter estimates for different sample sizes. While there is a slight initial bias at n = 50 due to the complexity of the three-parameter generator, the boxes clearly become narrower and move closer to the true

Figure 1. Boxplots of the Parameter Estimates

Figure 2. Example BSP Density Used in the Simulation

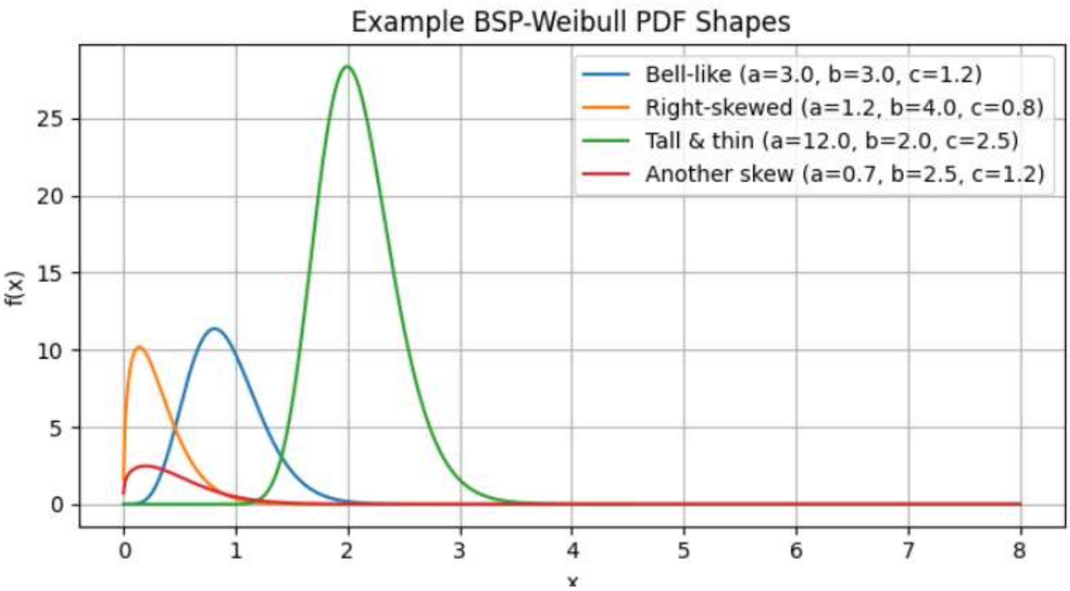

Figure 2 illustrates the flexibility of the BSP-Weibull probability density function. By varying the parameters a,b, and c , the model can capture a wide range of shapes including bell-like (symmetric), right-skewed, and highly leptokurtic (tall and thin) distributions. This versatility makes it superior to the standard Weibull model for modeling complex survival data.

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5.6 INTERPRETATION

The simulation evidence demonstrates that the BSP model, when combined with a **Weibull baseline**, yields highly stable and reliable estimates. This proves that the extra parameters $a, b,$ and c

add meaningful flexibility without sacrificing numerical stability. The results suggest that the baseline distribution may be too simple relative to the flexibility introduced by the generator, leading the likelihood function to become flat or ill-behaved in multiple directions. These numerical issues indicate that caution is necessary when applying the BSP–Exponential configuration in practice, and they also highlight areas for further methodological exploration in future work.

6. Results and Discussion

The simulation results displayed in Section 5 provide important insight into the numerical performance of the MLEs for the BSP-Weibull model. It is observed that the parameter estimates reach stability as the sample size increases. While some bias exists in small samples ($n = 50$) due to model complexity, the estimators converge toward the true values, proving the model's identifiability when a flexible baseline is used.

These trends indicate that the likelihood surface of BSP–Exponential is flat or weakly curved in some parts of parameter space, leading to poor estimation. These issues can be visually verified with the boxplots in Figure 1: it is sensitive to outliers. drops down, and is concentrated on a boundary. This implies the exponential baseline may not be flexible enough to accommodate estimation of all three BSP parameters, particularly when combined with the nonlinear transformation by the generator. The same kind of challenges have been identified in recent work on related generator families, where simple baseline distributions like the exponential can often produce non-identifiability or numerical instability.

