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حل معادلة فولتيرا– فريدهولم التكاملية الضبابية ذات ثلاثة متغيرات

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المستخلص:

يهدف هذا البحث إلى حل معادلات 'فولتيرا–فريدهولم' التكاملية غير الخطية الضبابية في ثلاثة متغيرات، حيث تُعد الصيغة قيد الدراسة نمطاً هاماً من معادلات 'فولتيرا–فريدهولم' غير الخطية. وقد تم إثبات المعادلة التكاملية الضبابية ذات النوى الضبابية غير الخطية مع إجراء تحليل التقارب، بالإضافة إلى استنباط خصائص معينة للحل. كما تُظهر النتائج، بالنسبة للحالات الوسيطية لصياغة المسألة الضبابية، استيفاء بعض الشروط الضرورية والكافية لنموذج 'ليبنز' لمختلف الدوال ومشتقاتها، فضلاً عن تطبيق قاعدة 'ليبنز' على صياغة المسألة المطروحة. وتثبت الدراسة أيضاً وجود ووحدانية الحلول في فضاء الدوال الضبابية المستمرة وفق معيار منتظم، والذي يُعد تعميماً لفضاء الأعداد الضبابية، وذلك بالاستناد إلى شرط الانكماش.

الكلمات المفتاحية : فولتيرا–فريدهولم غير الخطية الضبابية؛ المعادلات التكاملية؛ نظرية النقطة الثابتة لباناخ

Solution for the Fuzzy Volterra-Fredholm Integral Equation with Three Variables

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Abstract:
This paper aims to solve fuzzy nonlinear Volterra-Fredholm integral equations in three variables. The formula under study is an important type of nonlinear Volterra-Fredholm equation. The fuzzy integral equation with nonlinear fuzzy kernels is proven with convergence analysis. Furthermore, the study establishes certain properties of the solution. The results also demonstrate that, for the parametric cases of the fuzzy problem formulation, some necessary and sufficient conditions are met for the Lebenszian model of various functions and their derivatives, as well as the use of Lebenszian's rule for the presented problem formulation. The study also demonstrates the existence and uniqueness of solutions in the space of continuous fuzzy functions with a regular criterion, which generalises the space of fuzzy numbers and is based on the contraction condition.

Keywords: Fuzzy nonlinear Volterra-Fredholm; Integral equations; Three Variables; Banach fixed-point theorem (BFP).

1-Introduction:

One of the basic equations in the domains of engineering modelling, medical, and physical applications are arintriguinge fuzzy differential equation, whose solutions are constrained between an upper and a lower value since the majority of their models are defined in fuzzy sets. Volterra and Fredholm integral formulations may be used in these equations. Sometimes boundary or initial conditions are needed for these fuzzy integral equations.

The interesting issues in the analysis of fuzzy integral equations are the necessary and sufficient conditions of convergence and the constraint of boundedness. In light of this, we introduced the following scholars, some of whom looked at the necessary conditions for a bounded solution to fuzzy integral equations in (Abbasbandy, 2007).

The convergence of the consecutive approximations method and the approximate solution of nonlinear Hammerstein fuzzy integral equations have been introduced in (Al-Smadi, 2019). The parametric form of fuzzy numbers is used to guide the analytic solution of the second type of linear Fredholm fuzzy integral equations (Bica, 2014). Fuzzy Volterra integral equations with time lag, as well as the method's effective iterative process and corresponding uniqueness and convergence, were described in detail in (Yookesh, 2021). Analytical and approximation solutions were occasionally explored simultaneously, and a trustworthy numerical technique was presented. The fuzzy linear number parametric form is used to study the conversion of the linear fuzzy Fredholm integral equation to the linear system of the second kind integral equation (Nieto, 2006).

Contractive prosperity was attained, in addition to the novel and intriguing outcomes of the common fixed point utilised in applications on stochastic Volterra integral equations (Deng, 2022). Fuzzy Fredholm integral equations with respect to fuzzy-valued functions introduced by classical approximation are solved using the usual approximation technique in (Shiri, 2021).

In (Ullah, 2021), the hybrid approach for computing equation solutions with a degenerate kernel was developed. Using the homotopy analytic approach, the ideal solution values were found in (Fariborzi Araghi, 2021). Numerous papers that build analytical and approximate solutions in various ways for various methods with integrals of the Volterra and Fredholm types as their targets are interested in the existence and uniqueness of the solutions (Ghasemi, 2011), (Hussain, 2013) and (Samadpour Khalifeh Mahaleh, 2021).

In (Shahidi, 2023), fuzzy differentiability and integrability were investigated using a novel concept of correlated fuzzy processes on time scales. In (Sharif, 2022), the integral fuzzy problem under Hukuhara differentiability was forced by fuzzy Volterra and Fredholm integral equations with a convolution-type kernel using the fuzzy Laplace transform method. In order to solve linear and nonlinear fuzzy integral equations of two dimensions and some key findings for convergent systems, combines two technological approaches, such as fuzzy Lagrange interpolation and the fuzzy Gauss-Legendre quadrature formula, for establishing a collocation-based method.

In (Hasan, 2016), the Volterra-Fredholm nonlinear integral equation with fuzzy parameters was shown to exist and have a unique solution. Sharif et al. used the fuzzy

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Laplace transform method to solve the one-dimensional fuzzy integral equations in (Sharif, 2022), In (Alidema, 2019), Alidema and Georgieva introduced the double fuzzy Sumudu transform, which is used to solve Volterra fuzzy integral equations. In (Hussain, 2013), the existence and uniqueness of nonlinear solutions to the Volterra fuzzy integral equations contractions were examined. Analysis of convergence was introduced in (Hooshangian, 2017), and has been used to study fuzzy integral equations using nonlinear fuzzy kernels. The existence and uniqueness of solutions to fuzzy integral equations of product type have been examined in (Pathinathan, 2015), Furthermore, the integral-differential equations in (Ullah, 2021), are fuzzy and nonlinear. A study on the use of some basic properties of solutions of a general mixed fuzzy Volterra-Fredholm integral equation was conducted in (Pachpatte, 2010) and (Vu, 2021).

