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RESEARCH ARTICLE - MATHEMATICS

Complete Arcs of Various Degrees in the Projective Space of Dimension Seven over the Galois Field F_2

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Abstract

The aim of this paper is to use the action of some cyclic subgroups of a projective general linear group of order eight over the finite field of order two, PGL_2^8 , to construct complete arcs of degree higher than three in $P_2(7)$. Six arcs are found by group action, and from these arcs many complete arcs are constructed. As a result of thess actions a lower bound for maximum size of arcs of certain degree are found in $P_2(7)$.

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***Keywords**:* Action of group, Arcs, Finite field, Finite projective space

1. Introduction: Given $P_d(n)$ as a finite projective space of dimension n over a Galois field F_d, where d prime numbere. The a projective spaces are one of the simples ways to describe geometry, because the deal with lines, points and the shapes and resulting from them. Results on arcs and that rely on it were mostly results on arcs in $PG(2,q)$. The main reason for this since it is the smallest interesting space. For a projective space of higher degrees (greater than 2) study arcs and classified it become so hard. Therefore, appeared many works deals with special cases of spaces. An important goal in studying $(k;r)$-arcs in $P_d(n)$ is the determination of $m_r(n,d)$ which is the maximum number k such that an $(k;r)$-arc in $P_d(n)$ exists. Hirschfield in [1,2] gave an impressive historical and algebraic overview of finite a projective spaces, especially finite a projective plane and arcs of the second and higher degrees. Ball in [3] introduce new bounds of arcs in the a projective plane through codes. Daskalov did many works about bounds on $m_r(2,d)$ in the plane appear as in [4-6] and references therein. Braunin [7], gave lower bounds on the size of $(n;r)$-arcs in the a projective plane and then introduced an update to it in [8]. Also, Yahya in [9,10], and other papers focused on the bounds of $m_r(2,d)$ for a certain d. Arcs in the three dimensional projective space and bounds on $m_r(3,d)$ have been studied by Al-Rikabi in [11,12] for $d = 8$, Radhi in [13,14] for $d = 11$, and Attook in [15,16] for $d = 13$. For arcs of degree two and three in the a projective plane a cubic curves and some algebraic methods have used to construct arcs for a certain order of fields as in [17,18] and in a projective line as in [19]. See [20][21] for more details about links between algebra and cryptography.

The projective space $P_d(7)$ has $\theta_d^7 = d^7 + d^6 + d^5 + d^4 + d^3 + d^2 + d + 1$ points and hyperplane (the largest subspaces of $P_d(7)$).

Definition 1.1 [4]: A $(k;r)$-arc in $P_d(n)$, where $r \geq 3$, is a set of k points with at most r of which lie on any hyperplane. Here r is called the degree of thes arc is called complete if it is not contained in $(k+1;r)$-arc.

Definition 1.2[2]: (i) An i-secant of an $(k;r)$-arc K in $P_d(n)$ is a hyperplane H such that $|K \cap H| = i$. The number of i-secant of K denoted by T_i, and T_i-distribution is the vector $(T_r, T_{r-1}, \dots, T_0)$.
(ii) Let p a point of $P_d(n)$ not on the $(k;r)$-arc, K. Then the number of i-secant of K passing through p is denoted by $\sigma_i(p)$. The number of $\sigma_r(p)$ of r-secants is called the index of p with respect to K. The set of all points of index i will be denoted C_i and $c_i = |C_i|$, the cardinality of C_i. c_i-distribution of K is the vector $(c_l, c_{l-1}, \dots, c_0)$.

(iii) If $c_o = 0$, then the arc K is called complete. The maximum value of k for an $(k;r)$-arc is denoted by $m_r(n,d)$.

Definition 1.3[2]: The projective general linear group PGL_d^{n+1} of $P_d(n)$ which element are non-singular matrices of dimension $n+1$ and entries from the field F_d.

Definition 1.4[3]: Let G be a group with identity 1 and X be a non-empty set. We say that G acts of G on X if there is binary operation $X \times G \to X$ such that
(i) $x1 = x$ for all x in X,
(ii) $(xg_1)g_2 = x(g_1g_2)$ for all $g_1, g_2 \in G$, and $x \in X$.

Let $\mathcal{M}$ be 8×8 companion square matrix of degree 8 over F_2; that is, $\mathcal{M} \in PGL_2^8$,

$$\mathcal{M} = \begin{pmatrix} 0 & & & & & & & \\ \vdots & & & & I_7 & & & \\ 0 & & & & & & & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the points of $P_d(7)$ can be fined by using the formula $P_i = P_1\mathcal{M}^{i-1}, i = 1, \dots, \theta_2^7$, where $P_1 = [1,0,0,0,0,0,0,0]$. An i position will used to refer to the point P_i, $i = 1, \dots, 255$.

A GAP programming [22] has used to execute all algorithms in this article.

2. Construction Complete Arcs in $P_2(7)$.

In $P_2(7)$, the number 255 has six non-trivial divisors which are 3,5,15,17,51,85. Let Υ the set of these divisors. Thus, there are six cyclic subgroups of PGL_2^8, $\langle \mathcal{M}^i \rangle$, $i \in \Upsilon$ of order order m_i such that $i.m_i = 255 = \theta_2^7$.

