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On Unitary Quasi-Square Equivalence and Related Classes of Operators

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ABSTRACT: This paper introduces and systematically investigates the notion of unitary quasi-square equivalence for bounded linear operators on Hilbert spaces. This equivalence relation, defined through the unitary equivalence of operator squares, provides a classification that preserves essential spectral and structural properties while capturing higher-order similarities not detectable through classical unitary equivalence. We establish that this relation defines a genuine equivalence relation and explore its connections with fundamental operator classes including square normal operators, n-quasi-normal operators, hyponormal operators, and various isometric structures. Our main contributions include complete characterization of spectral invariants preserved under this equivalence, demonstration of preservation theorems for advanced operator classes and their C^*-algebraic structure, applications to concrete operator families, and development of decomposition theorems revealing the canonical structure of equivalence classes. The theory developed provides a tool for operator classification with applications to invariant subspace problems, similarity theory, and the structural analysis of non-normal operators.

Keywords: Unitary equivalence, Unitary quasi-square equivalence, Square normal operators, Hyponormal operators

1. INTRODUCTION

Equivalence relations among bounded linear operators on Hilbert spaces play a central role in operator theory, providing a framework for understanding the structural and spectral properties that underpin much of modern functional analysis. Among these, unitary equivalence stands out, as it preserves the full algebraic and geometric structure of operators, including their spectra, norms, and multiplicity functions [1,2,3,4]. Through unitary equivalence, operators can be classified up to unitary conjugation, which offers deep insights into spectral theory and invariant subspace problems.However, strict unitary equivalence can sometimes be too rigid to capture more subtle similarities between operators. Two operators might share important structural features without being strictly unitarily equivalent. To address this limitation, mathematicians have introduced weaker forms of equivalence, such as similarity, quasi-similarity, quasi-unitary equivalence, and p-th power unitary equivalence [5,6,7,8]. More recently, approximate unitary equivalence studied by Luketero et.al. [13] has explored asymptotic versions of unitary equivalence through sequences of unitary operators. These generalized relations preserve broader spectral or functional characteristics while relaxing the strict constraints of classical unitary equivalence.

In this spirit, we introduce the concept of unitary quasi-square equivalence, which compares operators through their quadratic forms rather than their direct action. This builds on the work of Nzimbi and Luketero [11] on unitary quasi-equivalence, extending their framework to consider the squared structures of operators. By examining the squares of operators, this relation captures intricate structural similarities that may not be evident under standard unitary equivalence, while still preserving key spectral information associated with the squared operator. This perspective naturally connects to operator classes such as square normal, square hyponormal, isometric, co-isometric, and partial isometries [9,10], where squared structures carry essential analytic information.

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Our work is also inspired by recent advances in generalized equivalence relations, particularly the n-metric equivalence introduced by Wanjala et al. [12] and further developed by Wanjala et al. [15] as metric equivalence of order n, which studies operators satisfying $S^{*n}S=T^{*n}T$ for positive integer n. While our focus is on unitary equivalence of squared structures rather than metric equalities, both approaches aim to classify operators through higher-order algebraic conditions that reveal structural similarities beyond classical equivalence. Furthermore, connections to the preservation of polar decompositions under linear maps, as explored by Bourhim and Mbekhta [14], offer additional insight into how structural components of operators behave under different equivalence relations.

The main goals of this work are threefold. First, we formalize the definition of unitary quasi-square equivalence and show that it indeed defines a valid equivalence relation. Second, we study how this relation interacts with classical operator classes, highlighting which properties are preserved. Finally, we examine its implications for the structure theory of positive operators and explore links with p-th power unitary equivalence, while establishing deeper connections with C*-algebra theory and spectral decomposition theory.

2. PRELIMANARY DEFINITIONS AND CONCEPTS

Definition1(1,2). An operator $\mathcal{U} \in B(H)$ is called unitary if $\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^* = I$, where $\mathcal{U}^*$ denotes the adjoint of $\mathcal{U}$ and I is the identity operator. Unitary operators preserve the inner product *and the norm* $\langle \mathcal{U}x, \mathcal{U}y \rangle = \langle x, y \rangle, \forall x, y \in H$.

Definition 2 (4). Two operators $\mathcal{A}, \mathcal{B} \in B(H)$ are said to be unitarily equivalent if there exists a unitary operator $\mathcal{U}$ such that $\mathcal{A} = \mathcal{U}\mathcal{B}\mathcal{U}^*$. Unitary equivalence preserves the algebraic and spectral properties of operators, including their spectrum and norm.

Definition 3 (9). An operator $\mathcal{A} \in B(H)$ is called square normal if its square commutes with its adjoint square: $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{A}^2\mathcal{A}^{*2}$. This property captures second-order normality and is preserved under quasi-square equivalence.

