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RESEARCH ARTICLE - MATHEMATICS

Reliability of a coherent multicomponent system with strength falling between two stresses following Chen distribution

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Abstract

This research presented reliability formula for multicomponent strengths falling between two stresses of a coherent system $P(T < X_i < Z)$, based on Chen distribution with unknown parameters $\gamma, \mu, \alpha_i; i = 1,2,3$ and known common parameter β for the two stress random variables T, Z and component strengths X_i, respectively. Three methods were discussed for estimating Chen distribution parameters that used to estimate the reliability function using maximum likelihood, least squares, and weighted least squares estimation methods. These estimates are compared via a simulation study using mean square error criteria to large, medium, and small samples across four different assumed experiments for the parameter values of the random variables. The comparison's most significant results are that the estimator's performance for maximum likelihood is better in most of these experiments that have been studied.

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1. Introduction

In reliability research, a stress-strength model represents life-time of a component with random strength X that subject to random stress Y. The classic stress-strength reliability concept involves estimating probability $P(X < Y)$ of component failure, when stress Y exceeds strength X. This model has been extensively investigated under different distributional hypotheses for X and Y [1], [2], [3], [4]. A significant case includes the estimate $R = P(T < X < Z)$, that illustrates a situation of the strength X must exceed a stress T but also remain below a stress Z. For instance, many devices fail to operate while the temperature is high or low. likewise, a person's blood pressure must fall within two ranges: diastolic and systolic [2], [5]. The stress-strength model, represented by $P(T < X < Z)$ has many applications in diverse fields such as engineering, psychology, genetics, and clinical trials [6]. The problem was in finding the reliability of the system in multicomponent stress-strength models because the strengths of various components may differ, and they may be subjected to the same or different stresses. Such situations have been discussed in coherent systems by many authors, J. D. Esary and F. Proscha in 1963, explored some general aspects relating to reliability for coherent systems that components are independent, but may not necessarily have the same reliability [7]. Richard E. Barlow in 1978, generalized the theory of binary coherent systems for multi-state components [8]. Jorge Navarroa et al. in 2005, using Samaniego's signatures, some ordering properties have been extended to coherent systems containing identically distributed components and independent to coherent systems containing (possibly) dependent components [9]. Serkan Eryilmaz in 2008, evaluated the reliability for coherent structures and established a multivariate stress–strength model depending on the conditional ordering of X_is and Y [10]. Debasis Bhattacharya and Soma Roychowdhury in 2013, showed how to derive stress-strength reliability - at least the lower bound - for multicomponent system. A system's stress-strength reliability was expressed to be a function for reliabilities of the stress-strength of its various components [11]. Mohammad Khanjari Sadegh in 2021, investigated stress-strength modeling for coherent systems, focusing on the effect of different stress levels on system reliability. He proposed ADDIN CSL_CITATION {"citationItems":[{"id":"ITEM-1","itemData":{"ISSN":"0196-6324","author":[{"dropping-particle":"","family":"Sadegh","given":"Mohammad Khanjari","non-dropping-particle":"","parse-names":false,"suffix":""}],"container-title":"American Journal of Mathematical and Management Sciences","id":"ITEM-1","issue":"4","issued":{"date-parts":[["2021"]]},"page":"336-339","publisher":"Taylor & Francis","title":"Erratum to: Reliability of a Coherent System in a Multicomponent Stress-Strength Model","type":"article-journal","volume":"40"},"uris":["http://www.mendeley.com/documents/?uuid=6425fcb1-6cca-4849-bb9a-586d406e7691"]}],"mendeley":{"formattedCitation":"[12]","plainTextFormattedCitation":"[12]","previouslyFormattedCita tion":"(Sadegh, 2021)"},"properties":{"noteIndex":0},"schema":"https://github.com/citation-style-

The main aim for this research is to find a mathematical formulation of the reliability R for the multicomponent stress-strength model for coherent system for $P(T < X_i < Z)$ depends on Chen distribution in section 5. Three different methods of estimation (Maximum Likelihood, Least Square, and Weighted Least Square methods) have been used to obtain estimates of the scale parameters $(\gamma, \mu, \alpha_1, \alpha_2, \alpha_3)$ for five random variables, and then estimate reliability parameters in section 6. To compare the performance of different reliability estimates, a simulation study was carried out in Section 7, via four experiments with values of scale parameter and different sample sizes (Small: 15; medium: 30; large: 90). The mean square error criteria are used to do this comparison, and discuss the conclusions in section 8.