Unlike existing studies that focus on one- or two-variable fuzzy integral equations, this work extends the analysis to fuzzy nonlinear Volterra–Fredholm integral equations with three variables. Moreover, the existence and uniqueness results are established under weaker contraction conditions, which generalize several known results in the literature.

2-Preliminaries:

This section presents fuzzy integral equations with intriguing definitions and conclusions that serve as a foundational idea of existence and uniqueness.

Definition 1: (Wu, 2001)

The collection of all nonempty convex and compact subsets of $\mathcal{R}$ is denoted as $\mathcal{K}_c(\mathcal{R})$.

The following is the definition of the Hausdorff metric $d_{\mathcal{H}}$ in $\mathcal{K}_c(R)$.

$$d_{\mathcal{H}}(\mathcal{A},\mathcal{B}) = max\{\sup_{a\in\mathcal{A}}\inf_{b\in\mathcal{B}} \|a-b\| \ , \ \sup_{b\in\mathcal{B}}\inf_{a\in\mathcal{A}} \|a-b\|\}$$

for every $\mathcal{A},\mathcal{B} \in \mathcal{K}_c(R)$ The fact that the space $(\mathcal{K}_c(R)\ ,d_{\mathcal{H}})$ is a complete metric space is readily apparent.

Definition 2: (Pathinathan, 2015)

Let $£ = \{u : \mathcal{R} \to [0,1]\}$ so that $x(z)$ satisfies (i)-(iv) as follows:

(i) x is normal, i.e $\exists\ z_0 \in \mathcal{R}$; $x(z_0) = 1$;

(ii) x is fuzzy convex, that is, for $0 \le \lambda \le 1$,

$x(\lambda a + (1-\lambda)b) \ge \min\{x(a), x(b)\}$, for any $a, b \in \mathcal{R}$

(iii) x is upper semicontinuous;

(iv) $[x]^0 = \mathcal{C}\{a \in \mathcal{R} : x(a) > 0\}$ is compact.

The integral has the following properties for further details: (Lakshmikantham, 2004)

(1) Let $\lambda > 0$ and $f, k : [a,b] \to £$ be integrable, Then:

I) $\int_a^b (f(t) + k(t))dt = \int_a^b f(t)dt + \int_a^b k(t)dt$.

II) $\int_a^b \lambda f(t)dt = \lambda \int_a^b f(t)dt$.

III) $D(f,k)$ is integrable and $D\left(\int_a^b f(t)dt, \int_a^b k(t)dt\right) \le \int_a^b D(f(s) + k(s))dt$

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(2) It is integrable if $\mathfrak{f}: [\mathfrak{a},\mathfrak{b}] \to \pounds$ is continuous.

(3) If $\mathfrak{f}: [\mathfrak{a},\mathfrak{b}] \to \pounds$ is integrable and $\mathfrak{c} \in [\mathfrak{a},\mathfrak{b}]$ then $\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{f}(\underline{t})d\underline{t} = \int_{\mathfrak{a}}^{\mathfrak{c}} \mathfrak{f}(\underline{t})d\underline{t} + \int_{\mathfrak{c}}^{\mathfrak{b}} \mathfrak{k}(\underline{t})d\underline{t}$

Let $\mathcal{J} = [\mathfrak{a},\mathfrak{b}] \times [\mathfrak{c},\mathfrak{d}] \subset \mathcal{R} \times \mathcal{R}$ and $C(\mathcal{J},\pounds)$ be a space of all continuous functions $\mathfrak{f}: \mathcal{J} \to \pounds$ with the supremum metric $\underline{D}^*$ defined by:

$$\underline{D}^*(\mathfrak{f},\mathfrak{k}) = \sup_{(\mathfrak{s},\underline{t})\in\mathcal{J}} \underline{D}(\mathfrak{f}(\mathfrak{s},\underline{t}),\mathfrak{k}(\mathfrak{s},\underline{t}))$$

The fact that $\underline{D}^*$ is a metric in £. is readily apparent, $(\pounds,\underline{D}^*)$ is a complete metric space.

Definition 3: (Lakshmikantham, 2004)

Let $\mathcal{T}$ be a mapping into itself from a set $\mathcal{X}$. If $x_0 \in \mathcal{X}$ such that $\mathcal{T}x_0 = x_0$.,
then the mapping $\mathcal{T}$ has a fixed point. The Contraction Mapping Principle, attributed to $\mathcal{S}$, is the most well-known fixed-point theorem.

Definition 4: (Lakshmikantham, 2004)

Let $(\mathcal{X},d)$ be the metric space. If there is a constant $\alpha > 0$ that makes a mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ Lipschitz continuous, then:

for every $x,y \in \mathcal{X}, d(\mathcal{T}x,\mathcal{T}y) \leq \alpha d(x,y)$.

When $0 \leq \alpha < 1$, $\mathcal{T}$ is referred to as a contraction mapping. α is referred to as $\mathcal{T}$'s contractively factor.

A fixed point is defined as the existence of $x \in \mathcal{X}$ such that $\mathcal{T}x = x$ for a (not necessarily Lipschitz) mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$.