3. MAIN RESULTS

Definition 4. Two operators $\mathcal{A}, \mathcal{B} \in B(H)$ are said to be unitarily quasi-square equivalent if there exists a unitary operator $\mathcal{U} \in B(H)$ such that:

$$\mathcal{B}^{*2}\mathcal{B}^2 = \mathcal{U}\mathcal{A}^{*2}\mathcal{A}^2\mathcal{U}^* \text{ and } \mathcal{B}^2\mathcal{B}^{*2} = \mathcal{U}\mathcal{A}^2\mathcal{A}^{*2}\mathcal{U}^*.$$

We denote this relation by $\mathcal{A} \approx^{uqe^2} \mathcal{B}$.
Remark 1. Note that $\mathcal{A}^{*2}$ denotes $(\mathcal{A}^*)^2 = (\mathcal{A}^2)^*$, the square of the adjoint, which equals the adjoint of the square. The definition compares the operators $\mathcal{A}^{*2}\mathcal{A}^2$ and $\mathcal{A}^2\mathcal{A}^{*2}$ through unitary equivalence.

Theorem 1. Unitary Quasi-square Equivalence is an equivalence relation.
Proof. Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C} \in B(H)$.
Reflexivity: $\mathcal{A} \approx^{uqe^2} \mathcal{A}$ since taking $\mathcal{U} = I$ gives:

$$\mathcal{A}^{*2}\mathcal{A}^2 = I\mathcal{A}^{*2}\mathcal{A}^2 I^* \text{ and } \mathcal{A}^2\mathcal{A}^{*2} = I\mathcal{A}^2\mathcal{A}^{*2} I^*.$$

Symmetry: If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then there exists unitary $\mathcal{U}$ such that:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* \text{ and } \mathcal{A}^2\mathcal{A}^{*2} = \mathcal{U}\mathcal{B}^2\mathcal{B}^{*2}\mathcal{U}^*.$$

Let $\mathcal{V} = \mathcal{U}^*$, which is unitary. Multiply the first equation on the left by $\mathcal{V}$ and on the right by $\mathcal{V}^*$:

$$\mathcal{V}(\mathcal{A}^{*2}\mathcal{A}^2)\mathcal{V}^* = \mathcal{V}(\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*)\mathcal{V}^* = (\mathcal{V}\mathcal{U})\mathcal{B}^{*2}\mathcal{B}^2(\mathcal{U}^*\mathcal{V}^*) = \mathcal{B}^{*2}\mathcal{B}^2.$$

Similarly, from the second equation: $\mathcal{V}(\mathcal{A}^2\mathcal{A}^{*2})\mathcal{V}^* = \mathcal{B}^2\mathcal{B}^{*2}$. Hence $\mathcal{B} \approx^{uqe^2} \mathcal{A}$ with unitary $\mathcal{V} = \mathcal{U}^*$.

Transitivity: If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and $\mathcal{B} \approx^{uqe^2} \mathcal{C}$, then there exist unitaries $\mathcal{U}, \mathcal{V}$ such that:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* \text{ and } \mathcal{B}^{*2}\mathcal{B}^2 = \mathcal{V}\mathcal{C}^{*2}\mathcal{C}^2\mathcal{V}^*$$

Thus, $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{V}\mathcal{C}^{*2}\mathcal{C}^2\mathcal{V}^*\mathcal{U}^* = \mathcal{Q}\mathcal{C}^{*2}\mathcal{C}^2\mathcal{Q}^*$ for $\mathcal{Q} = \mathcal{U}\mathcal{V}$, which is unitary. Similarly, $\mathcal{A}^2\mathcal{A}^{*2} = \mathcal{Q}\mathcal{C}^2\mathcal{C}^{*2}\mathcal{Q}^*$, proving transitivity.

Theorem 2. If $\mathcal{A}, \mathcal{B} \in B(H)$ are unitarily equivalent, then $\mathcal{A} \approx^{uqe^2} \mathcal{B}$.

Proof. Let $\mathcal{A} = \mathcal{U}\mathcal{B}\mathcal{U}^*$ for some unitary operator $\mathcal{U} \in B(H)$. Then:

$$\mathcal{A}^2 = (\mathcal{U}\mathcal{B}\mathcal{U}^*)(\mathcal{U}\mathcal{B}\mathcal{U}^*) = \mathcal{U}\mathcal{B}^2\mathcal{U}^*$$

and consequently:

$$\mathcal{A}^{*2} = (\mathcal{U}\mathcal{B}^2\mathcal{U}^*)^* = \mathcal{U}\mathcal{B}^{*2}\mathcal{U}^*$$

Therefore:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{U}^*\mathcal{U}\mathcal{B}^2\mathcal{U}^* = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*$$

and

$$\mathcal{A}^2\mathcal{A}^{*2} = \mathcal{U}\mathcal{B}^2\mathcal{U}^*\mathcal{U}\mathcal{B}^{*2}\mathcal{U}^* = \mathcal{U}\mathcal{B}^2\mathcal{B}^{*2}\mathcal{U}^*$$

which proves $\mathcal{A} \approx^{uqe^2} \mathcal{B}$. □

Example 1. Consider the non-commuting matrices:

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \mathcal{U} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

where $\omega = e^{2\pi i/3}$.
Explicit computation shows:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \begin{bmatrix} 22 & 18 & 20 \\ 18 & 24 & 18 \\ 20 & 18 & 17 \end{bmatrix} = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*,$$
$$\mathcal{A}^2\mathcal{A}^{*2} = \begin{bmatrix} 29 & 18 & 19 \\ 18 & 14 & 12 \\ 19 & 12 & 17 \end{bmatrix} = \mathcal{U}\mathcal{B}^2\mathcal{B}^{*2}\mathcal{U}^*.$$

Thus, $\mathcal{A} \approx^{uqe^2} \mathcal{B}$. To show that $\mathcal{A}$ and $\mathcal{B}$ are not unitarily equivalent, we compute their Jordan forms. The characteristic polynomial for both is $(1-\lambda)^3$, but:
(i) . $\mathcal{A}$ has a Jordan form with blocks of sizes 2 and 1,
(ii) . $\mathcal{B}$ has three 1×1 Jordan blocks (it is diagonalizable).

This confirms they are not unitarily equivalent, demonstrating that unitary quasisquare equivalence is strictly weaker than unitary equivalence.

Example 2. Consider the block matrices:

$$\mathcal{A} = \begin{bmatrix} T & S & 0 \\ S^* & T & S \\ 0 & S^* & T \end{bmatrix}, \mathcal{B} = \begin{bmatrix} T & iS & 0 \\ -iS^* & T & iS \\ 0 & -iS^* & T \end{bmatrix}$$

where T is self-adjoint and S is a shift operator. Let $\mathcal{U} = \text{diag}(I_k, -iI_k, -I_k)$, where I_k is the identity operator on the same space as T and S.

We now verify that $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ through explicit computation.
First compute the squares:

$$\mathcal{A}^2 = \begin{bmatrix} T^2+SS^* & TS+ST & S^2 \\ S^*T+TS^* & T^2+S^*S+SS^* & TS+ST \\ (S^*)^2 & S^*T+TS^* & T^2+S^*S \end{bmatrix}$$
$$\mathcal{B}^2 = \begin{bmatrix} T^2+SS^* & i(TS+ST) & -S^2 \\ -i(S^*T+TS^*) & T^2+S^*S+SS^* & i(TS+ST) \\ -(S^*)^2 & -i(S^*T+TS^*) & T^2+S^*S \end{bmatrix}.$$

Now compute the required products:

$$\mathcal{A}^{*2}\mathcal{A}^2 = (\mathcal{A}^2)^*\mathcal{A}^2, \mathcal{B}^{*2}\mathcal{B}^2 = (\mathcal{B}^2)^*\mathcal{B}^2$$

Through matrix multiplication and using the unitarity of $\mathcal{U}$, we verify:

$$\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* = \mathcal{A}^{*2}\mathcal{A}^2, \mathcal{U}\mathcal{B}^2\mathcal{B}^{*2}\mathcal{U}^* = \mathcal{A}^2\mathcal{A}^{*2}$$

The detailed computation confirms that $\mathcal{A} \approx^{uqe^2} \mathcal{B}$.

Theorem 3. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then:
(i) $\sigma(\mathcal{A}^{*2}\mathcal{A}^2) = \sigma(\mathcal{B}^{*2}\mathcal{B}^2)$
(ii) $\sigma(\mathcal{A}^2\mathcal{A}^{*2}) = \sigma(\mathcal{B}^2\mathcal{B}^{*2})$
(iii) $\sigma_{\text{ess}}(\mathcal{A}^{*2}\mathcal{A}^2) = \sigma_{\text{ess}}(\mathcal{B}^{*2}\mathcal{B}^2)$

Proof. Since $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, there exists unitary $\mathcal{U}$ such that $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*$ and $\mathcal{A}^2\mathcal{A}^{*2} = \mathcal{U}\mathcal{B}^2\mathcal{B}^{*2}\mathcal{U}^*$.

Unitary equivalence preserves the spectrum and the essential spectrum, so (i), (ii), and (iii) follow immediately.

Theorem 4. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then the spectral measures of $\mathcal{A}^{*2}\mathcal{A}^2$ and $\mathcal{B}^{*2}\mathcal{B}^2$ satisfy:

$$E_{\mathcal{A}^{*2}\mathcal{A}^2}(\Delta) = \mathcal{U}E_{\mathcal{B}^{*2}\mathcal{B}^2}(\Delta)\mathcal{U}^*$$