2. Coherent system

The system's reliability can be described in a number of ways depending on its structure. A binary indicator variable u_i represents the performance of n components in the system and can be defined as:[10], [13]

$$u_i = \begin{cases} 1 & \text{if the } i\text{th component operates} \\ 0 & \text{if the } i\text{th component fails} \end{cases}$$

In the same way, the binary variable $\emptyset$, which represents the system's state as a function of $\mathbf{u} = (u_1, u_2, \dots, u_n)$, can be defined as follows:

$$\emptyset(\mathbf{u}) = \begin{cases} 1 & \text{if the system operates} \\ 0 & \text{if the system fails} \end{cases}$$

This function is known as the system's structure function.

Any system's reliability can be examined using the coherent systems concept. The coherent system can be simply described with binary states u_i and $\emptyset(\mathbf{u})$.

The system is known to be coherent system if

1) The structure function $(\emptyset)$ is non-decreasing in all of its arguments
2) each component is relevant, which means that there is at least one vector u for which $\emptyset(1_i, \mathbf{u}) = 1$ and $\emptyset(0_i, \mathbf{u}) = 0$

Where $(0_i, \mathbf{u}) = (u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_n)$

$(1_i, \mathbf{u}) = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$

The minimal paths and cuts concept is used to express the structure function of a coherent system.[13], [14]

- Minimal path: is the minimal number of components whose operation guarantees the system's operation.

$$\emptyset(\mathbf{u}) = \coprod_{j=1}^{p} \rho_j(\mathbf{u}) = 1 - \prod_{j=1}^{p}\left[1 - \rho_j(\mathbf{u})\right]$$

Where $\rho_j(\mathbf{u})$ is the jth minimal path series structure of path A_j.

- Minimal cut: is the minimal number of components that, if they failed, the entire system will fail.

$$\emptyset(\mathbf{u}) = \prod_{k=1}^{s} \delta_k(\mathbf{u})$$

Where $\delta_k(\mathbf{u})$ is the kth minimal parallel cut structure of cut B_k.

The Reliability of a system is defined by:

$$R_s = P[\emptyset(\mathbf{X}) = 1]$$

Let A_j be the event in which the ith minimal path set is functioning (i.e., $\rho_j(\mathbf{u}) = 1$) and B_i be the event in which all component of the ith minimal path set are functioning, i.e., $B_j = \{T < X_i < Z\} = \{u_i = 1\}$. The system's reliability,

$$R = P(\varphi(\mathbf{u}) = 1)$$

$$= P(\text{at least one minimal path sets work}) = P\left(\cup_{j=1}^{p} A_j\right) = P\left\{\cup_{j=1}^{p}\left(\cap_{i\in P_j} B_i\right)\right\}$$

Because

$$A_j = \left\{j\text{th minimal path set } P_j \text{ is functioning}\right\}$$

$$A_j = \left\{\text{all } X_i \text{ is such that } T < X_i < Z, \text{for } i \in P_j\right\} = \cap_{i\in P_j} B_i$$

$$P(j\text{th minimal path set is working}) = P\left(A_j\right) = P\left(\cap_{i\in P_j}(T < X_i < Z)\right)$$

Result

In a multicomponent system, if the ith component with strength are falling between two stresses T_i and Z_i, then its reliability is given by:

$$R = P\left[\cup_{j=1}^{p}\left\{\cap_{i\in P_j}(T_i < X_i < Z_i)\right\}\right] = P\left[\cap_{k=1}^{s}\left\{\cup_{i\in C_k}(T_i < X_i < Z_i)\right\}\right]$$

Proof:

$$R = P(\text{system is working})$$
$$= P(\varphi(\mathbf{u}) = 1) = P\left[\coprod_{j=1}^{p} \rho_j(\mathbf{u}) = 1\right]$$
$$= P\left\{\coprod_{j=1}^{p}\left(\prod_{i\in P_j} u_j\right) = 1\right\}$$
$$= P\left[\cup_{i=1}^{p}\left\{\text{all } u_i \text{ belonging to } P_j \text{ is } 1\right\}\right]$$
$$= P\left[\cup_{j=1}^{p}\left\{\cap_{i\in P_j}\left(u_j = 1\right)\right\}\right]$$
$$= P\left[\cup_{j=1}^{p}\left\{\cap_{i\in P_j}(T_i < X_i < Z_i)\right\}\right]$$

Similarly, a second equality can also be proved.