Theorem 2.1: The space $P_2(7)$ partitioned into six projectively distinct set of length $m_i \in \Upsilon$, such that $i.m_i = \theta_2^7$.
Proof: For each $i \in \Upsilon$, the subgroup $\langle \mathcal{M}^i \rangle$ of PGL_2^8 will act from right on the space $P_2(7)$, as follows:

$$P_2(7) \times \langle \mathcal{M}^i \rangle \to P_2(7)$$
$$\left(P_i, \mathcal{M}^{i^k}\right) \to P_l\mathcal{M}^{i^k} = P_1\mathcal{M}^l\mathcal{M}^{i^k} = P_{l+ik} \quad \dots\dots\dots\dots\dots (*)$$

$1 \leq l + ik \leq \theta_2^7$, such that if $l + ik = \theta_2^7 + 1$, then reduced it to 1.
So, from the action we get i of orbits of length m_i. All these orbits will projectively equivalent by $\mathcal{M}$. Let denoted the represented orbit by $O[i, m_i]$, such that $i.m_i = \theta_2^7$. Therefore, there are six projectively inequivalent orbits in $P_2(7)$.■

Theorem 2.1: The six orbits $O[i, m_i]$ of $P_2(7)$ are complete arcs.

Proof: For each $i \in \Upsilon$, the action in (*) will gave the following results.

(i) If $i = 3$, then the orbit $O[3,85]$ ={1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, 88, 91, 94, 97, 100, 103, 106, 109, 112, 115, 118, 121, 124, 127, 130, 133, 136, 139, 142, 145, 148, 151, 154, 157, 160, 163, 166, 169, 172, 175, 178, 181, 184, 187,190, 193, 196, 199, 202, 205, 208, 211, 214, 217, 220, 223, 226, 229, 232, 235, 238, 241, 244, 247, 250, 253} which is complete $(85; 45)$-arc such that T_i-distribution of it is $(T_{37}, T_{45}) = (85,170)$, and $c_{82} = 170$, $c_i = 0$ for $i \neq 82$.

(ii) if $l = 5$, then the orbit $O[5,51]$ ={1, 6, 11, 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81, 86, 91, 96, 101, 106, 111, 116, 121, 126, 131, 136, 141, 146, 151, 156, 161, 166, 171, 176, 181, 186, 191, 196, 201, 206, 211, 216, 221, 226, 231,236, 241, 246, 251} which is complete $(51; 27)$-arc such that T_i-distribution of it is $(T_{19}, T_{27}) = (51,204)$, and $c_{100} = 204$, $c_i = 0$ for $i \neq 100$.

(iii) If $l = 15$, then the orbit $O[15,17]$ ={1, 16, 31, 46, 61, 76, 91, 106, 121, 136, 151, 166, 181, 196,211,226,241} which is complete $(17; 11)$-arc such that T_i-distribution of it is $(T_5, T_7, T_9, T_{11}) = (34,68,85,68)$, and $c_{30} = 68$, $c_{32} = 68$, $c_{34} = 68$, $c_{36} = 68$, $c_i = 0$ for $i \neq 30,32,34,36$.

(iv) If $l = 17$, then the orbit $O[17,15]$ ={ 1, 18, 35, 52, 69, 86, 103, 120, 137, 154, 171, 188, 205, 222, 239 } which is complete $(15; 15)$-arc such that T_i-distribution of it is $(T_7, T_{15}) = (240,15)$, and $c_7 = 240$, $c_i = 0$ for $i \neq 7$.

(v) If $l = 51$, then the orbit $O[51,5]$ ={1, 52, 103, 154, 205} which is complete $(5;5)$-arc such that T_i-distribution of it is $(T_1, T_3, T_5) = (80,160,15)$, and $c_7 = 204$, $c_{15} = 10$, $c_i = 0$ for $i \neq 7,15$.

(vi) If $l = 85$, then the orbit $O[85,3]$ ={1,86,171} which is complete $(3;3)$-arc such that T_i-distribution of it is $(T_1, T_3) = (192,63)$, and $c_{31} = 252$, $c_i = 0$ for $i \neq 31$.

Let i^j ≔ the base, i represent the ith coordinate of $T_i(c_i)$, and the power, j the value of $T_i(c_i)$ with condition that $T_i(c_i) \neq 0$.

Table1. Details about the complete arcs $O[i, m_i]$

i	Orbit	$(k;r)$-Arc	T_i-distribution	c_i-distribution
1	$O[3,85]$	$(85;45)$-Arc	$37^{85}, 45^{170}$	82^{170}
2	$O[5,51]$	$(51;27)$-Arc	$19^{51}, 27^{204}$	100^{204}
3	$O[15,17]$	$(17;11)$-Arc	$5^{34}, 7^{68}, 9^{85}, 11^{68}$	$30^{68}, 32^{34}, 34^{68}, 36^{68}$
4	$O[17,15]$	$(15;15)$-Arc	$7^{204}, 15^{15}$	7^{240}
5	$O[51,5]$	(5,5)-Arc	$1^{80}, 3^{160}, 5^{15}$	$7^{240}, 15^{10}$
6	$O[85,3]$	$(3;3)$-Arc	$1^{192}, 3^{63}$	31^{252}

Procedures of Extensions 2.2

To do an extension of size and degree of a complete $(k;r)$-arc in $P_2(7)$, the following procedures are used.

A- Size extension

1- Determined the set of points of all hyperplanes which meet the arc in r points, say W.

2- Determined the set C_0 which is equal to $P_2(7) \setminus (O[i, m_i] \cup W)$.