A (nonzero) Lipschitz continuous mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ with Lipschitz constant q is clearly continuous. This is true if we select $0 < \delta < \varepsilon/q \; for \; \varepsilon > 0$.

$$x,y \in \mathcal{X}, d(x,y) < \delta \Rightarrow d(\mathcal{T}x,\mathcal{T}y) \leq q d(x,y) < q.\frac{\varepsilon}{q} = \varepsilon$$

3-Results:

Let the provided subsets of R be $\mho = [a,b]$. "The class of continuous functions from the set $\mathcal{A}$ to the set $\mathcal{B}$ " is denoted by $\mathcal{C}: \mathcal{A} \to \mathcal{B}$, and $\pounds = [0,\mathfrak{X}] \times [0,\mathfrak{Y}] \times \mho$. (Pachpatte, 2010)

Examine the following solution to the fuzzy Volterra- Fredholm integral equation involving three variables:

$$\tilde{\mathfrak{r}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}') = \ddot{E}(\mathfrak{x},\mathfrak{y},\mathfrak{t}') + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{A}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\tilde{\mathfrak{r}}(\sigma,\tau,\varrho),\left(\acute{\Gamma}\tilde{\mathfrak{r}}\right)(\sigma,\tau,\varrho)\right) d\varrho d\tau d\sigma \qquad (3.1)$$

for all $(\mathfrak{x},\mathfrak{y},\mathfrak{t}') \in \pounds$ where

$$(\acute{\Gamma}\tilde{\mathrm{ư}})(\mathfrak{x},\mathfrak{y},\mathrm{t}') = \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \breve{H}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\check{\mathfrak{z}},6,▯,\tilde{\mathrm{ư}}(\check{\mathfrak{z}},6,▯))\,d▯\,d6\,d\check{\mathfrak{z}} \tag{3.2}$$

And $\tilde{\mathrm{ư}}, \ddot{\mathrm{E}} \in \mathcal{C}(\pounds,\hat{\mathrm{R}})$, $\ddot{\mathrm{A}} \in \mathcal{C}(\pounds\times\pounds\times\hat{\mathrm{R}}\times\hat{\mathrm{R}},\hat{\mathrm{R}})$, $\ddot{H} \in \mathcal{C}(\pounds\times\pounds\times\hat{\mathrm{R}},\hat{\mathrm{R}})$

The space of continuous function from $\pounds \in (\hat{\mathrm{R}}, \underline{\mathrm{D}})$ with $\hat{\mathrm{H}}(\tilde{\mathrm{ư}},\hat{\mathrm{o}})$ is denoted by $\mathfrak{H}$.

$$\mathfrak{H} = \{\,\tilde{\mathrm{ư}}|\tilde{\mathrm{ư}} : \pounds \to \hat{\mathrm{R}}\ \ is\ continuous\ \&\ \hat{\mathrm{H}}(\tilde{\mathrm{ư}},\hat{\mathrm{o}}) \le \delta\}$$

where

$$\hat{\mathrm{H}}(\tilde{\mathrm{ư}},\tilde{\mathrm{v}}) = \sup_{(\mathfrak{x},\mathfrak{y},\mathrm{t}')\in\pounds} \{\underline{\mathrm{D}}\,(\tilde{\mathrm{ư}}(\mathfrak{x},\mathfrak{y},\mathrm{t}'),\tilde{\mathrm{v}}(\mathfrak{x},\mathfrak{y},\mathrm{t}'))e^{\lambda(\mathfrak{x}+\mathfrak{y}+\mathrm{t}')}\}, \qquad \lambda > 0$$

It is easy to verify that $(\mathcal{C}(\pounds,\hat{\mathrm{R}}),\hat{\mathrm{H}})$ is also a complete metric space.

We now use the Banach fixed point theorem to demonstrate the existence and uniqueness of the solution to issue (3.1).

Theorem1. Let $\ddot{\mathrm{A}} \in \mathcal{C}(\pounds\times\pounds\times\hat{\mathrm{R}}\times\hat{\mathrm{R}},\hat{\mathrm{R}})$ *and* $\ddot{H} \in \mathcal{C}(\pounds\times\pounds\times\hat{\mathrm{R}},\hat{\mathrm{R}})$ be satisfy the following conditions :

(i) $\exists\ \ddot{\mathrm{w}} \in \mathcal{C}(\pounds^2,\hat{\mathrm{R}}_+)$ such that

$\mathrm{Đ}(\ddot{\mathrm{A}}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\tilde{\mathrm{ư}}_1,\tilde{\mathrm{v}}_1),\ddot{\mathrm{A}}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\tilde{\mathrm{ư}}_2,\tilde{\mathrm{v}}_2)) \le$
$\ddot{\mathrm{w}}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho)\{\mathfrak{D}(\tilde{\mathrm{ư}}_1,\tilde{\mathrm{ư}}_2),(\tilde{\mathrm{v}}_1,\tilde{\mathrm{v}}_2)\}$

For all $(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\tilde{\mathrm{ư}}_1,\tilde{\mathrm{v}}_1),(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\tilde{\mathrm{ư}}_2,\tilde{\mathrm{v}}_2) \in \pounds\times\pounds\times\hat{\mathrm{R}}\times\hat{\mathrm{R}}$

(ii) $\exists\ \dot{\mathrm{M}} \in \mathcal{C}(\pounds^2,\hat{\mathrm{R}}_+)$ such that

$\mathrm{Đ}(\ddot{H}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\check{\mathfrak{z}},6,▯,\tilde{\mathrm{ư}}),\ddot{H}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\check{\mathfrak{z}},6,▯,\tilde{\mathrm{v}}) \le \dot{\mathrm{M}}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\check{\mathfrak{z}},6,▯)\mathfrak{D}(\tilde{\mathrm{ư}},\tilde{\mathrm{v}})$

For all $(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\tilde{\mathrm{ư}}),(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\tilde{\mathrm{v}}) \in \pounds\times\pounds\times\hat{\mathrm{R}}$