3. Chen Distribution

In recent years, some probability distributions have been proposed for modeling the real life data with bathtub shape failure rates. A two-parameter life distribution with bathtub shape or increasing hazard function was proposed by Chen. Let X is a random variable follows Chen distribution then cumulative density function c.d.f is given by: [15]

$$F(x) = 1 - e^{\alpha\left(1-e^{x^{\beta}}\right)} \qquad ; x > 0; \alpha, \beta > 0 \tag{1}$$

Probability density function p.d.f:

$$\mathrm{f}(x) = \alpha\beta x^{\beta-1} e^{x^{\beta}} e^{\alpha\left(1-e^{x^{\beta}}\right)} \qquad ; x > 0; \alpha, \beta > 0 \tag{2}$$

Where α, β are scale and shape parameters respectively.

Since $\mathrm{f}(x)$ is p.d.f and $\int_0^{\infty} \mathrm{f}(x)dx = 1$, then equation (2) can be rewritten as:

$$\int_0^{\infty} \beta x^{\beta-1} e^{x^{\beta}} e^{\alpha\left(1-e^{x^{\beta}}\right)} dx = \frac{1}{\alpha} \tag{3}$$

4. Reliability Formulation

The reliability formula of the stress-strength model that the probability of a component strength falling in between two stresses, is given as follows: [16]

$$P(T < X < Z) = \int_{x=0}^{\infty} \int_{t=0}^{x} h(t)\, dt \left(\int_{z=x}^{\infty} g(z)\, dz \right) f(x) dx$$
$$= \int_{x=0}^{\infty} H_T(x) \left(1 - \int_{z=0}^{x} g(z)\, dz\right) f(x) dx$$
$$= \int_{x=0}^{\infty} H_T(x) \left(1 - G_z(x)\right) f(x) dx \qquad (4)$$

Consider T and Z as two independent stress r.v's having c.d.f $H_T(t)$ and $G_Z(z)$ from Chen distribution as $CD(\gamma, \beta)$ and $CD(\mu, \beta)$; respectively. Let X be a strength r.v's following $X_i \sim CD(\alpha_i, \beta), i = 1, 2, 3$ with c.d.f $F_{X_i}(x)$, suppose that X_i is independent from T and Z, therefore:

$$H_T(t) = 1 - e^{\gamma\left(1 - e^{t^{\beta}}\right)} \quad t > 0; \gamma, \beta > 0$$

$$G_Z(z) = 1 - e^{\mu\left(1 - e^{z^{\beta}}\right)} \quad z > 0; \mu, \beta > 0$$

So, from equation (4):

$$P(T < X_i < Z) = \int_{x_i=0}^{\infty} \left(1 - e^{\gamma\left(1 - e^{x_i^{\beta}}\right)}\right) \left(e^{\mu\left(1 - e^{x_i^{\beta}}\right)}\right) \alpha_i \beta x_i^{\beta-1} e^{x_i^{\beta}} e^{\alpha_i\left(1 - e^{x_i^{\beta}}\right)} dx_i$$
$$= \int_{x_i=0}^{\infty} \alpha_i \beta x_i^{\beta-1} e^{x_i^{\beta}} e^{\alpha_i + \mu\left(1 - e^{x_i^{\beta}}\right)} dx_i - \int_{x_i=0}^{\infty} \alpha_i \beta x_i^{\beta-1} e^{x_i^{\beta}} e^{\alpha_i + \mu + \gamma\left(1 - e^{x_i^{\beta}}\right)} dx_i$$
$$= \frac{\alpha_i}{\alpha_i + \mu} - \frac{\alpha_i}{\alpha_i + \mu + \gamma}$$