3- Adding points from C_0 to $O[i, m_i]$ gradually and check if it became complete or not.

B- Degree extension

1- Determined the first hyperplane, Λ which meet $O[i, m_i]$ in r points.

2- Find the set of complement of Λ with respect to $O[i, m_i]$, say Λ^*.

3- To increase the degree of $O[i, m_i]$ to $r + j$, we add j points from Λ^* to $O[i, m_i]$, and then the new arc $\Theta = O[i, m_i] \cup \{p_1, \dots, p_j\}$ will check every time if it is complete or not.

4- If $\Theta = O[i, m_i] \cup \{p_1, \dots, p_i\}$ incompete arc, then back to execute the steps in procedure **A**, else we printed out Θ.

Using the above algorithms and run it with GAP programming [17], many arcs (complete and incomplete) have founded bases on the arcs in Table 1. Since we cannot write these arcs so, we gave a summary to these results.

Theorem 2.3: Let $O[i, m_i]$ be the $(k;r)$-arcs in Table 1, which are complete arcs. Then the following lower bound are deduced for $m_r(7,2)$, $3 \leq r \leq 126$.

Table 2. Lower bound for $m_r(7,2)$

$m_3(7;2) \geq 3$	$m_4(7;2) \geq 4$	$m_5(7;2) \geq 5$	$m_6(7;2) \geq 6$
$m_7(7;2) \geq 7$	$m_8(7;2) \geq 8$	$m_9(7;2) \geq 11$	$m_{10}(7;2) \geq 14$
$m_{11}(7;2) \geq 17$	$m_{12}(7;2) \geq 18$	$m_{13}(7;2) \geq 19$	$m_{14}(7;2) \geq 22$
$m_{15}(7;2) \geq 23$	$m_{16}(7;2) \geq 25$	$m_{17}(7;2) \geq 27$	$m_{18}(7;2) \geq 30$
$m_{19}(7;2) \geq 31$	$m_{20}(7;2) \geq 32$	$m_{21}(7;2) \geq 35$	$m_{22}(7;2) \geq 36$
$m_{23}(7;2) \geq 37$	$m_{24}(7;2) \geq 38$	$m_{25}(7;2) \geq 41$	$m_{26}(7;2) \geq 42$
$m_{27}(7;2) \geq 51$	$m_{28}(7;2) \geq 52$	$m_{29}(7;2) \geq 53$	$m_{30}(7;2) \geq 54$
$m_{31}(7;2) \geq 55$	$m_{32}(7;2) \geq 56$	$m_{33}(7;2) \geq 59$	$m_{34}(7;2) \geq 60$
$m_{35}(7;2) \geq 61$	$m_{36}(7;2) \geq 64$	$m_{37}(7;2) \geq 65$	$m_{38}(7;2) \geq 66$
$m_{39}(7;2) \geq 69$	$m_{40}(7;2) \geq 70$	$m_{41}(7;2) \geq 73$	$m_{42}(7;2) \geq 74$
$m_{43}(7;2) \geq 77$	$m_{44}(7;2) \geq 78$	$m_{45}(7;2) \geq 85$	$m_{46}(7;2) \geq 86$
$m_{47}(7;2) \geq 87$	$m_{48}(7;2) \geq 88$	$m_{49}(7;2) \geq 89$	$m_{50}(7;2) \geq 90$
$m_{51}(7;2) \geq 93$	$m_{52}(7;2) \geq 94$	$m_{53}(7;2) \geq 97$	$m_{54}(7;2) \geq 96$
$m_{55}(7;2) \geq 99$	$m_{56}(7;2) \geq 102$	$m_{57}(7;2) \geq 105$	$m_{58}(7;2) \geq 106$
$m_{59}(7;2) \geq 109$	$m_{60}(7;2) \geq 110$	$m_{61}(7;2) \geq 113$	$m_{62}(7;2) \geq 114$
$m_{63}(7;2) \geq 115$	$m_{64}(7;2) \geq 118$	$m_{65}(7;2) \geq 119$	$m_{66}(7;2) \geq 120$
$m_{67}(7;2) \geq 123$	$m_{68}(7;2) \geq 126$	$m_{69}(7;2) \geq 129$	$m_{70}(7;2) \geq 128$
$m_{71}(7;2) \geq 131$	$m_{72}(7;2) \geq 132$	$m_{73}(7;2) \geq 133$	$m_{74}(7;2) \geq 136$
$m_{75}(7;2) \geq 139$	$m_{76}(7;2) \geq 140$	$m_{77}(7;2) \geq 143$	$m_{78}(7;2) \geq 146$
$m_{79}(7;2) \geq 147$	$m_{80}(7;2) \geq 148$	$m_{81}(7;2) \geq 151$	$m_{82}(7;2) \geq 152$
$m_{83}(7;2) \geq 155$	$m_{84}(7;2) \geq 158$	$m_{85}(7;2) \geq 159$	$m_{86}(7;2) \geq 162$
$m_{87}(7;2) \geq 163$	$m_{88}(7;2) \geq 166$	$m_{89}(7;2) \geq 167$	$m_{90}(7;2) \geq 170$
$m_{91}(7;2) \geq 171$	$m_{92}(7;2) \geq 174$	$m_{93}(7;2) \geq 175$	$m_{94}(7;2) \geq 176$
$m_{95}(7;2) \geq 183$	$m_{96}(7;2) \geq 182$	$m_{97}(7;2) \geq 183$	$m_{98}(7;2) \geq 188$
$m_{99}(7;2) \geq 188$	$m_{100}(7;2) \geq 190$	$m_{101}(7;2) \geq 191$	$m_{102}(7;2) \geq 196$
$m_{103}(7;2) \geq 195$	$m_{104}(7;2) \geq 198$	$m_{105}(7;2) \geq 202$	$m_{106}(7;2) \geq 204$
$m_{107}(7;2) \geq 205$	$m_{108}(7;2) \geq 208$	$m_{109}(7;2) \geq 209$	$m_{110}(7;2) \geq 212$
$m_{111}(7;2) \geq 217$	$m_{112}(7;2) \geq 216$	$m_{113}(7;2) \geq 219$	$m_{114}(7;2) \geq 222$
$m_{115}(7;2) \geq 223$	$m_{116}(7;2) \geq 226$	$m_{117}(7;2) \geq 227$	$m_{118}(7;2) \geq 230$
$m_{119}(7;2) \geq 233$	$m_{120}(7;2) \geq 234$	$m_{121}(7;2) \geq 235$	$m_{122}(7;2) \geq 240$
$m_{123}(7;2) \geq 241$	$m_{124}(7;2) \geq 244$	$m_{125}(7;2) \geq 247$	$m_{126}(7;2) \geq 252$