(iii) $\exists\ \beta > 0$ such that

$$\mathfrak{D}(\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathrm{t}'),\hat{\mathrm{o}}) + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}(\ddot{\mathrm{A}}(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho,\hat{\mathrm{o}},(\acute{\Gamma}\hat{\mathrm{o}})(\sigma,\tau,\varrho),\hat{\mathrm{o}})\,d\varrho\,d\tau\,d\sigma \le \beta\, e^{\lambda(\mathfrak{x}+\mathfrak{y}+\mathrm{t}')}$$

For all $(\mathfrak{x},\mathfrak{y},\mathrm{t}',\sigma,\tau,\varrho) \in \pounds\times\pounds$

(iv) there exists α∈ (0,1] such that

$$\int_0^{ꭓ}\int_0^{ꭚ}\int_{Ʊ} ẅ(ꭓ,ꭚ,ť,\sigma,\tau,\varrho)\left(e^{\lambda(\sigma,\tau,\varrho)}+\int_0^{\sigma}\int_0^{\tau}\int_{Ʊ} Ṁ(\sigma,\tau,\varrho,ǯ,6,⍰)e^{\lambda(ǯ,6,⍰)}d⍰\,d6\,dǯ\right)d\varrho\,d\tau\,d\sigma \le \alpha e^{\lambda(ꭓ+ꭚ+ť)}$$

For all $(ꭓ,ꭚ,ť,\sigma,\tau,\varrho),(ꭓ,ꭚ,ť,ǯ,6,⍰)\in £\times £$

Then, if $\beta+\alpha\,\delta<1$, the problem (3.1) has a unique solution on $\$$.

proof : Consider the operator ψ: $\$\rightarrow\$$, which is defined by

$$(\psi ũ)(ꭓ,ꭚ,ť)=Ë(ꭓ,ꭚ,ť)+\int_0^{ꭓ}\int_0^{ꭚ}\int_{Ʊ} Ä\left(ꭓ,ꭚ,ť,\sigma,\tau,\varrho,ũ(\sigma,\tau,\varrho),(Ѓũ)(\sigma,\tau,\varrho)\right)d\varrho\,d\tau\,d\sigma \qquad (3.3)$$

We will split the proof into two stages in order to demonstrate the theorem.

Step 1. Since the functions Ä and Ӥ are integrable on £×£ and Ë is a continuous function on £

Thus, ψ is a continuous operator. Applying the requirements from (3.3), we obtain

$$\mathfrak{D}((\psi ũ)(ꭓ,ꭚ,ť),ô)=\mathfrak{D}\left(Ë(ꭓ,ꭚ,ť)+\int_0^{ꭓ}\int_0^{ꭚ}\int_{Ʊ} Ä\left(ꭓ,ꭚ,ť,\sigma,\tau,\varrho,ũ(\sigma,\tau,\varrho),(Ѓũ)(\sigma,\tau,\varrho)\right)d\varrho\,d\tau\,d\sigma,ô\right)$$

$$\le \mathfrak{D}(Ë(ꭓ,ꭚ,ť),ô)+\mathfrak{D}\left(\int_0^{ꭓ}\int_0^{ꭚ}\int_{Ʊ} Ä\left(ꭓ,ꭚ,ť,\sigma,\tau,\varrho,ũ(\sigma,\tau,\varrho),(Ѓũ)(\sigma,\tau,\varrho)\right)d\varrho\,d\tau\,d\sigma,ô\right)$$

$$\le \mathfrak{D}(Ë(ꭓ,ꭚ,ť),ô)+\int_0^{ꭓ}\int_0^{ꭚ}\int_{Ʊ} \mathfrak{D}\left(Ä\left(ꭓ,ꭚ,ť,\sigma,\tau,\varrho,ũ(\sigma,\tau,\varrho),(Ѓũ)(\sigma,\tau,\varrho)\right),ô\right)d\varrho\,d\tau\,d\sigma$$

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$$\leq \mathfrak{D}(\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}'),\hat{o}) + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}\left(\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\hat{o},(\acute{\Gamma}\hat{o})(\sigma,\tau,\varrho)\right),\hat{o}\right) d\varrho d\tau d\sigma$$

$$+ \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{\mathrm{w}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho)$$

$$\left(\mathfrak{D}(\tilde{\mathrm{r}}(\sigma,\tau,\varrho),\hat{o}) + \mathrm{Đ}\left(\int_0^{\sigma}\int_0^{\tau}\int_{\mho} \ddot{\mathrm{H}}(\sigma,\tau,\varrho,\mathfrak{z},6,⍰,\tilde{\mathrm{r}}(\mathfrak{z},6,⍰))d⍰ d6 d\mathfrak{z},\right.\right.$$

$$\left.\left.\int_0^{\sigma}\int_0^{\tau}\int_{\mho} \ddot{\mathrm{H}}(\sigma,\tau,\varrho,\check{\mathfrak{z}},6,⍰,\hat{o})d⍰ d6 d\check{\mathfrak{z}}\right)\right) d\varrho d\tau d\sigma$$

$$\leq \mathfrak{D}(\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}'),\hat{o}) + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}\left(\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\hat{o},(\acute{\Gamma}\hat{o})(\sigma,\tau,\varrho)\right),\hat{o}\right) d\varrho d\tau d\sigma$$

$$+ \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{\mathrm{w}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho)$$

$$\left(e^{\lambda(\sigma,\tau,\varrho)} + \int_0^{\sigma}\int_0^{\tau}\int_{\mho} \dot{\mathrm{M}}(\sigma,\tau,\varrho,\check{\mathfrak{z}},6,⍰)e^{\lambda(\check{\mathfrak{z}},6,⍰)} d⍰ d6 d\check{\mathfrak{z}}\right) d\varrho d\tau d\sigma$$

$$\leq (\beta + \alpha\hat{\mathrm{H}}(\tilde{\mathrm{r}},\hat{o}))e^{\lambda(\mathfrak{x}+\mathfrak{y}+\mathfrak{t}')}$$

$$\mathfrak{D}((\psi\tilde{\mathrm{r}})(\mathfrak{x},\mathfrak{y},\mathfrak{t}'),\hat{o})e^{-\lambda(\mathfrak{x}+\mathfrak{y}+\mathfrak{t}')} \leq (\beta + \alpha\delta)$$

$$\hat{\mathrm{H}}(\psi\tilde{\mathrm{r}},\hat{o}) \leq (\beta + \alpha\delta)$$

for every $\tilde{\mathrm{r}} \in \mathfrak{H}$. Bounded sets are thus mapped onto themselves by ψ.

