



RESEARCH ARTICLE - MATHEMATICS

**The new search direction of four term conjugate gradient method for unconstrained optimization**

Ali Kassim Jabur <sup>1\*</sup>, Saad Shakir Mahmood <sup>2</sup>

<sup>1</sup> Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq

<sup>2</sup> Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq

\* Corresponding author E-mail: [alsrayly842@gmail.com](mailto:alsrayly842@gmail.com)

Article Info.	Abstract
<p><i>Article history:</i></p> <p>Received 27 October 2024</p> <p>Accepted 1 January 2025</p> <p>Publishing 30 March 2026</p>	<p>In this paper, we suggested a new four-term CG method, by expanding the three-term CG method to the four-term CG method. The search direction <math>p</math>, with some newly proposed parameters, is a four-term hybrid form for modified traditional CG methods. This hybridization yielded a newly created hybrid CG method with four terms as its ultimate result. The new method proposed ( the four-term CG method ) fulfills the conjugacy condition by the Dai-Laiao method and possesses sufficient descent property. Also, it was confirmed that the suggested new method would globally converge to the general Wolfe conditions beneath an adequate assumption. Finally, a comparison was made between the three terms CG that were adopted in the expansion and the proposed four-term formula, the computational results demonstrated the new method's robustness and effectiveness.</p>

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**1. Introduction**

Optimization is crucial, In many scientific fields, which have included various sectors and facets. It is an important part of the physical and mathematical sciences. additionally to their significance in the fields of engineering, mechanical engineering in particular, electricity, management, the economy, population growth, weather, and other natural phenomena. Knowing that many researchers in various fields and specializations have presented extremely important research to expand that field or specialization, including, for example:[1], [2]

The problem that emerges in many practical situations is the minimization of a given objective function, where is an n-dimensional real vector, which may be subject to constraints or without any constraints. The definition of the general unconstrained optimization problem is

$$\min f(x), x \in \mathbb{R}^n \tag{1}$$

Where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is twice continuously differentiable. The gradient of  $f(x)$  is defined by

$$g(x) = \nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1} \quad \frac{\partial f(x)}{\partial x_2} \quad \dots \quad \frac{\partial f(x)}{\partial x_n} \right)^T \tag{2}$$

The conjugate gradient method generates a sequence  $x_k$  given by

$$x_{k+1} = x_k + \alpha_k p_k \quad k = 1, 2, 3, \dots \tag{3}$$

Where  $\alpha_k$  is step length determined by line search (Wolfe conditions or strong Wolfe conditions) [4]

The Wolfe conditions

$$f(x_k + \alpha_k p_k) \leq f(x_k) + \rho \alpha_k g_k^T p_k \tag{4}$$

$$p_k^T g(x_k + \alpha_k p_k) \geq \sigma p_k^T g_k \tag{5}$$

The strong Wolfe conditions it is just becomes the second condition in this way

$$|p_k^T g(x_k + \alpha_k p_k)| \geq -\sigma p_k^T g_k \tag{6}$$

Such that  $0 < \sigma \leq \rho < 1$  and  $p_k$  is descent direction. The direction  $p_k$  is calculated as follows

$$p_{k+1} = -g_k \text{ if } k = 1, p_{k+1} = -g_{k+1} + \beta_k p_k \text{ if } k > 1 \tag{7}$$

The classical formula of CG method are:

$$\beta_k^{HS} = \frac{g_k^T y_{k-1}}{p_{k-1}^T y_{k-1}} \text{ that is HS formula [5]}$$

$$\beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\|g_{k+1}\|^2} \text{ that is PRP formula [6]}$$

$$\beta_k^{LS} = -\frac{g_k^T y_{k-1}}{p_{k-1}^T g_{k-1}} \text{ that is LS formula [7]}$$

$$\beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k+1}\|^2} \text{ that is FR formula [8]}$$

$$\beta_k^{DY} = \frac{\|g_k\|^2}{p_{k-1}^T g_{k-1}} \quad \text{that is DY formula [9]}$$

In addition to the above-mentioned formulas, there are researchers who have developed and presented new formulas[10], [11].