As an example the results of the extension of the first orbit are given in Table 3.

Let $o_i^j \coloneqq O[i,m]$ union j points; D.:=Degree of arc; S.:=Size of arc; S.t.:=Size of arc after extension; C:= Complete arc; NC:= Not complete arc.

Table 3. Results of the orbit $O[3,85]$

o_l^i	S. / S.t.	D.	T_i-distribution	c_i-distribution	C or NC
o_3^0	85	45	$37^{85}, 45^{170}$	0^0	C

o_3^1	86	46	$37^{40}, 38^{45}, 45^{88}, 46^{82}$	$38^{114}, 42^{55}$	C
o_3^2	87	47	$37^{20}, 38^{40}, 39^{25}, 45^{44}, 46^{88}, 47^{38}$	$16^{44}, 18^{90}, 20^{24}$	C
o_3^3	88	48	$37^{12}, 38^{24}, 39^{36}, 40^{13}, 45^{20}, 46^{72}, 47^{60}, 48^{18}$	$6^{8}, 7^{42}, 8^{42}, 9^{40}$ $10^{24}, 11^{6}, 12^{2}, 1$	C
o_3^4	89	49	$37^{6}, 38^{16}, 39^{32}, 40^{24}$, $41^{7}, 45^{10}, 46^{48}, 47^{64}, 48^{40}, 49^{8}$	$2^{14}, 3^{72}, 4^{50}, 5^{16}$ $, 6^{6}, 8^{8}$	C
o_3^5	90	50	$37^{4}, 38^{10}, 39^{20}, 40^{32}$, $41^{16}, 42^{3}, 45^{4}, 46^{30}, 47^{60}, 48^{48}, 49^{24}, 50^{4}$	$1^{72}, 2^{60}, 3^{16}, 4^{17}$	C
o_3^6	91	51	$37^{1}, 38^{9}, 39^{12}, 40^{28}$, $41^{23}, 42^{11}, 43^{1}, 45^{3}, 46^{15}, 47^{48}, 48^{52}, 49^{37}$, $50^{13}, 51^{2}$	$0^{48}, 1^{80}, 2^{36}$	NC
	93			0^{0}	C
o_3^7	92	52	$37^{1}, 38^{5}, 39^{9}, 40^{19}$, $41^{25}, 42^{21}, 43^{5}, 45^{1}, 46^{9}, 47^{33}, 48^{51}, 49^{45}$ $50^{21}, 51^{9}, 52^{1}$	$0^{88}, 1^{75}$	NC
	94			0^{0}	C
o_3^8	93	53	$38^{5}, 39^{6}, 40^{11}, 41^{21}$, $42^{29}, 43^{10}, 44^{3}, 46^{5}, 47^{24}, 48^{39}, 49^{49}, 50^{33}$, $51^{16}, 52^{3}, 53^{1}$	$0^{88}, 1^{74}$	NC
	95			0^{0}	C
o_3^9	94	54	$38^{2}, 39^{4}, 40^{11}, 41^{13}$, $42^{27}, 43^{22}, 44^{5}, 45^{1}, 46^{4}, 47^{16}, 48^{25}, 49^{47}$ $50^{45}, 51^{22}, 52^{7}, 53^{3}, 54^{1}$	$0^{88}, 1^{73}$	NC
	96			0^{0}	C
o_3^{10}	95	55	$38^{1}, 39^{3}, 40^{6}, 41^{13}$, $42^{20}, 43^{26}, 44^{12}, 45^{3}, 46^{2}, 47^{9}, 48^{24}, 49^{37}$ $50^{46}, 51^{32}, 52^{14}, 53^{3}, 54^{3}, 55^{1}$	$0^{88}, 1^{72}$	NC