Now, consider the system in figure.1

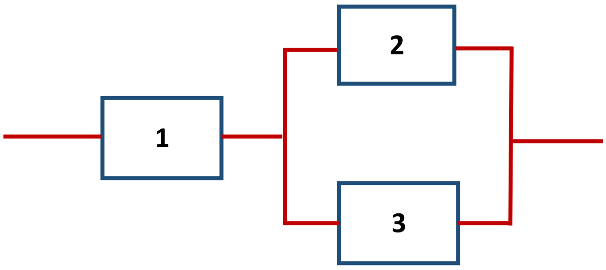

Fig. 1. A series-parallel system scheme

The minimal path sets for this system are {1,2} and {1,3}. Let the components 1, 2, and 3 have strengths X_1, X_2 and X_3, respectively, and the components are exposed to common stress variables T and Z, then the system's stress-strength reliability is given by:

$$R = P\left(\cup_{j=1}^{2}\left[T < \min_{i \in P_j} X_i < Z\right]\right)$$

Hence by using inclusion-exclusion formula we get:

$$R = P(T < min(X_1, X_2) < Z) + P(T < min(X_1, X_3) < Z)$$
$$- P(T < min(X_1, X_2, X_3) < Z) \qquad (5)$$

when X_i's are independent Chen distributed random variables, then $Y = min\{X_1, \dots, X_n\}$ is distributed as $ChD(\sum \alpha_i, \beta)$ and this shown as follows:

$$P(Y > x) = P(min\{X_1, \dots, X_n\} > x) = P(X_i > x \text{ for each } i = 1, \dots, n)$$
$$= \prod_{i=1}^{n} P(X_i > x) \qquad \text{from independency}$$
$$= \prod_{i=1}^{n} e^{\alpha_i\left(1-e^{x^\beta}\right)}$$
$$= exp\left\{\sum_{i=1}^{n} \alpha_i \left(1 - e^{x^\beta}\right)\right\}$$

So, the probability of $P(T < Y = min\{X_1, \dots, X_n\} < Z)$ can be given as:

$$P(T < Y < Z) = P(T < y, Z > y)$$
$$= \int_{y=0}^{\infty} H_T(y) \left(1 - G_z(y)\right) f(y) dy$$

$$P(T < Y < Z) = \int_{y=0}^{\infty} \left(1 - e^{\gamma\left(1-e^{y^\beta}\right)}\right) \times e^{\mu\left(1-e^{y^\beta}\right)} \times f(y) dy$$
$$= \int_{y=0}^{\infty} \left(1 - e^{\gamma\left(1-e^{y^\beta}\right)}\right) \times e^{\mu\left(1-e^{y^\beta}\right)} \times (\alpha_1 + \cdots + \alpha_n) e^{(\alpha_1+\cdots+\alpha_n)\left(1-e^{y^\beta}\right)} dy$$

$$P(T < Y < Z) = \int_{y=0}^{\infty} (\alpha_1 + \cdots + \alpha_n) e^{(\mu+\alpha_1+\cdots+\alpha_n)\left(1-e^{y^\beta}\right)} dy$$
$$- \int_{y=0}^{\infty} (\alpha_1 + \cdots + \alpha_n) e^{(\gamma+\mu+\alpha_1+\cdots+\alpha_n)\left(1-e^{y^\beta}\right)} dy$$
$$= \frac{\alpha_1+\cdots+\alpha_n}{\alpha_1+\cdots+\alpha_n+\mu} - \frac{\alpha_1+\cdots+\alpha_n}{\alpha_1+\cdots+\alpha_n+\mu+\gamma} \tag{6}$$

Hence, by substituting the expression of equation (6) into equation (5), the system's stress-strength reliability can be obtained as follows:

$$R = \left[\frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\mu} - \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\mu+\gamma}\right] + \left[\frac{\alpha_1+\alpha_3}{\alpha_1+\alpha_3+\mu} - \frac{\alpha_1+\alpha_3}{\alpha_1+\alpha_3+\mu+\gamma}\right] + \left[\frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3+\mu} - \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3+\mu+\gamma}\right] \tag{7}$$

5. Estimation Method

Three different estimation methods are used in this section to get the estimator for stress-strength model's reliability R, and the Chen unknown scale parameters; $\gamma, \mu, \alpha_1, \alpha_2$ and α_3. These three methods include maximum likelihood, least squares, and weighted least squares. The best reliability estimate is obtained using these methods.