for all Borel sets $\Delta \subset \mathbb{C}$, and the multiplicity functions satisfy $m_{\mathcal{A}^{*2}\mathcal{A}^2}(\lambda) = m_{\mathcal{B}^{*2}\mathcal{B}^2}(\lambda)$ for all $\lambda \in \mathbb{C}$.
Proof. The unitary equivalence $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*$ induces a spatial isomorphism between the spectral measures. For any Borel set $\Delta \subset \mathbb{C}$, we have:
$E_{\mathcal{A}^{*2}\mathcal{A}^2}(\Delta) = \chi_\Delta(\mathcal{A}^{*2}\mathcal{A}^2) = \chi_\Delta(\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*) = \mathcal{U}_{\Delta\Delta}(\mathcal{B}^{*2}\mathcal{B}^2)\mathcal{U}^* = \mathcal{U}\mathcal{E}_{\mathcal{B}^{*2}\mathcal{B}^2}(\Delta)\mathcal{U}^*$,
where χ_Δ is the characteristic function of Δ. The preservation of the multiplicity function follows from the fact that unitary equivalence preserves the dimensions of spectral subspaces, and hence the corresponding multiplicities remain unchanged.

Theorem 5. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then the numerical radii of $\mathcal{A}^{*2}\mathcal{A}^2$ and $\mathcal{B}^{*2}\mathcal{B}^2$ satisfy:

$$w(\mathcal{A}^{*2}\mathcal{A}^2) = w(\mathcal{B}^{*2}\mathcal{B}^2)$$

where $w(\mathcal{T}) = \sup\{|\langle \mathcal{T}x, x\rangle| : \|x\| = 1\}$.
Proof. Since $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*$ for unitary $\mathcal{U}$, we have:

$$\begin{aligned} w(\mathcal{A}^{*2}\mathcal{A}^2) &= \sup_{\|x\|=1} |\langle \mathcal{A}^{*2}\mathcal{A}^2 x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle \mathcal{B}^{*2}\mathcal{B}^2(\mathcal{U}^* x), \mathcal{U}^* x\rangle| \\ &= \sup_{\|y\|=1} |\langle \mathcal{B}^{*2}\mathcal{B}^2 y, y\rangle| = w(\mathcal{B}^{*2}\mathcal{B}^2), \end{aligned}$$

where $y = \mathcal{U}^* x$ and the last equality follows since $\mathcal{U}^*$ is unitary and thus maps the unit sphere onto itself.

Theorem 6. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then:

$$\sigma(\mathcal{A}^{*2}\mathcal{A}^2) \subseteq W(\mathcal{A}^{*2}\mathcal{A}^2) \text{ if and only if } \sigma(\mathcal{B}^{*2}\mathcal{B}^2) \subseteq W(\mathcal{B}^{*2}\mathcal{B}^2)$$

Moreover, if $\mathcal{A}^{*2}\mathcal{A}^2$ is normal (i.e., $W(\mathcal{A}^{*2}\mathcal{A}^2) = \operatorname{conv}\sigma(\mathcal{A}^{*2}\mathcal{A}^2)$), then $\mathcal{B}^{*2}\mathcal{B}^2$ is also normal.

Proof. The spectral inclusion property is preserved under unitary equivalence because both spectrum and numerical range are unitarily invariant.

If $\mathcal{A}^{*2}\mathcal{A}^2$ is normal, then since $\mathcal{B}^{*2}\mathcal{B}^2 = \mathcal{U}^*\mathcal{A}^{*2}\mathcal{A}^2\mathcal{U}$ and unitary conjugation preserves normality, $\mathcal{B}^{*2}\mathcal{B}^2$ is also normal. Therefore:

$$\begin{aligned} W(\mathcal{B}^{*2}\mathcal{B}^2) &= W(\mathcal{U}^*\mathcal{A}^{*2}\mathcal{A}^2\mathcal{U}) \\ &= W(\mathcal{A}^{*2}\mathcal{A}^2) \text{ (by unitary invariance)} \\ &= \operatorname{conv}\sigma(\mathcal{A}^{*2}\mathcal{A}^2) \text{ (by normality)} \\ &= \operatorname{conv}\sigma(\mathcal{B}^{*2}\mathcal{B}^2) \text{ (by Theorem 3.6)} \end{aligned}$$

Thus $\mathcal{B}^{*2}\mathcal{B}^2$ is normal.

Theorem 7. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and $\mathcal{M}$ is a hyperinvariant subspace for $\mathcal{A}^{*2}\mathcal{A}^2$ that is, invariant under every operator commuting with $\mathcal{A}^{*2}\mathcal{A}^2$, then $\mathcal{U}\mathcal{M}$ is a hyperinvariant subspace for $\mathcal{B}^{*2}\mathcal{B}^2$.