	97			0^{0}	C
o_3^{11}	96	56	$39^{3}, 40^{5}, 41^{9}, 42^{18}$, $43^{18}, 44^{20}, 45^{9}, 46^{2}, 47^{4}, 48^{17}, 49^{35}, 50^{42}$ $51^{38}, 52^{20}, 53^{11}, 54^{2}, 55^{1}, 56^{1}$	$0^{88}, 1^{71}$	NC
	100			0^{0}	C
o_3^{12}	97	57	$39^{2}, 40^{3}, 41^{3}, 42^{19}$, $43^{19}, 44^{15}, 45^{18}, 46^{3}, 47^{5}, 48^{15}, 49^{13}, 50^{47}$ $51^{43}, 52^{23}, 53^{20}, 54^{3}, 55^{3}, 57^{1}$	$0^{88}, 1^{70}$	NC
	103			0^{0}	C
o_3^{13}	98	58	$39^{1}, 40^{3}, 41^{2}, 42^{10}$, $43^{21}, 44^{17}, 45^{12}, 46^{15}, 47^{5}, 48^{9}, 49^{14}, 50^{28}$ $51^{47}, 52^{35}, 53^{20}, 54^{9}, 55^{6}, 58^{1}$	$0^{88}, 1^{69}$	NC
	104			0^{0}	C
o_3^{14}	99	59	$39^{1}, 40^{3}, 42^{7}, 43^{12}$, $44^{21}, 45^{18}, 46^{9}, 47^{15}, 48^{5}, 49^{6}, 50^{31}, 51^{32}$, $52^{41}, 53^{32}, 54^{9}, 55^{10}, 56^{2}, 59^{1}$	$0^{88}, 1^{68}$	NC
	107			0^{0}	C
o_3^{15}	100	60	$40^{2}, 41^{2}, 42^{2}, 43^{11}$, $44^{18}, 45^{14}, 46^{18}, 47^{13}, 48^{9}, 49^{6}, 50^{16}, 51^{35}$ $52^{38}, 53^{32}, 54^{20}, 55^{13}, 56^{3}, 57^{2}, 60^{1}$	$0^{88}, 1^{67}$	NC
	108			0^{0}	C
o_3^{16}	101	61	$41^{3}, 42^{3}, 43^{5}, 44^{17}$, $45^{12}, 46^{19}, 47^{13}, 48^{11}, 49^{9}, 50^{9}, 51^{31}, 52^{31}$, $53^{34}, 54^{33}, 55^{15}, 56^{5}, 57^{4}, 61^{1}$	$0^{88}, 1^{66}$	NC
	112			0^{0}	C
o_3^{17}	102	62	$41^{1}, 42^{5}, 43^{2}, 44^{11}$, $45^{15}, 46^{14}, 47^{14}, 48^{15}, 49^{13}, 50^{9}, 51^{16}, 52^{31}$, $53^{39}, 54^{26}, 55^{24}, 56^{15}, 57^{4}, 62^{1}$	$0^{88}, 1^{65}$	NC
	112			0^{0}	C
o_3^{18}	103	63	$41^{1}, 42^{1}, 43^{5}, 44^{7}$, $45^{10}, 46^{18}, 47^{12}, 48^{13}, 49^{20}, 50^{4}, 51^{17}, 52^{27}$, $53^{32}, 54^{32}, 55^{20}, 56^{25}, 57^{9}, 58^{1}, 63^{1}$	$0^{88}, 1^{64}$	NC
	113			0^{0}	C
o_3^{19}	104	64	$42^{1}, 43^{5}, 44^{3}$, $45^{10}, 46^{10}, 47^{18}, 48^{16}, 49^{16}, 50^{10}, 51^{8}, 52^{27}$, $53^{28}, 54^{32}, 55^{24}, 56^{24}, 57^{18}, 58^{3}, 59^{1}, 64^{1}$	$0^{88}, 1^{63}$	NC
	116			0^{0}	C