Numerous more formulations of  $\beta_k$  created by researchers exist as well. Three-term CG - methods are another kind of conjugate gradient method. We take some formulas presented by researchers for example:

$$d^{k+1} = -g_{k+1} + \beta_k p_k + \gamma_k p_t \quad [8] \tag{8}$$

Where  $\beta_k$  is ( $\beta_k^{HS}$  or  $\beta_k^{FR}$  ... etc.)

$$\text{and } \gamma_k = 0 \text{ if } k = t + 1, \quad \gamma_k = \frac{g_{k+1}^T y_t}{p_t^T y_t} \text{ if } k > t + 1$$

$$p^{k+1} = -y_k + \frac{y_k^T y_k}{y_k^T p_k} p_k + \frac{y_{k-1}^T y_k}{y_{k-1}^T p_k} p_{k-1} \quad [8] \tag{9}$$

Also, Anderi in 2012 [12] Suggest another formula as follows:

$$p_{k+1} = -g_{k+1} - S_k S_k - \eta_k y_k \tag{10}$$

$$\text{Where } S_k = \left(1 + \frac{\|y_k\|^2}{y_k^T S_k}\right) \frac{S_k^T g_{k+1}}{y_k^T S_k} - \frac{y_k^T g_{k+1}}{y_k^T S_k} \tag{11}$$

$$\text{An } \eta_k = \frac{S_k^T g_{k+1}}{y_k^T S_k} \tag{12}$$

And Also, Jinkui Liu and Xuesha Wu, et al in 2014 [13] Suggest another formula about al Three–Term CG Method as follows:

$$p_{k+1} = -g_{k+1} - \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) y_k \tag{13}$$

many researchers have formulated trinomial formulas in recent years [14], [15], [16]. Recently, some formula was proposed for a new search direction consisting of four terms for example Azzam, M. et al [17] as follows:

$$p_{k+1} = -g_{k+1} + \beta_k^{HS} p_k - t \frac{g_{k+1}^T S_k}{|p_k^T y_k|} p_k - \frac{g_{k+1}^T p_k}{p_k^T y_k} y_k \tag{14}$$

In addition, in recent years and at the present time, this field has begun to expand and some researchers have delved into it, including:[18], [19].

## 2. The new extended search direction

We introduced the new formula, which we extended to four-term CG as follows, based on (13)

$$p_{k+1} = -g_{k+1} - 2 \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) y_k + \vartheta_k p_k \tag{15}$$

$$\text{Where } \vartheta_k = -\frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2 y_k^T p_k} \tag{16}$$

We write the new formula in detail as follows:

$$p_{k+1} = -g_{k+1} - 2 \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) y_k - \left(\frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2 y_k^T p_k}\right) p_k \tag{17}$$

Now, we will present how to fulfill the Conjugacy condition for  $p_{k+1}$  given by (17) when satisfies the condition  $y_k^T p_{k+1} = 0$ , in Dai-Liao method modified condition to  $y_k^T p_{k+1} = -\omega g_{k+1}^T S_k$  in the following proposition

**Proposition 2.1.** If the line search satisfy the conditions (4) and (5) , then  $p_{k+1}$  given by (17) satisfies the conjugacy condition that is  $y_k^T p_{k+1} = -\omega g_{k+1}^T S_k$  where  $\omega > 0$  .