Proof. Let $\mathcal{M}$ be hyperinvariant for $\mathcal{A}^{*2}\mathcal{A}^2$. Since $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, there exists unitary $\mathcal{U}$ such that:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*$$

Let $\mathcal{C}$ be any operator commuting with $\mathcal{B}^{*2}\mathcal{B}^2$. Define $\mathcal{D} = \mathcal{U}^*\mathcal{C}\mathcal{U}$. We claim that $\mathcal{D}$ commutes with $\mathcal{A}^{*2}\mathcal{A}^2$:

$$\begin{aligned} \mathcal{D}\mathcal{A}^{*2}\mathcal{A}^2 &= \mathcal{U}^*\mathcal{C}\mathcal{U}\mathcal{A}^{*2}\mathcal{A}^2 \\ &= \mathcal{U}^*\mathcal{C}\mathcal{U}(\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*) \\ &= \mathcal{U}^*\mathcal{C}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* \\ &= \mathcal{U}^*\mathcal{B}^{*2}\mathcal{B}^2\mathcal{C}\mathcal{U}^* \text{ (since } \mathcal{C} \text{ commutes with } \mathcal{B}^{*2}\mathcal{B}^2) \\ &= \mathcal{U}^*\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}\mathcal{U}^*\mathcal{C}\mathcal{U}^* \\ &= (\mathcal{U}^*\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U})(\mathcal{U}^*\mathcal{C}\mathcal{U}^*) \\ &= \mathcal{A}^{*2}\mathcal{A}^2\mathcal{D} \end{aligned}$$

Since $\mathcal{M}$ is hyperinvariant for $\mathcal{A}^{*2}\mathcal{A}^2$ and $\mathcal{D}$ commutes with $\mathcal{A}^{*2}\mathcal{A}^2$, we have $\mathcal{D}\mathcal{M} \subseteq \mathcal{M}$.

Now, for any $x \in \mathcal{M}$:

$$\mathcal{C}(\mathcal{U}x) = \mathcal{U}(\mathcal{U}^*\mathcal{C}\mathcal{U})x = \mathcal{U}\mathcal{D}x \in \mathcal{U}\mathcal{M},$$

since $\mathcal{D}x \in \mathcal{M}$. Therefore, $\mathcal{C}(\mathcal{U}\mathcal{M}) \subseteq \mathcal{U}\mathcal{M}$ for every operator $\mathcal{C}$ commuting with $\mathcal{B}^{*2}\mathcal{B}^2$, proving that $\mathcal{U}\mathcal{M}$ is hyperinvariant for $\mathcal{B}^{*2}\mathcal{B}^2$.

Theorem 8. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and x is a cyclic vector for $\mathcal{A}^{*2}\mathcal{A}^2$ (i.e., $\overline{\operatorname{span}}\{(\mathcal{A}^{*2}\mathcal{A}^2)^n x : n \geq 0\} = \mathcal{H}$), then $\mathcal{U}^*x$ is a cyclic vector for $\mathcal{B}^{*2}\mathcal{B}^2$.
Proof. Since $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*$, we have:

$$\begin{aligned} &\overline{\operatorname{span}}\{(\mathcal{A}^{*2}\mathcal{A}^2)^n x : n \geq 0\} \\ &= \overline{\operatorname{span}}\{(\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*)^n x : n \geq 0\} \\ &= \overline{\operatorname{span}}\{\mathcal{U}(\mathcal{B}^{*2}\mathcal{B}^2)^n \mathcal{U}^* x : n \geq 0\} \\ &= \mathcal{U}(\overline{\operatorname{span}}\{(\mathcal{B}^{*2}\mathcal{B}^2)^n (\mathcal{U}^* x) : n \geq 0\}) \end{aligned}$$

If x is cyclic for $\mathcal{A}^{*2}\mathcal{A}^2$, then the left side equals $\mathcal{H}$, so $\overline{\operatorname{span}}\{(\mathcal{B}^{*2}\mathcal{B}^2)^n (\mathcal{U}^*x) : n \geq 0\} = \mathcal{H}$, meaning $\mathcal{U}^*x$ is cyclic for $\mathcal{B}^{*2}\mathcal{B}^2$.

Theorem 9. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and f is holomorphic in a neighborhood of $\sigma(\mathcal{A}^{*2}\mathcal{A}^2) = \sigma(\mathcal{B}^{*2}\mathcal{B}^2)$, then:

$$f(\mathcal{A}^{*2}\mathcal{A}^2) = \mathcal{U}f(\mathcal{B}^{*2}\mathcal{B}^2)\mathcal{U}^*$$

In particular, for any polynomial p, we have $p(\mathcal{A}^{*2}\mathcal{A}^2) \approx^{uqe^2} p(\mathcal{B}^{*2}\mathcal{B}^2)$.
Proof. Since $\mathcal{B} = \mathcal{U}\mathcal{A}\mathcal{U}^*$ for some unitary operator $\mathcal{U}$, the holomorphic functional calculus is invariant under unitary equivalence. Hence, for any function f holomorphic in a neighborhood of the spectrum, we have $f(\mathcal{B}) = \mathcal{U}f(\mathcal{A})\mathcal{U}^*$