6.1. Maximum Likelihood Estimator (MLE)

The most popular method for estimating parameters is the maximum likelihood method [17]. Suppose that $x_{I1}, x_{I2}, \dots, x_{In_I}$ are random strength samples with size $n_I, I = 1, 2, 3$ following $ChD(\alpha_I, \beta), I = 1, 2, 3$ in which β is known and α_I is an unknown parameter. The MLE function is therefore given by: [18]

$$L\left(x_{I1}, x_{I2}, \dots, x_{In_I}; \alpha_I, \beta\right) = (\alpha_I\beta)^{n_I} \prod_{Ii=1}^{n_I} x_{Ii}{}^{\beta-1}\, e^{\sum_{Ii=1}^{n_I} x_{Ii}{}^{\beta}}\, e^{\alpha_I \sum_{Ii=1}^{n_I}\left(1-e^{x_{Ii}{}^{\beta}}\right)} \tag{8}$$

Then, for equation (8), the expression for the natural logarithm function as:

$$\ln L = n_I \ln \alpha_I + n_I \ln \beta + (\beta - 1) \sum_{Ii=1}^{n_I} \ln x_{Ii} + \sum_{Ii=1}^{n_I} x_{Ii}{}^{\beta} + \alpha_I \sum_{Ii=1}^{n_I} \left(1 - e^{x_{Ii}{}^{\beta}}\right) \tag{9}$$

By differentiating equation (9) w.r.t the unknown parameter α_I, and equating its result to zero, to obtain:

$$\frac{\partial \ln L}{\partial \alpha_I} = \frac{n_I}{\alpha_I} + \sum_{Ii=1}^{n_I}\left(1 - e^{x_{Ii}{}^{\beta}}\right) \Longrightarrow \frac{n_I}{\hat{\alpha}_I} + \sum_{Ii=1}^{n_I}\left(1 - e^{x_{Ii}{}^{\beta}}\right) = 0$$

$$\hat{\alpha}_{IMLE} = \frac{-n_I}{\sum_{Ii=1}^{n_I}\left(1 - e^{x_{Ii}{}^{\beta}}\right)} \qquad I = 1, 2, 3 \tag{10}$$

Where β is known parameter

In similar manner, let $t_1, t_2, \dots, t_m$ and $z_1, z_2, \dots, z_h$ are two random stresses sample of size n and m from $CD(\gamma, \beta)$ and $CD(\mu, \beta)$, respectively. Then, the unknown parameter's MLE estimators of γ and μ are:

$$\hat{\gamma}_{MLE} = \frac{-m}{\sum_{j=1}^{n}\left(1 - e^{t_j{}^{\beta}}\right)} \tag{11}$$

$$\hat{\mu}_{MLE} = \frac{-h}{\sum_{k=1}^{m}\left(1 - e^{z_k{}^{\beta}}\right)} \tag{12}$$

Substitution equations (10), (11) and (12) in equation (7) thus produce the MLE estimator for Reliability R, using an invariant property of that method as follows:

$$\hat{R}_{MLE} = \left[\frac{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}}{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}+\hat{\mu}_{MLE}} - \frac{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}}{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}+\hat{\mu}_{MLE}+\hat{\gamma}_{MLE}}\right] + \left[\frac{\hat{\alpha}_{1MLE}+\hat{\alpha}_{3MLE}}{\hat{\alpha}_{1MLE}+\hat{\alpha}_{3MLE}+\hat{\mu}_{MLE}} - \frac{\hat{\alpha}_{1MLE}+\hat{\alpha}_{3MLE}}{\hat{\alpha}_{1MLE}+\hat{\alpha}_{3MLE}+\hat{\mu}_{MLE}+\hat{\gamma}_{MLE}}\right] - \left[\frac{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}+\hat{\alpha}_{3MLE}}{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}+\hat{\alpha}_{3MLE}+\hat{\mu}_{MLE}} - \frac{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}+\hat{\alpha}_{3MLE}}{\hat{\alpha}_{1MLE}+\hat{\alpha}_{2MLE}+\hat{\alpha}_{3MLE}+\hat{\mu}_{MLE}+\hat{\gamma}_{MLE}}\right]$$

6.2. Least Square Method (LS)

Swain, Venkatraman, and Wilson originally proposed the least squares method in 1988 to estimate the beta distribution's parameters. The process for minimizing the sum between the value and its expected value gives the least squares estimators [19]. Suppose that $x_{I(1)}, x_{I(2)}, \dots, x_{I(n_I)}$ is an order statistics strength random sample of size n_I from $ChD(\alpha_I, \beta), I = 1, 2, 3$. The following equation can be minimized to get least square estimator:

$$S = \sum_{Ii=1}^{n}\left[F\left(x_{I(i)}\right) - E\left(F\left(x_{I(i)}\right)\right)\right]^2 \tag{13}$$