**Proof:** From (17) we have

$$p_{k+1} = -g_{k+1} - 2 \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) y_k - \left(\frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2 y_k^T p_k}\right) p_k$$

Multiplying by  $y_k^T$  we get

$$y_k^T p_{k+1} = -y_k^T g_{k+1} - 2 \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) y_k^T S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) y_k^T y_k - \left(\frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2 y_k^T p_k}\right) y_k^T p_k$$

$$\begin{aligned}
 &= -y_k^T g_{k+1} - 2 g_{k+1}^T S_k + y_k^T g_{k+1} - \frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2} \\
 &= -2 g_{k+1}^T S_k - \frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2} \\
 &= -\left(2 + \frac{1}{2 (y_k^T y_k)^2}\right) g_{k+1}^T S_k
 \end{aligned}$$

Since  $(y_k^T y_k)^2 > 0$ , so  $\left(2 + \frac{1}{2 (y_k^T y_k)^2}\right) > 0$

Letting  $\omega = 2 + \frac{1}{2 (y_k^T y_k)^2}$ , we get

$$y_k^T p_{k+1} = -\omega g_{k+1}^T S_k$$

Therefore the Conjugacy condition is hold ■

Now, we will present how to fulfill the descent direction for  $p_{k+1}$  given by (17) in the following proposition

**Proposition 2.2.** If the line search satisfy the conditions (4) and (5) then  $p_{k+1}$  given by (17) is descent direction.

**Proof:** since the line search direction satisfies the conditions (4) and (5), so  $y_k^T S_k > 0$ . From (17) we have

$$p_{k+1} = -g_{k+1} - 2 \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) y_k - \left(\frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2 y_k^T p_k}\right) p_k$$

Multiplying by  $g_{k+1}^T$  we get

$$\begin{aligned}
 g_{k+1}^T p_{k+1} &= -g_{k+1}^T g_{k+1} - 2 \left(\frac{g_{k+1}^T S_k}{S_k^T y_k}\right) g_{k+1}^T S_k + \left(\frac{y_k^T g_{k+1}}{y_k^T y_k}\right) g_{k+1}^T y_k - \left(\frac{g_{k+1}^T S_k}{2 (y_k^T y_k)^2 y_k^T p_k}\right) g_{k+1}^T p_k \\
 &= -\|g_{k+1}\|^2 - 2 \frac{(g_{k+1}^T S_k)^2}{S_k^T y_k} + \frac{(y_k^T g_{k+1})^2}{y_k^T y_k} \\
 &= -\|g_{k+1}\|^2 - 2\alpha_k \frac{(g_{k+1}^T p_k)^2}{p_k^T y_k} + \frac{(y_k^T g_{k+1})^2}{y_k^T y_k} \\
 &\leq -\|g_{k+1}\|^2 + \frac{(y_k^T g_{k+1})^2}{y_k^T y_k}
 \end{aligned}$$

Now, using the following relationship:  $y_k^T g_{k+1} = \|y_k\| \|g_{k+1}\| \cos(y_k, g_{k+1})$  we get

$$g_{k+1}^T p_{k+1} \leq -\|g_{k+1}\|^2 + \frac{(\|y_k\| \|g_{k+1}\| \cos(y_k, g_{k+1}))^2}{y_k^T y_k}$$

If  $\cos(y_k, g_{k+1}) = 0$ , then  $g_{k+1}^T p_{k+1} \leq -\|g_{k+1}\|^2$

Now, if  $\cos(y_k, g_{k+1}) \neq 0$ , then

$$g_{k+1}^T p_{k+1} \leq -\|g_{k+1}\|^2 + (\|g_{k+1}\| \cos(y_k, g_{k+1}))^2$$

Since there exists  $u \in (0,1)$  such that

$$\cos^2(y_k, g_{k+1}) \leq 1 - u$$

So, we have

$$\begin{aligned}
 g_{k+1}^T p_{k+1} &\leq \|g_{k+1}\|^2 (-1 + \cos^2(y_k, g_{k+1})) \\
 &\leq \|g_{k+1}\|^2 (-1 + 1 - u) \\
 &\leq -u \|g_{k+1}\|^2
 \end{aligned}$$

Therefore the descent direction is hold ( $p_{k+1}^T g_{k+1} < 0$ ) ■

### The New extended four-term CG Algorithm

Stage 1: set  $k = 0$ , select the initial point  $x_0 \in R^n$  and  $\epsilon > 0$