Step 2. A contraction of $\mathfrak{H}$ is the operator ψ. In fact, given the criteria and letting $\tilde{\mathrm{r}},\tilde{\mathrm{v}} \in \mathfrak{H}$, we have

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$$\mathfrak{D}((\psi\tilde{\mathfrak{r}})(\mathfrak{x},\mathfrak{y},\mathfrak{t}'),(\psi\tilde{\mathrm{v}})(\mathfrak{x},\mathfrak{y},\mathfrak{t}'))$$

$$= \mathfrak{D}\Bigg(\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}') + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\tilde{\mathfrak{r}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathfrak{r}})(\sigma,\tau,\varrho)\right) d\varrho d\tau d\sigma,$$

$$\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}') + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\tilde{\mathrm{v}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathrm{v}})(\sigma,\tau,\varrho)\right) d\varrho d\tau d\sigma\Bigg)$$

$$\leq \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}(\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\tilde{\mathfrak{r}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathfrak{r}})(\sigma,\tau,\varrho)\right),\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho,\tilde{\mathrm{v}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathrm{v}})(\sigma,\tau,\varrho)\right)\Big) d\varrho d\tau d\sigma$$

$$\leq \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{\mathrm{w}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho)\Bigg(\mathfrak{D}\big(\tilde{\mathfrak{r}}(\sigma,\tau,\varrho),\tilde{\mathrm{v}}(\sigma,\tau,\varrho)\big) + \mathfrak{D}\Bigg(\int_0^{\sigma}\int_0^{\tau}\int_{\mho} \ddot{\mathrm{H}}\big(\sigma,\tau,\varrho,\check{\mathfrak{z}},6,\square,\tilde{\mathfrak{r}}(\check{\mathfrak{z}},6,\square)\big)d\square d6 d\check{\mathfrak{z}},$$

$$\int_0^{\sigma}\int_0^{\tau}\int_{\mho} \ddot{\mathrm{H}}\big(\sigma,\tau,\varrho,\check{\mathfrak{z}},6,\square,\tilde{\mathrm{v}}(\sigma,\tau,\varrho)\big)d\square d6 d\check{\mathfrak{z}}\Bigg)\Bigg) d\varrho d\tau d\sigma$$

$$\leq \hat{\mathrm{H}}(\tilde{\mathfrak{r}},\tilde{\mathrm{v}})\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{\mathrm{w}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}',\sigma,\tau,\varrho)\Bigg(e^{\lambda(\sigma,\tau,\varrho)} + \int_0^{\sigma}\int_0^{\tau}\int_{\mho} \dot{\mathrm{M}}(\sigma,\tau,\varrho,\check{\mathfrak{z}},6,\square)e^{\lambda(\check{\mathfrak{z}},6,\square)} d\square d6 d\check{\mathfrak{z}}\Bigg) d\varrho d\tau d\sigma$$

$$\leq \alpha\ \hat{\mathrm{H}}(\tilde{\mathfrak{r}},\tilde{\mathrm{v}})e^{\lambda(\mathfrak{x}+\mathfrak{y}+\mathfrak{t}')}$$

$$\hat{\mathrm{H}}(\psi\tilde{\mathfrak{r}},\psi\tilde{\mathrm{v}}) \leq \alpha\ \hat{\mathrm{H}}(\tilde{\mathfrak{r}},\tilde{\mathrm{v}})$$

Since ψ is a contraction operator, for every $\tilde{\mathfrak{r}},\tilde{\mathrm{v}} \in \mathcal{B}$.

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Therefore, we find that ψ has a fixed point, which is a solution to problem (3.1), by applying (BFP) theorem. This proof is finished.

Using the inequality in Theorem 1 (Pachpatte, 2010), we shall investigate some basic estimates of the answers to (3.1) in the follow-up.

Theorem 2. Let $\ddot{w}, \dot{M}: £ \times £ \to R_+$ be a continuous function, provided that all of Theorem 1's criteria are met.

$$\ddot{w}(𝔵, 𝔶, ť, \sigma, \tau, \varrho) = f(𝔵, 𝔶, ť)\, g(\sigma, \tau, \varrho)$$

$$\dot{M}(𝔵, 𝔶, ť, ǯ, 6, \square) = f(𝔵, 𝔶, ť)\, q(ǯ, 6, \square)$$

Where $f, g, q \in \mathcal{C}(£, \hat{R}_+)$. Let

$$Ḉ = \sup_{(𝔵,𝔶,ť)\in £} \left\{ \mathfrak{D}(\ddot{E}(𝔵, 𝔶, ť), \hat{o}) + \int_0^{𝔵}\int_0^{𝔶}\int_{Ʊ} \mathfrak{D}\left(\ddot{A}\left(𝔵, 𝔶, ť, \sigma, \tau, \varrho, \hat{o}, \left(Ɣ\hat{o}\right)(\sigma, \tau, \varrho), \hat{o}\right) d\varrho\, d\tau\, d\sigma \right\}$$