Where $E\left(F\left(x_{I(i)}\right)\right) = P_{Ii}$ the plotting position and $P_{Ii} = \frac{Ii}{n_I+1}$, $Ii = 1, 2, \dots, n_I$

using the c.d.f of ChD in equation (13), to get:

$$S = \sum_{Ii=1}^{n_I}\left[1 - e^{\alpha_I\left(1 - e^{x_{I(i)}^{\beta}}\right)} - P_{Ii}\right]^2$$

So then,

$$S = \sum_{Ii=1}^{n_I}\left[\alpha_I\left(1 - e^{x_{I(i)}^{\beta}}\right) - q_{Ii}\right]^2 \tag{14}$$

Where $q_{Ii} = \ln\left(1 - F\left(x_{I(i)}\right)\right) = \ln(1 - P_{Ii})$

By the partial derivative of equation (14) w.r.t an unknown scale parameter α_I and then equating its result to zero, we will obtain:

$$\frac{\partial S}{\partial \alpha_I} = 2\sum_{Ii=1}^{n_I}\left[\alpha_I\left(1-e^{x_{I(i)}^{\beta}}\right)-q_{Ii}\right].\left(1-e^{x_{I(i)}^{\beta}}\right)$$

$$\Rightarrow \hat{\alpha}_I\sum_{Ii=1}^{n_I}\left(1-e^{x_{I(i)}^{\beta}}\right)^2-\sum_{Ii=1}^{n_I}q_{Ii}\left(1-e^{x_{I(i)}^{\beta}}\right)=0$$

$$\hat{\alpha}_{ILS}=\sum_{Ii=1}^{n_I}q_{Ii}\left(1-e^{x_{I(i)}^{\beta}}\right)\Big/\sum_{Ii=1}^{n_I}\left(1-e^{x_{I(i)}^{\beta}}\right)^2 \qquad I=1,2,3 \tag{15}$$

Similarly, the lest square estimators of unknown parameters γ and μ are given by:

$$\hat{\gamma}_{LS}=\sum_{j=1}^{m}q_j\left(1-e^{t_{(j)}^{\beta}}\right)\Big/\sum_{j=1}^{m}\left(1-e^{t_{(j)}^{\beta}}\right)^2 \tag{16}$$

$$\hat{\mu}_{LS}=\sum_{k=1}^{h}q_j\left(1-e^{z_{(k)}^{\beta}}\right)\Big/\sum_{k=1}^{h}\left(1-e^{z_{(k)}^{\beta}}\right)^2 \tag{17}$$

Substitution equations (15), (16) and (17) in equation (7), we get the LS estimator of the reliability R, approximately will be as:

$$\hat{R}_{LS}=\left[\frac{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}}{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}+\hat{\mu}_{LS}}-\frac{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}}{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}+\hat{\mu}_{LS}+\hat{\gamma}_{LS}}\right]+\left[\frac{\hat{\alpha}_{1LS}+\hat{\alpha}_{3LS}}{\hat{\alpha}_{1LS}+\hat{\alpha}_{3LS}+\hat{\mu}_{LS}}-\frac{\hat{\alpha}_{1LS}+\hat{\alpha}_{3LS}}{\hat{\alpha}_{1LS}+\hat{\alpha}_{3LS}+\hat{\mu}_{LS}+\hat{\gamma}_{LS}}\right]$$
$$-\left[\frac{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}+\hat{\alpha}_{3LS}}{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}+\hat{\alpha}_{3LS}+\hat{\mu}_{LS}}-\frac{\hat{\alpha}_{1LS}+\hat{\alpha}_{3LS}+\hat{\alpha}_{5LS}}{\hat{\alpha}_{1LS}+\hat{\alpha}_{2LS}+\hat{\alpha}_{3LS}+\hat{\mu}_{LS}+\hat{\gamma}_{LS}}\right]$$

6.3. Weighted Least Square Method (WLS)

The following equation can be minimized to obtain the weighted least square estimator:

$$WS=\sum_{Ii=1}^{n}w_{Ii}.\left[F\left(x_{I(i)}\right)-E\left(F\left(x_{I(i)}\right)\right)\right]^2 \tag{18}$$

Where $w_{Ii}=\frac{1}{var\left[F\left(x_{I(i)}\right)\right]}=\frac{(n_I+1)^2+(n_I+2)}{Ii(n_I-Ii+1)}$