Stage 2: Compute  $f_0 = f(x_0)$  and  $g_0 = \nabla f(x_0)$

Stage 3: If  $\|g_0\| \leq \epsilon$  stop, else

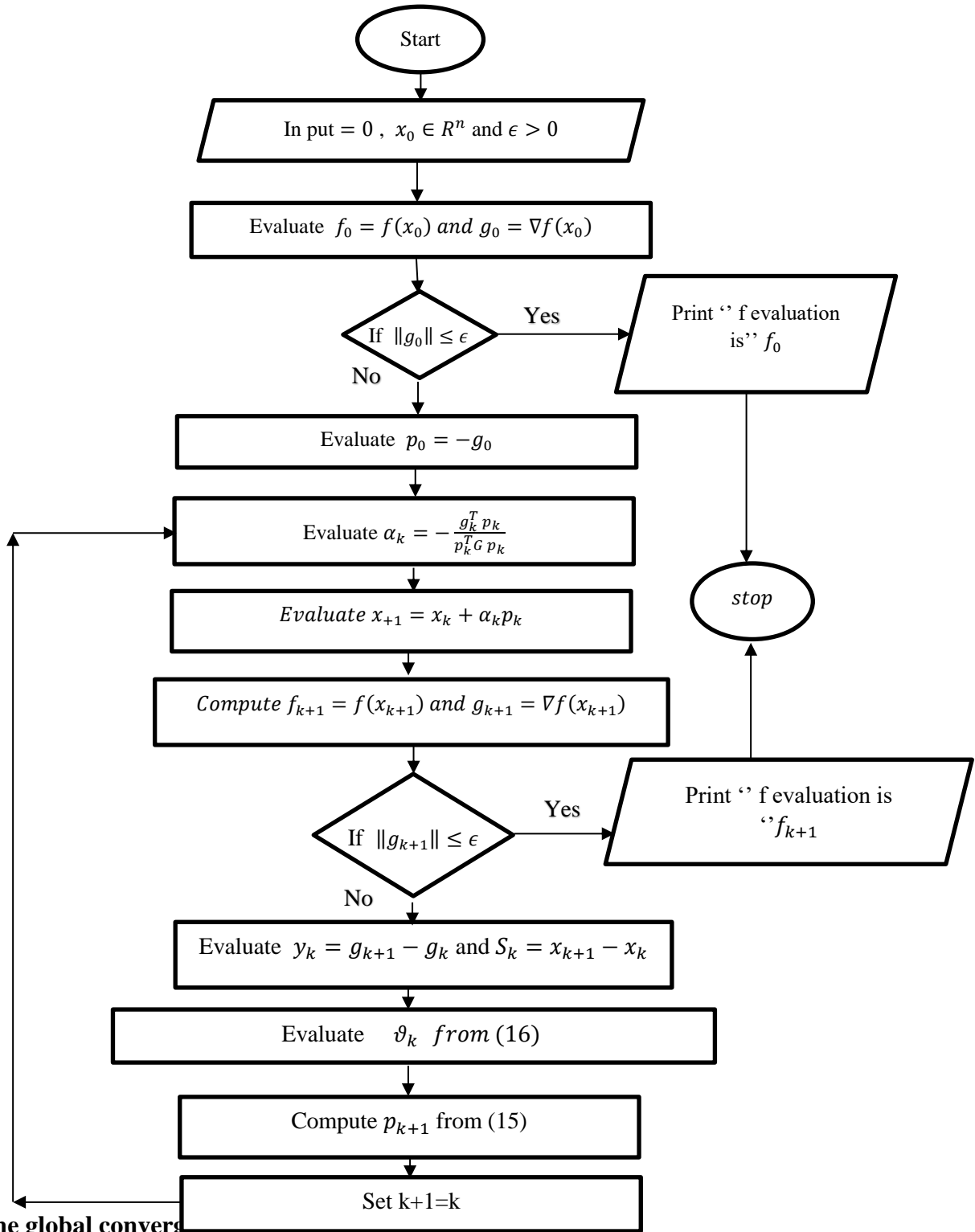
Stage 4: Set  $p_0 = -g_0$

Stage 5: Evaluate  $\alpha_k = -\frac{g_k^T p_k}{p_k^T G p_k}$  satisfy Wolfe condition