o_3^{20}	105	65	$43^2, 44^6, 45^6, 46^{10}, 47^{18}, 48^{14}, 49^{11}, 50^{14}, 51^{10}, 52^{22}, 53^{22}, 54^{34}, 55^{30}, 56^{22}, 57^{23}, 58^6, 59^4, 65^1$	$0^{88}, 1^{62}$	NC
	117			0^0	C
o_3^{21}	106	66	$43^2, 44^4, 45^3, 46^6, 47^{17}, 48^{20}, 49^9, 50^{13}, 51^{13}, 52^{12}, 53^{25}, 54^{34}, 55^{23}, 56^{28}, 57^{27}, 58^9, 59^9, 66^1$	$0^{88}, 1^{61}$	NC
	118			0^0	C
o_3^{22}	107	67	$43^1, 44^3, 45^2, 46^5, 47^{11}, 48^{22}, 49^{12}, 50^9, 51^{20}, 52^{10}, 53^{10}, 54^{35}, 55^{33}, 56^{26}, 57^{24}, 58^{15}, 59^{13}, 60^3, 67^1$	$0^{88}, 1^{60}$	NC
	122			0^0	C
o_3^{23}	108	68	$44^2, 45^4, 46^2, 47^9, 48^{14}, 49^{18}, 50^{12}, 51^{11}, 52^{21}, 53^6, 54^{20}, 55^{45}, 56^{20}, 57^{28}, 58^{22}, 59^7, 60^{13}, 68^1$	$0^{88}, 1^{59}$	NC
	124			0^0	C
o_3^{24}	109	69	$45^3, 46^4, 47^6, 48^{14}, 49^{11}, 50^{16}, 51^{10}, 52^{20}, 53^{12}, 54^{10}, 55^{40}, 56^{28}, 57^{23}, 58^{26}, 59^{16}, 60^{10}, 61^5, 69^1$	$0^{88}, 1^{58}$	NC
	125			0^0	C
o_3^{25}	110	70	$45^1, 46^5, 47^4, 48^{11}, 49^8, 50^{19}, 51^{14}, 52^9, 53^{18}, 54^{10}, 55^{24}, 56^{33}, 57^{32}, 58^{25}, 59^{22}, 60^{11}, 61^5, 62^3, 70^1$	$0^{88}, 1^{57}$	NC
	126			0^0	C
o_3^{26}	111	71	$46^4, 47^4, 48^9, 49^9, 50^7, 51^{23}, 52^7, 53^{13}, 54^{21}, 55^{14}, 56^{31}, 57^{35}, 58^{25}, 59^{21}, 60^{17}, 61^7, 62^7, 71^1$	$0^{88}, 1^{56}$	NC
	127			0^0	C
o_3^{27}	112	72	$46^2, 47^4, 48^{10}, 49^3, 50^{10}, 51^{15}, 52^{13}, 53^{13}, 54^{16}, 55^{15}, 56^{24}, 57^{37}, 58^{26}, 59^{29}, 60^{15}, 61^{11}, 62^{10}, 63^1, 72^1$	$0^{88}, 1^{55}$	NC
	130			0^0	C
o_3^{28}	113	73	$47^4, 48^7, 49^7, 50^6, 51^7, 52^{23}, 53^{12}, 54^{10}, 55^{15}, 56^{19}, 57^{37}, 58^{32}, 59^{23}, 60^{23}, 61^{14}, 62^8, 63^7, 73^1$	$0^{88}, 1^{54}$	NC
	133			0^0	C
o_3^{29}	114	74	$47^1, 48^7, 49^8, 50^5, 51^7, 52^{11}, 53^{20}, 54^{12}, 55^{13}, 56^{13}, 57^{30}, 58^{39}, 59^{27}, 60^{23}, 61^{14}, 62^{14}, 63^8, 64^2, 74^1$	$0^{88}, 1^{53}$	NC
	136			0^0	C
o_3^{30}	115	75	$47^1, 48^1, 49^{11}, 50^5, 51^7, 52^9, 53^{14}, 54^{17}, 55^{12}, 56^{11}, 57^{20}, 58^{43}, 59^{31}, 60^{23}, 61^{18}, 62^{15}, 63^{11}, 64^4, 65^1, 75$	$0^{88}, 1^{52}$	NC
	139			0^0	C
o_3^{31}	116	76	$48^1, 49^5, 50^9, 51^5, 52^8, 53^{11}, 54^{19}, 55^{15}, 56^9, 57^{19}, 58^{35}, 59^{31}, 60^{22}, 61^{25}, 62^{17}, 63^{13}, 64^6, 65^4, 76$	$0^{88}, 1^{51}$	NC
	140			0^0	C
o_3^{32}	117	77	$48^1, 50^{11}, 51^5, 52^7, 53^{10}, 54^{13}, 55^{15}, 56^{15}, 57^{15}, 58^{25}, 59^{41}, 60^{19}, 61^{26}, 62^{21}, 63^{11}, 64^{14}, 65^3, 6$	$0^{88}, 1^{50}$	NC
	143			0^0	C
o_3^{33}	118	78	$48^1, 50^4, 51^8, 52^7, 53^9, 54^{10}, 55^{16}, 56^{17}, 57^{13}, 58^{17}, 59^{40}, 60^{25}, 61^{19}, 62^{28}, 63^{16}, 64^{14}, 65^7, 6$	$0^{88}, 1^{49}$	NC
	146			0^0	C
o_3^{34}	119	79	$48^1, 50^1, 51^6, 52^8, 53^8, 54^{10}, 55^{11}, 56^{20}, 57^{14}, 58^{14}, 59^{30}, 60^{30}, 61^{22}, 62^{28}, 63^{21}, 64^{13}, 65^{12}, 66^3, 67^2, 79^1$	$0^{88}, 1^{48}$	NC
	147			0^0	C
o_3^{35}	120	80	$49^1, 51^3, 52^7, 53^8, 54^{12}, 55^9, 56^{14}, 57^{16}, 58^{10}, 59^{35}, 60^{27}, 61^{22}, 62^{26}, 63^{21}, 64^{22}, 65^9, 66^8, 67^4, 80^1$	$0^{88}, 1^{47}$	NC
	148			0^0	C
o_3^{36}	121	81	$50^1, 52^5, 53^{10}, 54^{11}, 55^{10}, 56^{10}, 57^{13}, 58^{13}, 59^{18}, 60^{36}, 61^{32}, 62^{23}, 63^{20}, 64^{20}, 65^{15}, 66^8, 67^8, 68^1, 81$	$0^{88}, 1^{46}$	NC
	151			0^0	C