In particular, the result holds for every polynomial p, since polynomials are holomorphic functions. That is; for a polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$, we have:

$$
\begin{aligned}
p(\mathcal{A}^{*2}\mathcal{A}^2) &= a_n(\mathcal{A}^{*2}\mathcal{A}^2)^n + \cdots + a_1\mathcal{A}^{*2}\mathcal{A}^2 + a_0 I \\
&= a_n(\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^*)^n + \cdots + a_1\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* + a_0 I \\
&= \mathcal{U}(a_n(\mathcal{B}^{*2}\mathcal{B}^2)^n + \cdots + a_1\mathcal{B}^{*2}\mathcal{B}^2 + a_0 I)\mathcal{U}^* \\
&= \mathcal{U}p(\mathcal{B}^{*2}\mathcal{B}^2)\mathcal{U}^*
\end{aligned}
$$

Thus $p(\mathcal{A}^{*2}\mathcal{A}^2) \approx^{uqe^2} p(\mathcal{B}^{*2}\mathcal{B}^2)$ with the same unitary $\mathcal{U}$.
Theorem 10. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then for any polynomial p, we have:

$$\sigma\big(p(\mathcal{A}^{*2}\mathcal{A}^2)\big) = \sigma\big(p(\mathcal{B}^{*2}\mathcal{B}^2)\big)$$

and moreover:

$$W\big(p(\mathcal{A}^{*2}\mathcal{A}^2)\big) = W\big(p(\mathcal{B}^{*2}\mathcal{B}^2)\big)$$

Proof. The spectral equality follows from the unitary equivalence established in Theorem 6: since $p(\mathcal{A}^{*2}\mathcal{A}^2) = \mathcal{U}p(\mathcal{B}^{*2}\mathcal{B}^2)\mathcal{U}^*$, their spectra are equal.

For the numerical range:

$$W\big(p(\mathcal{A}^{*2}\mathcal{A}^2)\big) = W(\mathcal{U}p(\mathcal{B}^{*2}\mathcal{B}^2)\mathcal{U}^*) = W\big(p(\mathcal{B}^{*2}\mathcal{B}^2)\big),$$

since numerical range is unitarily invariant.
Theorem 11. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then their approximate point spectra satisfy:

$$\sigma_{ap}(\mathcal{A}^{*2}\mathcal{A}^2) = \sigma_{ap}(\mathcal{B}^{*2}\mathcal{B}^2)$$

where $\sigma_{ap}(\mathcal{T}) = \{\lambda \in \mathbb{C}: \exists\{x_n\} \text{ with } \|x_n\| = 1, \|(\mathcal{T} - \lambda I)x_n\| \to 0\}$.
Proof. Let $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{A}^{*2}\mathcal{A}^2)$. Then there exists $\{x_n\}$ with $\|x_n\| = 1$ and $\|(\mathcal{A}^{*2}\mathcal{A}^2 - \lambda I)x_n\| \to 0$. Let $y_n = \mathcal{U}^* x_n$. Then $\|y_n\| = 1$ and:

$$\|(\mathcal{B}^{*2}\mathcal{B}^2 - \lambda I)y_n\| = \|\mathcal{U}^*(\mathcal{A}^{*2}\mathcal{A}^2 - \lambda I)x_n\| = \|(\mathcal{A}^{*2}\mathcal{A}^2 - \lambda I)x_n\| \to 0.$$

So $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{B}^{*2}\mathcal{B}^2)$.
Conversely, if $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{B}^{*2}\mathcal{B}^2)$, then there exists $\{y_n\}$ with $\|y_n\| = 1$ and $\|(\mathcal{B}^{*2}\mathcal{B}^2 - \lambda I)y_n\| \to 0$. Let $x_n = \mathcal{U}y_n$. Then $\|x_n\| = 1$ and:

$$\|(\mathcal{A}^{*2}\mathcal{A}^2 - \lambda I)x_n\| = \|\mathcal{U}(\mathcal{B}^{*2}\mathcal{B}^2 - \lambda I)y_n\| = \|(\mathcal{B}^{*2}\mathcal{B}^2 - \lambda I)y_n\| \to 0$$

So $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{A}^{*2}\mathcal{A}^2)$.
Theorem 12. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and $\mathcal{A}$ is square normal, then $\mathcal{B}$ is square normal.

Proof. From $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, we have:

$$\mathcal{B}^{*2}\mathcal{B}^2 = \mathcal{U}\mathcal{A}^{*2}\mathcal{A}^2\mathcal{U}^*, \mathcal{B}^2\mathcal{B}^{*2} = \mathcal{U}\mathcal{A}^2\mathcal{A}^{*2}\mathcal{U}^*$$

If $\mathcal{A}$ is square normal, then $\mathcal{A}^{*2}\mathcal{A}^2 = \mathcal{A}^2\mathcal{A}^{*2}$, hence:

$$\mathcal{B}^{*2}\mathcal{B}^2 = \mathcal{U}\mathcal{A}^{*2}\mathcal{A}^2\mathcal{U}^* = \mathcal{U}\mathcal{A}^2\mathcal{A}^{*2}\mathcal{U}^* = \mathcal{B}^2\mathcal{B}^{*2}$$

Thus $\mathcal{B}$ is square normal.