Stage 6: Evaluate  $x_{k+1} = x_k + \alpha_k p_k$

- Stage 7: Compute  $f_{k+1} = f(x_{k+1})$  and  $g_{k+1} = \nabla f(x_{k+1})$
- Stage 8: If  $\|g_{k+1}\| \leq \epsilon$  stop ,else
- Stage 9: Evaluate  $y_k = g_{k+1} - g_k$  and  $S_k = x_{k+1} - x_k$
- Stage 10: Evaluate  $\vartheta_k$  from (16)
- Stage 11: Compute  $p_{k+1}$  from (15)
- Stage 12: Set  $k+1=1$
- Stage 13: go to stage 5

Now, one can write flow chart of a new four term CG – method



### 3. The global convergence

In this section we will present how to prove the global convergence of this new direction, starting from Assume the following:

I. let  $\mathcal{L} = \{x \in R^n : f(x) < f(x_0)\}$  be level set is bounded i.e.  $\exists B > 0$  such that,  $\forall x \in S, \|x\| \leq B$

II. let  $\mathcal{N}$  be a neighborhood of  $S$  and  $f$  be a function is differentiable continuously and  $\nabla f$  is Lipschitz continuous, i.e.  $\exists L > 0$  such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \text{ for every } x, y \in \mathcal{N}.$$

These assumptions on  $f$ ,  $\exists$  a constant  $\ell \geq 0$  such that  $\|\nabla f(x)\| \leq \ell, \forall x \in \mathcal{L}$ .

In order to demonstrate the worldwide convergence of nonlinear conjugate gradient algorithms, the Zoutendijk condition is employed.

**Proposition 3.1.** If the assumption (I) and (II) are hold, set  $x_{k+1} = x_k + \alpha_k p_k$  and  $p_k$  from equation (15) is descent direction and  $\alpha_k$  is computed from (4) and (5) then

$$\sum_{k=0}^{\infty} \frac{(g_k^T p_k)^2}{\|p_k\|^2} < +\infty. \quad (18)$$

**Proof:** from (5) we have  $g_{k+1}^T p_k \geq \sigma g_k^T p_k$

$$g_{k+1}^T p_k - g_k^T p_k \geq \sigma g_k^T p_k - g_k^T p_k$$

$$(\sigma - 1)g_k^T p_k \leq (g_{k+1} - g_k)^T p_k$$

$$\leq \|(g_{k+1} - g_k)^T\| \|p_k\|$$

By using Lipschitz continuity we have

$$(\sigma - 1)g_k^T p_k \leq \alpha_k L \|p_k\|^2$$

$$\text{Hence } \alpha_k \geq \frac{(1-\sigma)|g_k^T p_k|}{L\|p_k\|^2} \quad (19)$$

Now, from  $f(x_{k+1}) - f(x_k) \leq \rho \alpha_k g_k^T p_k$  and (19) we get

$$f(x_{k+1}) - f(x_k) \geq \rho \frac{(1-\sigma)|g_k^T p_k|}{L\|p_k\|^2} g_k^T p_k$$

$$\geq \rho \frac{(1-\sigma)(g_k^T p_k)}{L\|p_k\|^2}$$

By assumptions (I) we have  $\sum_{k=0}^{\infty} \frac{(g_k^T p_k)^2}{\|p_k\|^2} < +\infty$  this Zoutendijk condition ■

**Proposition 3.2.** If the assumptions (I) and (II) hold, set  $x_{k+1} = x_k + \alpha_k p_k$  and  $p_k$  from equation (15) is descent direction and  $\alpha_k$  is computed from (4) and (6). If  $\sum_{k \geq 1} \frac{1}{\|p_k\|^2} = \infty$  then  $\lim_{k \rightarrow \infty} (\inf \|g_k\|) = 0$ .