o_3^{37}	122	82	$51^1, 52^3, 53^3, 54^{15}, 55^{10}, 56^{12}, 57^8, 58^{12}, 59^{22}, 60^{22}, 61^{36}, 62^{25}, 63^{22}, 64^{24}, 65^{16}, 66^{10}, 67^9, 68^3, 6$	$0^{88}, 1^{45}$	NC
	152			0^0	C
o_3^{38}	123	83	$52^2, 53^3, 54^9, 55^{12}, 56^{15}, 57^8, 58^5, 59^{21}, 60^{23}, 61^{24}, 62^{33}, 63^{26}, 64^{27}, 65^{18}, 66^9, 67^{11}, 68^5, 69^3, 83^1$	$0^{88}, 1^{44}$	NC
	155			0^0	C
o_3^{39}	124	84	$53^3, 54^9, 55^7, 56^{14}, 57^{10}, 58^8, 59^9, 60^{25}, 61^{30}, 62^{24}, 63^{31}, 64^{24}, 65^{24}, 66^{14}, 67^9, 68^7, 69^5, 70^1, 84^1$	$0^{88}, 1^{43}$	NC
	158			0^0	C
o_3^{40}	125	85	$53^1, 54^5, 55^7, 56^{13}, 57^{15}, 58^7, 59^7, 60^{19}, 61^{22}, 62^{31}, 63^{25}, 64^{27}, 65^{33}, 66^{17}, 67^9, 68^5, 69^7, 70^4, 85^1$	$0^{88}, 1^{42}$	NC
	159			0^0	C
o_3^{41}	126	86	$54^3, 55^4, 56^{10}, 57^{17}, 58^{10}, 59^8, 60^{10}, 61^{27}, 62^{25}, 63^{28}, 64^{28}, 65^{21}, 66^{28}, 67^{14}, 68^8, 69^7, 70^4, 71^2, 86^1$	$0^{88}, 1^{41}$	NC
	162			0^0	C
o_3^{42}	127	87	$55^6, 56^4, 57^{13}, 58^{18}, 59^6, 60^7, 61^{23}, 62^{22}, 63^{33}, 64^{27}, 65^{19}, 66^{34}, 67^{14}, 68^9, 69^9, 70^6, 71^3, 72^1, 87^1$	$0^{88}, 1^{40}$	NC
	163			0^0	C
o_3^{43}	128	88	$55^4, 56^3, 57^6, 58^{20}, 59^{10}, 60^8, 61^{10}, 62^{26}, 63^{34}, 64^{25}, 65^{22}, 66^{28}, 67^{26}, 68^8, 69^{10}, 70^6, 71^6, 72^2, 88^1$	$0^{88}, 1^{39}$	NC
	166			0^0	C
o_3^{44}	129	89	$56^5, 57^5, 58^{12}, 59^{20}, 60^7, 61^7, 62^{16}, 63^{26}, 64^{35}, 65^{20}, 66^{32}, 67^{28}, 68^{13}, 69^{13}, 70^4, 71^6, 72^4, 73^1, 89^1$	$0^{88}, 1^{38}$	NC
	167			0^0	C
o_3^{45}	130	90	$56^2, 57^8, 58^6, 59^{12}, 60^{19}, 61^5, 62^9, 63^{24}, 64^{25}, 65^{37}, 66^{24}, 67^{24}, 68^{29}, 69^{11}, 70^7, 71^4, 72^5, 73^3, 90^1$	$0^{88}, 1^{37}$	NC
	170			0^0	C
o_3^{46}	131	91	$56^1, 57^3, 58^9, 59^9, 60^{17}, 61^9, 62^7, 63^{20}, 64^{19}, 65^{31}, 66^{35}, 67^{29}, 68^{23}, 69^{15}, 70^{13}, 71^4, 72^4, 73^6, 91^1$	$0^{88}, 1^{36}$	NC
	171			0^0	C
o_3^{47}	132	92	$56^1, 57^2, 58^6, 59^7, 60^{10}, 61^{18}, 62^8, 63^{11}, 64^{19}, 65^{32}, 66^{32}, 67^{25}, 68^{28}, 69^{22}, 70^{16}, 71^5, 72^4, 73^6, 74^2, 92^1$	$0^{88}, 1^{35}$	NC
	174			0^0	C
o_3^{48}	133	93	$57^2, 58^4, 59^7, 60^4, 61^{21}, 62^{13}, 63^7, 64^{10}, 65^{24}, 66^{39}, 67^{25}, 68^{32}, 69^{25}, 70^{19}, 71^9, 72^2, 73^6, 74^5, 93^1$	$0^{88}, 1^{34}$	NC
	175			0^0	C
o_3^{49}	134	94	$58^6, 60^{11}, 62^{34}, 64^{17}, 66^{63}, 68^{57}, 70^{44}, 72^{11}, 74^{11}, 94^1$	$0^{88}, 1^{33}$	NC
	176			0^0	C
o_3^{50}	135	95	$58^1, 59^5, 60^5, 61^6, 62^{14}, 63^{20}, 64^{15}, 65^2, 66^{22}, 67^{41}, 68^{35}, 69^{22}, 70^{22}, 71^{22}, 72^9, 73^2, 74^5, 75^6, 95^1$	$0^{88}, 1^{32}$	NC
	181			0^0	C
o_3^{51}	136	96	$59^3, 60^4, 61^7, 62^9, 63^{15}, 64^{20}, 65^9, 66^{13}, 67^{35}, 68^{33}, 69^{29}, 70^{33}, 71^{21}, 72^{18}, 73^3, 74^3, 75^6, 76^3, 96^1$	$0^{88}, 1^{31}$	NC
	182			0^0	C
o_3^{52}	137	97	$59^3, 60^4, 61^7, 62^9, 63^{15}, 64^{20}, 65^9, 66^{13}, 67^{35}, 68^{33}, 69^{29}, 70^{33}, 71^{21}, 72^{18}, 73^3, 74^3, 75^6, 76^3, 77\ \ , 97$	$0^{88}, 1^{30}$	NC
	183			0^0	C
o_3^{53}	138	98	$60^2, 61^5, 62^5, 63^{13}, 64^{14}, 65^{14}, 66^{12}, 67^{21}, 68^{36}, 69^{20}, 70^{23}, 71^{43}, 72^{22}, 73^6, 74^6, 75^3, 76^6, 77^3, 98^1$	$0^{88}, 1^{29}$	NC
	184			0^0	C