By minimizing the following equation, the weighted least squares estimator for an unknown scale parameter α_I can be given as:

$$WS=\sum_{Ii=1}^{n_I}w_{Ii}.\left[1-e^{\alpha_I\left(1-e^{x_{I(i)}^{\beta}}\right)}-P_{Ii}\right]^2$$

So then,

$$WS=\sum_{Ii=1}^{n_I}w_{Ii}.\left[\alpha_I\left(1-e^{x_{I(i)}^{\beta}}\right)-q_{Ii}\right]^2 \tag{19}$$

By the partial derivative of equation (19) w.r.t an unknown scale parameter α_I and then equating its result to zero, we will obtain:

$$\frac{\partial WS}{\partial \alpha_I}=2\sum_{Ii=1}^{n_I}w_{Ii}.\left[\alpha_I\left(1-e^{x_{I(i)}^{\beta}}\right)-q_{Ii}\right].\left(1-e^{x_{I(i)}^{\beta}}\right)$$

$$\Rightarrow \hat{\alpha}_I\sum_{Ii=1}^{n_I}w_{Ii}.\left(1-e^{x_{I(i)}^{\beta}}\right)^2-\sum_{Ii=1}^{n_I}w_{Ii}.q_{Ii}.\left(1-e^{x_{I(i)}^{\beta}}\right)=0$$

$$\hat{\alpha}_{IWLS} = \sum_{Ii=1}^{n_I} w_{Ii} . q_{Ii} . \left(1 - e^{x_{I(i)}^{\beta}}\right) \Big/ \sum_{Ii=1}^{n_I} w_{Ii} . \left(1 - e^{x_{I(i)}^{\beta}}\right)^2 \quad (20)$$

Similarly, weighted lest square estimators for unknown parameters γ and μ are given by:

$$\hat{\gamma}_{WLS} = \sum_{j=1}^{m} w_j . q_j . \left(1 - e^{t_{(j)}^{\beta}}\right) \Big/ \sum_{j=1}^{m} w_j . \left(1 - e^{t_{(j)}^{\beta}}\right)^2 \quad (21)$$

$$\hat{\mu}_{WLS} = \sum_{k=1}^{h} w_k . q_k . \left(1 - e^{z_{(k)}^{\beta}}\right) \Big/ \sum_{k=1}^{h} w_k . \left(1 - e^{z_{(k)}^{\beta}}\right)^2 \quad (22)$$

By substitute equations (20), (21) and (22) into equation (7), the WLS estimator of the reliability R, approximately will be as follows:

$$\hat{R}_{WLS} = \left[\frac{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}}{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}+\hat{\mu}_{WLS}} - \frac{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}}{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}+\hat{\mu}_{WLS}+\hat{\gamma}_{WLS}}\right] + \left[\frac{\hat{\alpha}_{1WLS}+\hat{\alpha}_{3WLS}}{\hat{\alpha}_{1WLS}+\hat{\alpha}_{3WLS}+\hat{\mu}_{WLS}} - \frac{\hat{\alpha}_{1WLS}+\hat{\alpha}_{3WLS}}{\hat{\alpha}_{1WLS}+\hat{\alpha}_{3WLS}+\hat{\mu}_{WLS}+\hat{\gamma}_{WLS}}\right] - \left[\frac{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}+\hat{\alpha}_{3WLS}}{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}+\hat{\alpha}_{3WLS}+\hat{\mu}_{LS}} - \frac{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}+\hat{\alpha}_{3WLS}}{\hat{\alpha}_{1WLS}+\hat{\alpha}_{2WLS}+\hat{\alpha}_{3WLS}+\hat{\mu}_{WLS}+\hat{\gamma}_{WLS}}\right]$$

6. Simulation Study

The best estimate for the reliability of unknown parameters for the Chen distribution is determined in this section using a simulated experiment. All three estimates via the maximum likelihood, least squares, and weighted least squares estimation method are performed and evaluated according to mean square error criteria (MSE) for four distinct experiments in each instance for the value of the parameter β, using various sample sizes (15, 30, 90) and ($\beta = 2$).