. *If* $\tilde{ȓ}(𝔵, 𝔶, ť)$ is a solution to the issue (4.1), then we have

$$\mathfrak{D}(\tilde{ȓ}(𝔵, 𝔶, ť), \hat{o}) \le Ḉ \left\{ 1 + f(𝔵, 𝔶, ť) \int_0^{𝔵}\int_0^{𝔶}\int_{Ʊ} [f(\sigma, \tau, \varrho) + g(\sigma, \tau, \varrho)] \right.$$

$$\left. \times\, exp\left(\int_{\sigma}^{𝔵}\int_{\tau}^{𝔶}\int_{Ʊ} f(ǯ, 6, \square)[g(ǯ, 6, \square) + q(ǯ, 6, \square)]\, d\square\, d6\, dǯ \right) d\varrho\, d\tau\, d\sigma \right\}$$

Proof. Since the solution to problem (3.1) is $\tilde{ȓ}(𝔵, 𝔶, ť)$, and from (3.1), we have

$$\mathfrak{D}(\tilde{ȓ}(𝔵, 𝔶, ť), \hat{o})$$

$$= \mathfrak{D}\left(\ddot{E}(𝔵, 𝔶, ť) + \int_0^{𝔵}\int_0^{𝔶}\int_{Ʊ} \ddot{A}\left(𝔵, 𝔶, ť, \sigma, \tau, \varrho, \tilde{ȓ}(\sigma, \tau, \varrho), \left(Ɣ\tilde{ȓ}\right)(\sigma, \tau, \varrho)\right) d\varrho\, d\tau\, d\sigma, \hat{o} \right)$$

$$\le \mathfrak{D}(\ddot{E}(𝔵, 𝔶, ť), \hat{o}) + \mathfrak{D}\left(\int_0^{𝔵}\int_0^{𝔶}\int_{Ʊ} \ddot{A}\left(𝔵, 𝔶, ť, \sigma, \tau, \varrho, \tilde{ȓ}(\sigma, \tau, \varrho), \left(Ɣ\tilde{ȓ}\right)(\sigma, \tau, \varrho)\right) d\varrho\, d\tau\, d\sigma, \hat{o} \right)$$

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$$\leq \mathfrak{D}(\ddot{E}(\mathfrak{x},\mathfrak{y},\acute{t}),\hat{o}) + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}\left(\ddot{A}\left(\mathfrak{x},\mathfrak{y},\acute{t},\sigma,\tau,\varrho,\tilde{u}(\sigma,\tau,\varrho),\left(\acute{\Gamma}\tilde{u}\right)(\sigma,\tau,\varrho)\right),\hat{o}\right) d\varrho d\tau d\sigma$$

$$+ \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}(\ddot{A}\left(\mathfrak{x},\mathfrak{y},\acute{t},\sigma,\tau,\varrho,\tilde{u}(\sigma,\tau,\varrho),\left(\acute{\Gamma}\tilde{u}\right)(\sigma,\tau,\varrho)\right),\ddot{A}\left(\mathfrak{x},\mathfrak{y},\acute{t},\sigma,\tau,\varrho,\hat{o},\left(\acute{\Gamma}\hat{o}\right)(\sigma,\tau,\varrho)\right)\Big) d\varrho d\tau d\sigma$$

$$\leq \mathfrak{D}(\ddot{E}(\mathfrak{x},\mathfrak{y},\acute{t}),\hat{o}) + \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \mathfrak{D}\left(\ddot{A}\left(\mathfrak{x},\mathfrak{y},\acute{t},\sigma,\tau,\varrho,\hat{o},\left(\acute{\Gamma}\hat{o}\right)(\sigma,\tau,\varrho)\right),\hat{o}\right) d\varrho d\tau d\sigma$$

$$+ \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} \ddot{w}(\mathfrak{x},\mathfrak{y},\acute{t},\sigma,\tau,\varrho)$$

$$\Bigg(\mathfrak{D}(\tilde{u}(\sigma,\tau,\varrho),\hat{o}) + \mathfrak{D}\Bigg(\int_0^{\sigma}\int_0^{\tau}\int_{\mho} \ddot{H}(\sigma,\tau,\varrho,\mathfrak{z},6,?,\tilde{u}(\mathfrak{z},6,?))d?d6d\mathfrak{z},$$

$$\int_0^{\sigma}\int_0^{\tau}\int_{\mho} \ddot{H}(\sigma,\tau,\varrho,\check{\mathfrak{z}},6,?,\hat{o})d?d6d\check{\mathfrak{z}}\Bigg)\Bigg) d\varrho d\tau d\sigma$$

$$\leq \acute{Ç} + f(\mathfrak{x},\mathfrak{y},\acute{t}) \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} g(\sigma,\tau,\varrho)\,(\mathfrak{D}(\tilde{u}(\sigma,\tau,\varrho),\hat{o})$$

$$+f(\sigma,\tau,\varrho)\int_0^{\sigma}\int_0^{\tau}\int_{\mho} q(\check{\mathfrak{z}},6,?)\,(\mathfrak{D}(\tilde{u}(\check{\mathfrak{z}},6,?),\hat{o})d?d6d\check{\mathfrak{z}}\Bigg) d\varrho d\tau d\sigma \qquad (3.4)$$

Now applying the Theorem 1 to (Pachpatte, 2010), it results (3.4).

$$\mathfrak{D}(\tilde{u}(\mathfrak{x},\mathfrak{y},\acute{t}),\hat{o}) \leq \acute{Ç}\Bigg\{1 + f(\mathfrak{x},\mathfrak{y},\acute{t})\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} [f(\sigma,\tau,\varrho) + g(\sigma,\tau,\varrho)]$$

$$\times exp\left(\int_{\sigma}^{\mathfrak{x}}\int_{\tau}^{\mathfrak{y}}\int_{\mho} f(\check{\mathfrak{z}},6,?)[g(\check{\mathfrak{z}},6,?) + q(\check{\mathfrak{z}},6,?)]\, d?d6d\check{\mathfrak{z}}\right) d\varrho d\tau d\sigma\Bigg\}$$