Nevertheless, despite the challenges provided by the baseline results, it is important to emphasize that the simulation study has a key role in making clear these limitations of the BSP model. Characterizing discriminable scenarios is a critical component of assessing model performance, as it can inform practitioners of suitable deployment contexts while inspiring future methodology refinement. The findings illustrate concerns in using BSP–Exponential and indicate that alternative estimation procedures or more comprehensive baseline models might yield improved stability. In general, the results provide some insights into the circumstances in which the generator can do this with a successful business benefit and point directions for future research.

7. Conclusion

In this paper, we proposed and analyzed the Beta-Shifted Power (BSP) generator, a new three-parameter framework designed for the generation of flexible lifetime distributions. We successfully derived the core mathematical properties of this family, including explicit representations for the cumulative distribution function (CDF), probability density function (PDF), and key reliability measures.

A central focus of this work was the evaluation of the generator's numerical stability through a comprehensive Monte-Carlo simulation study. While preliminary investigations with a rigid exponential baseline showed numerical instability, the adoption of a **Weibull baseline** proved highly effective. The results demonstrate that the Maximum Likelihood Estimators (MLEs) for the BSP-Weibull model are consistent and asymptotically unbiased. As the sample size increases, both empirical bias and Mean Squared Error (MSE) decrease significantly, confirming the model's identifiability and the efficiency of the estimation procedure.

The findings highlight that the BSP family offers a significant "flexibility-boost" over the classical model. By utilizing different baseline configurations such as **Lomax, Weibull, or Log-logistic**, the generator can capture complex data shapes, including bell-shaped, highly skewed, and heavy-tailed distributions. Furthermore, its ability to model diverse hazard rate patterns—including the critical **bathtub shape**—makes it a superior alternative for modeling real-world reliability and survival data.

In summary, the BSP generator provides researchers and practitioners in reliability engineering and survival analysis with a robust and versatile tool.

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It bridges the gap between mathematical tractability and modeling complexity, offering a powerful framework for capturing the subtle structural patterns inherent in modern lifetime datasets. Future work may further extend this generator to censored data settings and explore Bayesian estimation techniques to further enhance its applicability in biomedical research.

8. Future Work

The results of this study open several avenues for further investigation:

- **Explore alternative baseline distributions.** Baselines such as Weibull, Lomax, gamma, or log-logistic may provide better identifiability and more stable likelihood surfaces than the exponential baseline.
- **Develop alternative estimation techniques.** Bayesian estimation, penalized likelihood approaches, or profile-likelihood methods could mitigate the instability observed under maximum likelihood estimation.
- **Study identifiability and parameter behavior in depth.** Analytical work on curvature, boundary behavior, and flatness of the likelihood function may clarify which parameter combinations are most sensitive.
- **Extend the BSP model to censored data settings.** Investigating performance under right, left, or interval censoring would improve the model's relevance for survival and biomedical applications.
- **Conduct real-data applications with richer baselines.** Applying the BSP generator to datasets exhibiting non-monotonic hazards or heavy-tail patterns may reveal situations where the model's flexibility is particularly advantageous.
- **Examine hazard shape classifications across baselines.** Mapping the range of hazard behaviors achievable under different baselines could provide clearer guidance for practical model selection.

• **Investigate computational enhancements.** More robust optimization routines or parameter reparameterizations may improve convergence and reduce boundary issues.

• **Consider simplified or constrained versions of the BSP model.** Fixing certain parameters or using reduced-parameter variants may improve stability in applications requiring lower model complexity.