Using MATLAB 2023a, a simulation study is carried out for the four experiments in order to compare the reliability estimator's performance through the steps given below:

Step1: we use the inverse function of equation (1) for generating random values for the r.v's by using the following formula: $x = \left[\ln\left(1 - \left(\ln\left(1 - F(x)\right)/\alpha\right)\right)\right]^{1/\beta}$

Step2: The mean square error criteria are used to compare estimate methods: MSE$= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{R}_i - R\right)^2$, where N represents the 500 replications for each experiment.

The results are listed in Table 1, based on four specific experiments with the parameter values of the random variables mentioned in the research ($\gamma, \mu, \alpha_I; I = 1,2,3$), and calculate the real reliability for each of these four experiments.

The MSE values are used to compare the performance of these estimators; for each experiment, the MSE value decreases as sample sizes increases for MLE, LS, and WLS. In experiments $1, 3$ and 4, the MLE estimator has the best MSE value, followed by LS and WLS; in experiment 2, the estimated LS has the best MSE value, followed by MLE and WLS for the sample size $(15, 90)$, while the MLE estimator has the best MSE value, followed by LS and WLS for other sample size. As a result, estimator performance for the maximum likelihood is lot better than that for least squares, and weighted least squares for all sample sizes and most of the experiments.

Table 1: Estimate for Reliability

Exp. 1: $\gamma = 1.8, \mu = 1.8, \alpha_1 = 0.5, \alpha_2 = 0.51, \alpha_3 = 0.52, R = 0.1201$				
n, n_I		MLE	LS	WLS
15,15	MSE	0.0021	0.0026	0.0031
30,30	MSE	0.0010	0.0011	0.0016

90,90	MSE	3.2380e-04	4.0130e-04	8.7880e-04
30,15	MSE	0.0009	0.0012	0.0018
15,90	MSE	0.0018	0.0029	0.0035
30,90	MSE	0.0008	0.0012	0.0017

$\mathbf{Exp.2}$: $\gamma = 0.3, \mu = 0.3, \alpha_1 = 1.9, \alpha_2 = 1.91, \alpha_3 = 1.92, R = 0.0810$				
15,15	MSE	5.0460e-04	5.8890e-04	7.1090e-04
30,30	MSE	2.8290e-04	3.4100e-04	4.8960e-04
90,90	MSE	0.8360e-04	1.0290e-04	2.2190e-04
30,15	MSE	3.0130e-04	4.3020e-04	5.7870e-04
15,90	MSE	5.4900e-04	5.3490e-04	6.9140e-04
30,90	MSE	2.3920e-04	2.5260e-04	3.6860e-04
$\mathbf{Exp.3}$: $\gamma = 2.2, \mu = 0.6, \alpha_1 = 2.2, \alpha_2 = 2.21, \alpha_3 = 2.22, R = 0.3229$				
15,15	MSE	0.0034	0.0039	0.0047
30,30	MSE	0.0019	0.0022	0.0031
90,90	MSE	0.0007	0.0008	0.0017
30,15	MSE	0.0024	0.0027	0.0034
15,90	MSE	0.0036	0.0041	0.0048
30,90	MSE	0.0020	0.0024	0.0035
$\mathbf{Exp.4}$: $\gamma = 0.2, \mu = 1.7, \alpha_1 = 1.7, \alpha_2 = 1.71, \alpha_3 = 1.72, R = 0.0289$				
15,15	MSE	0.9620e-04	1.1140e-04	1.3640e-04
30,30	MSE	5.3010e-05	6.5480e-05	9.0630e-05
90,90	MSE	1.4580e-05	1.8330e-05	3.7730e-05
30,15	MSE	4.3630e-05	5.6980e-05	8.3250e-05
15,90	MSE	1.0620e-05	1.1220e-05	1.3880e-05
30,90	MSE	4.2700e-05	5.1490e-05	7.5930e-05

7. Conclusion

In this study, the reliability formula for the coherent system of multicomponent strength falling between two stresses has been reached, and calculating this formula for the parallel-series system scheme is based on the Chen distribution. Three methods for estimating that reliability with different parameter values have been presented. Due to the very small MSE values of the maximum likelihood method in all experiments, the estimator performance for the maximum likelihood is much better than that for least squares and weighted least squares of all sample sizes and all of the experiments, based on simulation results that have appeared.

8. Recommendations

Using the reliability formula we have reached and the possibility of applying it by conducting a study on real-life systems and analyzing their results. Additionally, different estimation methods can be used for the same system to estimate the parameters and thus estimate the system's reliability.

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