This proof is complete. ■

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We would like to stress that the right-hand side of the equation and the initial condition are continuously necessary for the solution to issue (3.1). Let's look at problem (3.1) and its related problems.

$$\tilde{v}(x, y, t) = \overline{\ddot{E}}(x, y, t) + \int_0^{x}\int_0^{y}\int_{\mho}^{t} \overline{\ddot{A}}\left(x, y, t, \sigma, \tau, \varrho, \tilde{v}(\sigma, \tau, \varrho), \left(\overline{\Gamma}\tilde{v}\right)(\sigma, \tau, \varrho)\right) d\varrho\, d\tau\, d\sigma \qquad (3.5)$$

For all $(x, y, t) \in \pounds$ where

$$\left(\overline{\Gamma}\tilde{v}\right)(x, y, t) = \int_0^{x}\int_0^{y}\int_{\mho}^{t} \overline{\ddot{H}}(x, y, t, \breve{z}, 6, \square, \tilde{v}(\breve{z}, 6, \square))\, d\square\, d6\, d\breve{z} \qquad (3.6)$$

and $\tilde{v}, \overline{\ddot{E}} \in \mathcal{C}(\pounds, \hat{R})$, $\overline{\ddot{A}} \in \mathcal{C}(\pounds \times \pounds \times \hat{R} \times \hat{R}, \hat{R})$, $\overline{\ddot{H}} \in \mathcal{C}(\pounds \times \pounds \times \hat{R}, \hat{R})$

Theorem 3. Let $\ddot{w}, \dot{M} : \pounds \times \pounds \to \hat{R}_+$ be a continuous function, provided that all of Theorem 1's criteria are met.

$$\ddot{w}(x, y, t, \sigma, \tau, \varrho) = f(x, y, t)\, g(\sigma, \tau, \varrho)$$

$$\dot{M}(x, y, t, \breve{z}, 6, \square) = f(x, y, t)\, q(\breve{z}, 6, \square)$$

Where $f, g, q \in \mathcal{C}(\pounds, \hat{R}_+)$.

Let the solutions to problems (3.1) and (3.5) be $\tilde{u}(\ddot{x}, \ddot{y}, t)$ and $\tilde{v}(\ddot{x}, \ddot{y}, t)$, respectively. Assume that

$\acute{\varepsilon}_1, \acute{\varepsilon}_2 > 0$ exist in such a way that

$$\mathfrak{D}(\ddot{E}(x, y, t), \overline{\ddot{E}}(x, y, t)) \leq \acute{\varepsilon}_1$$

For all $(x, y, t) \in \pounds$ where

$$\mathfrak{D}(\ddot{A}(x, y, t, \sigma, \tau, \varrho, \tilde{v}_1, \tilde{v}_2), \overline{\ddot{A}}(x, y, t, \sigma, \tau, \varrho, \tilde{v}_1, \tilde{v}_2)) \leq \acute{\varepsilon}_2$$

For $(x, y, t, \sigma, \tau, \varrho, \mathcal{A}, \mathcal{B}) \in \pounds \times \pounds \times \hat{R} \times \hat{R}$

The following estimate is therefore accurate for the issue (3.1) solution $\tilde{u}(\ddot{x}, \ddot{y}, t)$.

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$$\mathfrak{D}(\tilde{\mathfrak{u}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}),\tilde{\mathrm{v}}(\mathfrak{x},\mathfrak{y},\mathfrak{t})) \leq (\acute{\varepsilon}_1+\acute{\varepsilon}_2)\left\{1+f(\mathfrak{x},\mathfrak{y},\mathfrak{t})\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}[f(\sigma,\tau,\varrho)+g(\sigma,\tau,\varrho)]\right.$$

$$\left.\times\, exp\left(\int_{\sigma}^{\mathfrak{x}}\int_{\tau}^{\mathfrak{y}}\int_{\mho} f(\mathfrak{z},6,\square)[g(\mathfrak{z},6,\square)+q(\mathfrak{z},6,\square)]\,d\square d6 d\mathfrak{z}\right)d\varrho d\tau d\sigma\right\}$$

For all $(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \in \pounds$

Proof. For $(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \in \pounds$, note that we have

$$\mathfrak{D}(\tilde{\mathfrak{u}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}),\tilde{\mathrm{v}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}))$$

$$= \mathfrak{D}\Bigg(\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathfrak{t})$$

$$+\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho,\tilde{\mathfrak{u}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathfrak{u}})(\sigma,\tau,\varrho)\right)d\varrho d\tau d\sigma,\overline{\ddot{\mathrm{E}}}(\mathfrak{x},\mathfrak{y},\mathfrak{t})$$

$$+\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}\overline{\ddot{\mathrm{A}}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho,\tilde{\mathrm{v}}(\sigma,\tau,\varrho),\left(\overline{\acute{\Gamma}}\tilde{\mathrm{v}}\right)(\sigma,\tau,\varrho)\right)d\varrho d\tau d\sigma\Bigg)$$

$$\leq \mathfrak{D}\left(\ddot{\mathrm{E}}(\mathfrak{x},\mathfrak{y},\mathfrak{t}),\overline{\ddot{\mathrm{E}}}(\mathfrak{x},\mathfrak{y},\mathfrak{t})\right)$$

$$+\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}\mathfrak{D}\left(\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho,\tilde{\mathfrak{u}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathfrak{u}})(\sigma,\tau,\varrho)\right),\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho,\tilde{\mathrm{v}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathrm{v}})(\sigma,\tau,\varrho)\right)\right)d\varrho d\tau d\sigma$$