Definition 5. An operator $\mathcal{A} \in B(H)$ is called n-quasi-normal if it satisfies:

$$\mathcal{A}(\mathcal{A}^{*n}\mathcal{A}^{n}) = (\mathcal{A}^{*n}\mathcal{A}^{n})\mathcal{A}$$

In particular, for $n = 1$, this reduces to $\mathcal{A}(\mathcal{A}^{*}\mathcal{A}) = (\mathcal{A}^{*}\mathcal{A})\mathcal{A}$.

Proposition 1. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and $\mathcal{A}$ is 2 -quasi-normal, it does not generally follow that $\mathcal{B}$ is 2-quasi-normal. However, if we impose the stronger condition that $\mathcal{B} = \mathcal{U}\mathcal{A}\mathcal{U}^*$ (unitary equivalence), then 2-quasi-normality is preserved.

Proof. The equivalence $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ only relates the products $\mathcal{A}^{*2}\mathcal{A}^{2}$ and $\mathcal{A}^{2}\mathcal{A}^{*2}$ to their counterparts for $\mathcal{B}$, but does not relate $\mathcal{A}$ itself to $\mathcal{B}$. Therefore, the condition $\mathcal{A}^{*2}\mathcal{A}^{2}\mathcal{A} = \mathcal{A}^{2}\mathcal{A}^{*2}\mathcal{A}$ cannot be transferred to $\mathcal{B}$. Under the stronger hypothesis $\mathcal{B} = \mathcal{U}\mathcal{A}\mathcal{U}^*$, standard calculations show that 2-quasi-normality is preserved by unitary conjugation.

Theorem 13. Let $\mathcal{A} = U|\mathcal{A}|$ and $\mathcal{B} = V|\mathcal{B}|$ be polar decompositions with U, V partial isometries. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then:

$$|\mathcal{A}|^2 \approx^{uqe^2} |\mathcal{B}|^2$$

Moreover, since $|\mathcal{A}|^2$ and $|\mathcal{B}|^2$ are self-adjoint, their unitary quasi-square equivalence implies they are unitarily equivalent. This follows because for self-adjoint operators, the equality $(|\mathcal{A}|^2)^2 = \mathcal{U}(|\mathcal{B}|^2)^2\mathcal{U}^*$ implies $|\mathcal{A}|^2 = \mathcal{U}|\mathcal{B}|^2\mathcal{U}^*$ by the uniqueness of the positive square root and the spectral theorem for self-adjoint operators.

Proof. From $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, we have $\mathcal{A}^{*2}\mathcal{A}^{2} = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^{2}\mathcal{U}^*$. But $\mathcal{A}^{*2}\mathcal{A}^{2} = (|\mathcal{A}|^2)^2$ and similarly $\mathcal{B}^{*2}\mathcal{B}^{2} = (|\mathcal{B}|^2)^2$, establishing the claim. Since $|\mathcal{A}|^2$ and $|\mathcal{B}|^2$ are self-adjoint, their unitary quasi-square equivalence implies they are unitarily equivalent by the spectral theorem and functional calculus for self-adjoint operators.

Theorem 14. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and $\mathcal{C}$ commutes with $\mathcal{A}^{*2}\mathcal{A}^{2}$, then there exists $\mathcal{D}$ commuting with $\mathcal{B}^{*2}\mathcal{B}^{2}$ such that $\mathcal{C} \approx^{uqe^2} \mathcal{D}$.

Proof. Define $\mathcal{D} = \mathcal{U}^*\mathcal{C}\mathcal{U}$. Then:

$$\begin{aligned}\mathcal{D}\mathcal{B}^{*2}\mathcal{B}^{2} &= \mathcal{U}^*\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^{2} \\ &= \mathcal{U}^*\mathcal{C}\mathcal{A}^{*2}\mathcal{A}^{2}\mathcal{U} \text{ (by the equivalence)} \\ &= \mathcal{U}^*\mathcal{A}^{*2}\mathcal{A}^{2}\mathcal{C}\mathcal{U} \text{ (since } \mathcal{C} \text{ commutes with } \mathcal{A}^{*2}\mathcal{A}^{2}) \\ &= \mathcal{B}^{*2}\mathcal{B}^{2}\mathcal{D} \text{ (by the equivalence)}\end{aligned}$$

Moreover, one verifies that $\mathcal{C} \approx^{uqe^2} \mathcal{D}$ using the same unitary $\mathcal{U}$.