We can demonstrate that the norm of the direction  $p_{k+1}$  given by (15) is bounded above for uniformly convex functions. Therefore, we may demonstrate the following Theorem using Proposition 2.

**Theorem 3.3.** If the assumptions (I) and (II) hold, set  $x_{k+1} = x_k + \alpha_k p_k$  and  $p_k$  from equation (15) is descent direction and  $\alpha_k$  is computed from (4) and (6). let  $f$  be uniformly convex function on  $\mathcal{L}$ , i.e.  $\exists m > 0$  such that  $(\nabla f(x) - \nabla f(z))^T (x - z) \geq m \|x - z\|^2, \forall x, z \in \mathcal{N}$ , then  $\lim_{k \rightarrow \infty} \|g_k\| = 0$

**Proof:** from (17) we get

$$p_{k+1} = -g_{k+1} - 2 \left( \frac{g_{k+1}^T S_k}{S_k^T y_k} \right) S_k + \left( \frac{y_k^T g_{k+1}}{y_k^T y_k} \right) y_k - \left( \frac{g_{k+1}^T S_k}{2(y_k^T y_k)^2 y_k^T p_k} \right) p_k$$

$$\|p_{k+1}\| \leq \|g_{k+1}\| + 2 \left\| \left( \frac{g_{k+1}^T S_k}{S_k^T y_k} \right) S_k \right\| + \left\| \left( \frac{y_k^T g_{k+1}}{y_k^T y_k} \right) y_k \right\| + \left\| \left( \frac{g_{k+1}^T S_k}{2(y_k^T y_k)^2 y_k^T p_k} \right) p_k \right\|$$

$$\leq \|g_{k+1}\| + 2 \frac{|g_{k+1}^T S_k|}{|S_k^T y_k|} \|S_k\| + \frac{|y_k^T g_{k+1}|}{|y_k^T y_k|} \|y_k\| + \frac{|g_{k+1}^T S_k|}{2|(y_k^T y_k)^2| |y_k^T p_k|} \|p_k\|$$

$$\leq \|g_{k+1}\| + 2 \frac{\|g_{k+1}\| \|S_k\|}{|S_k^T y_k|} \|S_k\| + \frac{\|y_k\| \|g_{k+1}\|}{\|y_k\|^2} \|y_k\| + \frac{\|g_{k+1}\| \|S_k\|}{2\|y_k\|^4 \|y_k\| \|p_k\|} \|p_k\|$$

Since assumptions (II) hold we have  $\|y_k\| \leq L \|S_k\|$  and  $\exists \ell \geq 0$  such that

$\|\nabla f(x)\| \leq \ell, \forall x \in \mathcal{L}$ . and also uniformly convexity  $y_k^T S_k \geq m \|S_k\|^2$ , and also assumptions (I) hold we have,  $\forall x \in S, \|x\| \leq B$

therefore we get

$$\begin{aligned} \|p_{k+1}\| &\leq \ell + 2 \frac{\ell \|S_k\|}{m \|S_k\|^2} \|S_k\| + \frac{\|y_k\| \ell}{\|y_k\|^2} \|y_k\| + \frac{\ell \|S_k\|}{2L^4 \|S_k\|^4 L \|S_k\| \|p_k\|} \|p_k\| \\ &\leq \ell + 2 \frac{\ell}{m} + \ell + \frac{\ell}{2L^4 B^4 L} \end{aligned}$$

Hence  $\|p_{k+1}\| \leq \ell \left( 2 + 2 \frac{1}{m} + \frac{1}{2L^4 B^4 L} \right)$