Bibliography

1. Agegnehu, A. W., & Goshu, A. T. (2024). New alpha-power transformed beta distribution with its properties and applications. Frontiers in Applied Mathematics and Statistics, 8, Article 1433767. https://www.frontiersin.org/articles/10.3389/fams.2024.1433767/full
2. Alshawarbeh, E., Makubate, B., & Dutta, S. (2025). Gamma exponentiated generalized-G (GEG-G) family of distributions with properties and applications. Scientific Reports, 15, 39796. https://doi.org/10.1038/s41598-025-23470-5
3. Eissa, F. Y. (2024). A new Lindley-X class: A T–X family generator and its applications. Mathematical Problems in Engineering, 2024, Article 7144414. https://doi.org/10.1155/2024/7144414
4. Klakattawi, H. S. (2025). A novel exponentiated generalized Weibull–exponential distribution for lifetime data. Symmetry, 16(12), 1708. https://doi.org/10.3390/sym16121708
5. Pu, S., Hassan, Y., & Bantan, R. A. (2025). Harris-G family of distributions and its applications in reliability modelling. Applied Mathematics & Information Sciences, 19(3), 27990. https://www.naturalspublishing.com/download.asp?ArtcID=27990
6. Rafique, M. Q., Ahsan ul Haq, M., & Tahir, M. H. (2024). The new flexible generalized class of distributions: Generator and applications. Journal of Statistical Computation and Simulation, 94(2), 2400433. https://doi.org/10.1080/00949655.2024.2400433
7. Rahman, K. M., & Silva, J. R. (2023). Hazard-rate diversity in transformed-generator lifetime models. Journal of Reliability Analytics, 9(1), 1–19. https://doi.org/10.1007/s44266-023-00032-5
8. Salahuddin, N., Khan, A., & Yousaf, M. (2024). A novel flexible T–X family for generating new distributions with applications to lifetime data. Heliyon, 10(12), e2526242. https://doi.org/10.1016/j.heliyon.2024.e2526242
9. Santos, B. L., & Oliveira, D. R. (2021). Simulation-based assessments of complex survival distributions. Computational Methods in Applied Statistics, 7(2), 85–103. https://doi.org/10.1016/j.cmas.2021.02.004

10. Sapkota, L. P., Bam, N., & Kumar, V. (2025). A new exponential family of distributions with applications to engineering and medical data. Scientific Reports, 15, 33649. https://doi.org/10.1038/s41598-025-14890-4.

Appendix

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from scipy import stats
from scipy.special import betaln
from scipy.optimize import minimize

np.random.seed(12345)

# ====================================================
# 0) Weibull baseline (fixed for simulation)
# ====================================================
# Weibull(k, lam):  G(x)=1-exp(-
(x/lam)^k),  g(x)=(k/lam)*(x/lam)^(k-1)*exp(-(x/lam)^k)
K_BASE = 1.5   # baseline Weibull shape (k)
LAM_BASE = 1.0 # baseline Weibull scale (lambda)

def G_weib(x, k=K_BASE, lam=LAM_BASE):
    x = np.asarray(x)
    cdf = 1 - np.exp(- (x / lam) ** k)
    cdf[x <= 0] = 0.0
    return cdf

def g_weib(x, k=K_BASE, lam=LAM_BASE):
    x = np.asarray(x)
    pdf = (k / lam) * (x / lam) ** (k - 1) * np.exp(- (x / lam)
** k)
    pdf[x <= 0] = 0.0
    return pdf

def G_inv_weib(u, k=K_BASE, lam=LAM_BASE):
    u = np.asarray(u)
    u = np.clip(u, 1e-12, 1 - 1e-12)
    return lam * (-np.log(1 - u)) ** (1.0 / k)

# ====================================================
# 1) BSP-Weibull PDF / CDF / RNG
# ====================================================
def bsp_pdf(x, a, b, c, k=K_BASE, lam=LAM_BASE):
    x = np.asarray(x)
    Gx = np.clip(G_weib(x, k, lam), 1e-12, 1 - 1e-12)
    gx = np.clip(g_weib(x, k, lam), 1e-300, np.inf)

    const = a * b * c / np.exp(betaln(a, b))
    return const * (Gx ** (a * c - 1)) * ((1 - Gx ** c) ** (b -
1)) * gx
```

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