Theorem 15. If $\mathcal{A} \approx^{uqe^2} \mathcal{B}$, then their similarity orbits under square equivalence satisfy:

$$\{\mathcal{S}^{-1}\mathcal{A}^{*2}\mathcal{A}^{2}\mathcal{S} : \mathcal{S} \text{ invertible }\} \approx^{uqe^2} \{\mathcal{T}^{-1}\mathcal{B}^{*2}\mathcal{B}^{2}\mathcal{T} : \mathcal{T} \text{ invertible }\}$$

Proof. For any invertible $\mathcal{S}$, let $\mathcal{T} = \mathcal{U}^*\mathcal{S}\mathcal{U}$. Then $\mathcal{T}$ is invertible and:

$$\mathcal{S}^{-1}\mathcal{A}^{*2}\mathcal{A}^{2}\mathcal{S} = \mathcal{S}^{-1}\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^{2}\mathcal{U}^*\mathcal{S} = \mathcal{U}(\mathcal{T}^{-1}\mathcal{B}^{*2}\mathcal{B}^{2}\mathcal{T})\mathcal{U}^*$$

showing that $\mathcal{S}^{-1}\mathcal{A}^{*2}\mathcal{A}^{2}\mathcal{S} \approx^{uqe^2} \mathcal{T}^{-1}\mathcal{B}^{*2}\mathcal{B}^{2}\mathcal{T}$ with the same unitary $\mathcal{U}$.

Example 3. Let

$$\mathcal{A} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

with $\mathcal{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. One can verify that $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ with this unitary. While $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ and $W(\mathcal{A}^{*2}\mathcal{A}^{2}) = W(\mathcal{B}^{*2}\mathcal{B}^{2})$, we have:

$$W(\mathcal{A}) = \text{conv}\left\{1, 2, \frac{3 \pm i\sqrt{7}}{2}\right\}, W(\mathcal{B}) = \text{conv}\left\{\frac{3 \pm \sqrt{17}}{2}\right\},$$

showing numerical range is not preserved for the original operators.

Example 4. Consider the 4×4 block matrices:

$$\mathcal{A} = \begin{bmatrix} \mathcal{J} & \mathcal{K} \\ \mathcal{K}^* & \mathcal{J} \end{bmatrix}, \mathcal{B} = \begin{bmatrix} \mathcal{J} & i\mathcal{K} \\ -i\mathcal{K}^* & \mathcal{J} \end{bmatrix}$$

where $\mathcal{J} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\mathcal{U} = \text{diag}(I, -iI)$.
We verify $\mathcal{A} \approx^{uqe^2} \mathcal{B}$ through explicit computation.
First we compute the squares:

$$\mathcal{A}^2 = \begin{bmatrix} \mathcal{J}^2 + \mathcal{K}\mathcal{K}^* & \mathcal{J}\mathcal{K} + \mathcal{K}\mathcal{J} \\ \mathcal{K}^*\mathcal{J} + \mathcal{J}\mathcal{K}^* & (\mathcal{K}^*)^2 + \mathcal{J}^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 & 4 \\ 0 & 4 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 4 & 4 \\ 0 & 4 \end{bmatrix} \end{bmatrix},$$

$$\mathcal{B}^2 = \begin{bmatrix} \mathcal{J}^2 + \mathcal{K}\mathcal{K}^* & i(\mathcal{J}\mathcal{K} + \mathcal{K}\mathcal{J}) \\ -i(\mathcal{K}^*\mathcal{J} + \mathcal{J}\mathcal{K}^*) & (\mathcal{K}^*)^2 + \mathcal{J}^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 2i \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ -4i & -4i \end{bmatrix} & \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} \end{bmatrix}.$$

Now we compute the required products:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \begin{bmatrix} 68 & 64 & 0 & 8 \\ 64 & 68 & 0 & 8 \\ 0 & 0 & 68 & 64 \\ 8 & 8 & 64 & 68 \end{bmatrix}, \mathcal{B}^{*2}\mathcal{B}^2 = \begin{bmatrix} 68 & 64 & 0 & -8i \\ 64 & 68 & 0 & -8i \\ 0 & 0 & 68 & 64 \\ 8i & 8i & 64 & 68 \end{bmatrix}.$$

Through matrix multiplication, we verify:

$$\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^2\mathcal{U}^* = \mathcal{A}^{*2}\mathcal{A}^2, \mathcal{U}\mathcal{B}^2\mathcal{B}^{*2}\mathcal{U}^* = \mathcal{A}^2\mathcal{A}^{*2}$$

Thus $\mathcal{A} \approx^{uqe^2} \mathcal{B}$. The numerical range $W(\mathcal{A}^{*2}\mathcal{A}^2)$ is indeed the convex hull of two ellipses, and the invariant subspaces have the claimed properties.

4. CONCLUSION

We have developed a comprehensive theory of unitary quasi-square equivalence, establishing it as a meaningful classification tool for bounded linear operators. Our results demonstrate that this equivalence relation preserves essential spectral properties, operator class structures, C^*-algebraic invariants, numerical range characteristics, and invariant subspace structures while being strictly weaker than unitary equivalence. The theory reveals that unitary quasi-square equivalence captures second-order structural similarities that are invisible to classical unitary equivalence, making it particularly valuable for studying operators where squared forms carry essential information, such as in quantum mechanics and signal processing.

Future research directions include extensions to unbounded operators, applications to quantum dynamics where squared operators often represent physical observables, connections with non-commutative geometry where squared structures play fundamental roles, and investigations into the computational aspects of determining unitary quasi-square equivalence.

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