$\|p_{k+1}\| \leq C$ , where  $\ell \left( 2 + 2 \frac{1}{m} + \frac{1}{2L^4 B^4 L} \right) = C$  is positive number

One can have  $\frac{1}{\|p_{k+1}\|^2} \geq \frac{1}{C^2}$

$$\sum_{k \geq 1} \frac{1}{\|p_{k+1}\|^2} \geq \sum_{k \geq 1} \frac{1}{C^2}$$

$$\sum_{k \geq 1} \frac{1}{\|p_{k+1}\|^2} \geq \frac{1}{C^2} \sum_{k \geq 1} 1 = \infty$$

Therefore  $\sum_{k \geq 1} \frac{1}{\|p_{k+1}\|^2} = \infty$ , and by using Proposition 3.2. we get  $\lim_{k \rightarrow \infty} (\inf \|g_k\|) = 0$  which for uniformly convex is equivalent to

$$\lim_{k \rightarrow \infty} (\|g_k\|) = 0 \quad \blacksquare$$

#### 4. Numerical results

This section is devoted to numerical experiments. MATLAB is used to write the computer applications. The rationale behind their choice is that the issues seem to have been utilized in typical problems in the majority of the literature; these functions are the outcome of their use in the industry and technological field.

The performance of the four-term conjugate gradient method's new search direction has been evaluated. and compared with standard the three terms that were adopted in the expansion method using the following test functions. For the sake of uniformity in comparison. Table 1 presents the findings.

The test functions are chosen as follows:

1. Brown's badly scaled function[20]
2. Rosen brock's function [20]
3. A quartic function [20]
4. Extended Cliff function[21]
5. quadratic QF1 function[21]
6. Extended Himmelblau function[21]
7. DQDRTIC function (CUTE) [21]
8. Perturbed Tridiagonal Quadratic function[21]
9. Extended Maratos function[21]
10. Almost Perturbed Quadratic function[21]
11. QUARTC function(CUTE) [21]
12. Generalized Quartic function[21]
13. Rosen rock's function[21]
14. Powell singular function[21]

Table 1 Explains the four-term CG - method's new search direction, for our selected test functions and compares with standard the three terms that were adopted in the new expansion method.

At first, we give the number of iterations and then the F evaluation (Feval) of every function starting from different points.

Then after that, we give all the results of this analysis, and what is the best in each case.

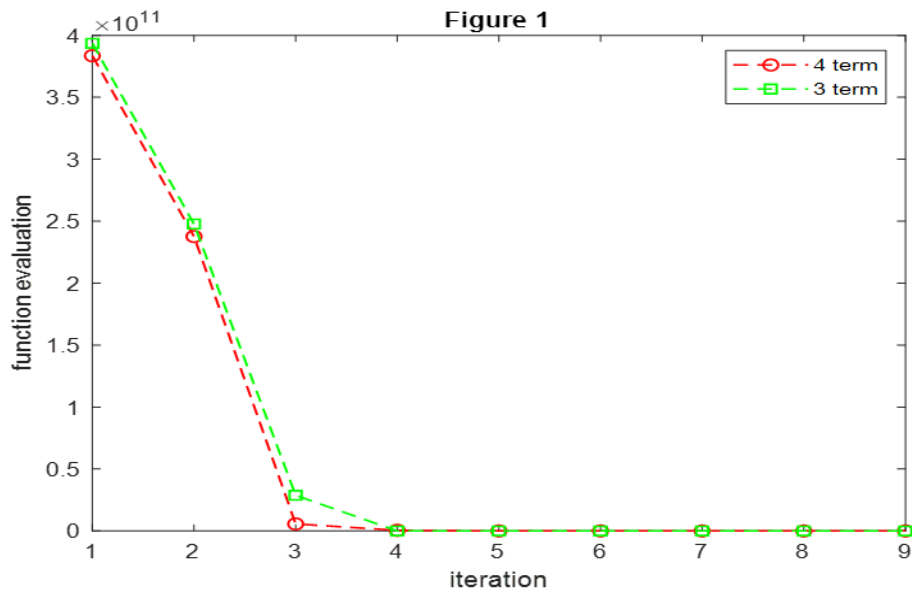
So, in comparison, preference is mainly based on Feval. In other cases, if Feval is equal in the two methods, we resort to the number of iterations.