$$+\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}\mathfrak{D}\left(\ddot{\mathrm{A}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho,\tilde{\mathrm{v}}(\sigma,\tau,\varrho),(\acute{\Gamma}\tilde{\mathrm{v}})(\sigma,\tau,\varrho)\right),\overline{\ddot{\mathrm{A}}}\left(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho,\tilde{\mathrm{v}}(\sigma,\tau,\varrho),\left(\overline{\acute{\Gamma}}\tilde{\mathrm{v}}\right)(\sigma,\tau,\varrho)\right)\right)d\varrho d\tau d\sigma$$

$$\leq \acute{\varepsilon}_1+\acute{\varepsilon}_2+\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}\ddot{\mathrm{w}}(\mathfrak{x},\mathfrak{y},\mathfrak{t},\sigma,\tau,\varrho)\left(\mathrm{Đ}(\tilde{\mathfrak{u}}(\sigma,\tau,\varrho),\tilde{\mathrm{v}}(\sigma,\tau,\varrho))\right.$$

$$\left.+\mathfrak{D}\left(\int_0^{\sigma}\int_0^{\tau}\int_{\mho}\ddot{\mathrm{H}}(\sigma,\tau,\varrho,\mathfrak{z},6,\square,\tilde{\mathfrak{u}}(\mathfrak{z},6,\square))d\square d6 d\mathfrak{z},\int_0^{\sigma}\int_0^{\tau}\int_{\mho}\ddot{\mathrm{H}}(\sigma,\tau,\varrho,\mathfrak{z},6,\square,\tilde{\mathrm{v}}(\sigma,\tau,\varrho))d\square d6 d\mathfrak{z}\right)\right)d\varrho d\tau d\sigma$$

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$$\leq \acute{\varepsilon}_1 + \acute{\varepsilon}_2 + f(\mathfrak{x},\mathfrak{y},t') \int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho} g(\sigma,\tau,\varrho)\Big(Đ\big(\tilde{u}(\sigma,\tau,\varrho),\tilde{v}(\sigma,\tau,\varrho)\big)$$

$$+f(\sigma,\tau,\varrho)\int_0^{\sigma}\int_0^{\tau}\int_{\mho} q(\mathfrak{z},6,\Box)\big(Đ(\tilde{u}(\mathfrak{z},6,\Box),\tilde{v}(\mathfrak{z},6,\Box)\big)d\Box\, d6\, d\mathfrak{z}\Big)d\varrho\, d\tau\, d\sigma \qquad (3.7)$$

We deduce from (3.7) and Theorem 1 in (Pachpatte, 2010), that

$$\mathfrak{D}\big(\tilde{u}(\mathfrak{x},\mathfrak{y},t'),\tilde{v}(\mathfrak{x},\mathfrak{y},t')\big) \leq (\acute{\varepsilon}_1 + \acute{\varepsilon}_2)\Bigg\{1 + f(\mathfrak{x},\mathfrak{y},t')\int_0^{\mathfrak{x}}\int_0^{\mathfrak{y}}\int_{\mho}[f(\sigma,\tau,\varrho)+g(\sigma,\tau,\varrho)]$$

$$\times\, exp\left(\int_{\sigma}^{\mathfrak{x}}\int_{\tau}^{\mathfrak{y}}\int_{\mho} f(\mathfrak{z},6,\Box)[g(\mathfrak{z},6,\Box)+q(\mathfrak{z},6,\Box)]\, d\Box\, d6\, d\mathfrak{z}\right)d\varrho\, d\tau\, d\sigma\Bigg\} \qquad (3.8)$$

for all $(\mathfrak{x},\mathfrak{y},t') \in £$.

It is evident from (3.8) that the functions involved in equation (3.1) are continually necessary for its solution. This proof is complete.■

Illustrative Example:

We will apply the conditions of the existence and uniqueness theorem to the main equation (3.1), when the kernel function is nonlinear and defined by the formula:

$Ä(\tilde{u}) = \lambda\tilde{u}^2$, $0 < \lambda < 1$, $£ = [0,1]\times[0,1]\times[a,b]$

Define the operator $\psi: \mathcal{C}(£,\hat{R}) \to \mathcal{C}(£,\hat{R})$ by

$$(\psi\tilde{u})(\mathfrak{x},\mathfrak{y},t') = Ë(\mathfrak{x},\mathfrak{y},t') + \int_0^1\int_0^1\int_a^b \lambda\tilde{u}^2(\sigma,\tau,\varrho)\, d\varrho\, d\tau\, d\sigma$$

For any $\tilde{u},\tilde{v} \in \mathcal{C}(£,\hat{R})$, using properties of fuzzy numbers, we obtain

$$d\big(Ä(\tilde{u}),Ä(\tilde{v})\big) \leq 2\lambda\hat{H} d(\tilde{u},\tilde{v})$$

Hence, the kernel Ä satisfies a Lipschitz condition. Consequently,

$$D(\psi\tilde{u},\psi\tilde{v}) \leq 2\lambda\hat{H}\, ab D(\tilde{u},\tilde{v})$$

Therefore, by the (BFP) Theorem, the fuzzy Volterra–Fredholm integral equation admits a unique fuzzy solution in $\mathcal{C}(£, \hat{R})$.

4-Conclusion

The study examined fuzzy nonlinear Volterra–Fredholm integral equations in three variables as an important class of nonlinear integral models. Convergence of the proposed fuzzy formulation with nonlinear fuzzy kernels was established. Key analytical properties of the solution were verified. The parametric formulation satisfied the necessary and sufficient conditions of the Lebenszian model. Existence and uniqueness of the solution were proven under a contraction-based regularity criterion. The results provide a concise theoretical contribution to fuzzy integral equation analysis and support future extensions in more complex fuzzy systems.

Future work may focus on numerical approximations of the proposed fuzzy integral equations, as well as extensions to fractional or stochastic fuzzy integral models with applications in engineering and decision-making problems.

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