**TABLE 1.** Comparison between New update (4 terms) and Old update (3 terms)

Fun	Starting point	Dim.	New update 4 terms	Old update 3 terms	The bast
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			iter	feval	iter	feval	
1	[2 2]'	2	9	8.1909e-33	9	5.8847e-26	4 terms
2	[-0.5 1.5]'	2	2	0	2	0	both
3	[0 2 0 2]'	4	11	5.7079e-21	10	1.3805e-13	4 terms
3	[1 1 0 1]'	4	42	4.8416e-10	62	9.714e-10	4 terms
4	[9 9 9 9 9]'	6	7	2.3196	7	2.3196	both
5	[13 13 13 13 13 13 13]'	7	11	-0.071429	11	-0.071429	both
6	[1.0005 1 1 1.005 1 1 1 1.0005]'	8	6	8.8385e-19	6	9.5739e-18	4 terms
7	[40 0 40 40 40 40 40 40 0]'	9	10	3.8837e-13	12	3.8872e-13	4 terms
7	[0 11 0 0 13 13 0 0 11 11]'	10	7	2.2059e-13	7	2.2059e-13	both
8	[0.002 0 0 0.002 0 0 0 0.002 0 0 0]'	11	16	1.1206e-09	16	1.1206e-09	both
9	[-0.5 -0.5 -0.5 -0.5 -0.5 -0.5 -0.5 -0.5 -0.5 -0.5 -0.5]'	12	23	-6.0037	33	-6.0037	4 terms
10	[ 0 0.25 0 0.05 0 0.25 0 0 0 0.25 0 0.05 0 0.25 0]'	15	25	1.8952e-10	25	1.9005e-10	4 terms
11	[10 20 30 40 50 60 70 80 90 10 10 20 30 40]'	14	15	5.579e-08	15	6.7777e-08	4 terms
12	[5 5 5 5 5 5 5 5 5 5 5 5 6 6 6 6]'	19	11	2.9611e-11	11	3.357e-11	4 terms
13	[10 10 10 10 10 10 10]'	7	2	1.7245e-21	2	1.7245e-21	both
14	[0 0.2 0.01 0]'	4	6	5.272e-09	6	5.3052e-09	4 terms

We will take, for example, Brown’s badly scaled function and draw it in both ways to show more clearly that the red color represents New update (4 terms) which has a function evaluation 8.1909e-33 at iteration 9 while the green color represents Old update (3 terms) which has a function evaluation 5.8847e-26 at iteration 9, as shown below.



**Figure 1.** Brown's badly scaled function

## 5. Conclusions

Intriguing computational innovations that yield effective conjugate gradient algorithms are represented by four-term conjugate gradient approaches. Numerous CG- algorithms with three terms can be produced, and in this regard, we propose the following project. These project-related algorithms' search direction is calculated as

$$p_{k+1} = -g_{k+1} - 2 \left( \frac{g_{k+1}^T S_k}{S_k^T y_k} \right) S_k + \left( \frac{y_k^T g_{k+1}}{y_k^T y_k} \right) y_k + \vartheta_k p_k$$

as modified for the three-term CG - method. Where the descent requirement and/or the conjugacy condition are to be satisfied by the determination of the scalar parameters  $\vartheta_k$ . Naturally, it is possible to apply more requirements on  $p_{k+1}$ . Numerous four-term CG - algorithms can be created with this concept in mind.

The paper presented a new four-term CG - algorithm. It has been demonstrated in this situation the conjugacy and descent conditions are both satisfied. It was confirmed that the new method would globally converge.

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