o_3^{54}	139	99	$61^4, 62^4$ $, 63^9, 64^{16}, 65^{14}, 66^{11}, 67^{22}, 68^{18}, 69^{32}, 70^{27}$ $, 71^{27}, 72^{40}, 73^{10}, 74^5, 75^4, 76^6, 77^4, 78^1, 99^1$	$0^{88}, 1^{28}$	NC
	188			0^0	C
o_3^{55}	140	100	$61^2, 62^4$ $, 63^6, 64^{13}, 65^{14}, 66^{11}, 67^{16}, 68^{22}, 69^{32}, 70^{23}$ $, 71^{26}, 72^{35}, 73^{26}, 74^9, 76^8, 77^6, 78^1, 100^1$	$0^{88}, 1^{27}$	NC
	189			0^0	C
o_3^{56}	141	101	$61^1, 62^2$ $, 63^7, 64^7, 65^{15}, 66^9, 67^{15}, 68^{23}, 69^{26}, 70^{27}$ $, 71^{25}, 72^{29}, 73^{29}, 74^{23}, 75^1, 76^5, 77^7, 78^3, 101^1$	$0^{88}, 1^{26}$	NC
	191			0^0	C
o_3^{57}	142	102	62^1 $, 63^6, 64^6, 65^{13}, 66^{11}, 67^8, 68^{22}, 69^{21}, 70^{26}$ $, 71^{34}, 72^{24}, 73^{33}, 74^{27}, 75^6, 76^4, 77^5, 78^5, 79^2, 102^1$	$0^{88}, 1^{25}$	NC
	196			0^0	C
o_3^{58}	143	103	$63^3, 64^7, 65^7, 66^{14}, 67^{10}, 68^{15}, 69^{23}, 70^{18}$ $, 71^{35}, 72^{31}, 73^{23}, 74^{36}, 75^{16}, 76^3, 77^3, 78^4, 79^6, 103$	$0^{88}, 1^{24}$	NC
	193			0^0	C
o_3^{59}	144	104	$64^6, 65^9, 66^8, 67^{12}, 68^{11}, 69^{23}, 70^{20}$ $, 71^{22}, 72^{34}, 73^{31}, 74^{32}, 75^{24}, 76^9, 77^1, 78^4, 79^6, 80^2$	$0^{88}, 1^{23}$	NC
	198			0^0	C
o_3^{60}	145	105	$64^1, 65^7, 66^{11}, 67^9, 68^{13}, 69^{18}, 70^{21}, 71^{19}, 72^{23}, 73^{3}$ $, 74^{39}, 75^{21}, 76^{17}, 77^4, 78^1, 79^7, 80^2, 81^2, 105^1$	$0^{88}, 1^{22}$	NC
	201			0^0	C
o_3^{61}	146	106	$65^6, 66^4, 67^{11}, 68^{13}, 69^{12}, 70^{28}, 71^{15}, 72^{17}, 73^{42}$ $, 74^{35}, 75^{27}, 76^{21}, 77^{10}, 78^2, 79^3, 80^5, 81^2, 82^1, 106^1$	$0^{88}, 1^{21}$	NC
	202			0^0	C
o_3^{62}	147	107	$65^2, 66^4, 67^8, 68^{15}, 69^{10}, 70^{18}, 71^{27}, 72^{15}, 73^{24}$ $, 74^{44}, 75^{32}, 76^{23}, 77^{16}, 78^4, 79^3, 80^3, 81^4, 82^2, 107^1$	$0^{88}, 1^{20}$	NC
	205			0^0	C
o_3^{63}	148	108	$66^5, 67^3, 68^{14}, 69^{12}, 70^{14}, 71^{21}, 72^{23}, 73^{14}$ $, 74^{32}, 75^{53}, 76^{24}, 77^{16}, 78^{10}, 79^3, 80^1, 81^6, 82^3, 108$	$0^{88}, 1^{19}$	NC
	206			0^0	C
o_3^{64}	149	109	$66^2, 67^5, 68^5, 69^{15}, 70^{14}, 71^{13}, 72^{29}, 73^{16}$ $, 74^{22}, 75^{47}, 76^{37}, 77^{17}, 78^{16}, 79^5, 80^1, 81^6, 82^2, 83^2$	$0^{88}, 1^{18}$	NC
	209			0^0	C
o_3^{65}	150	110	$67^5, 68^4, 69^{10}, 70^{14}, 71^{11}, 72^{22}, 73^{24}$ $, 74^{23}, 75^{37}, 76^{36}, 77^{26}, 78^{20}, 79^9, 80^2, 81^4, 82^5, 83^2$	$0^{88}, 1^{17}$	NC
	210			0^0	C
o_3^{66}	151	111	$67^2, 68^3, 69^6, 70^{14}, 71^{16}, 72^{15}, 73^{20}$ $, 74^{26}, 75^{31}, 76^{35}, 77^{28}, 78^{28}, 79^{16}, 80^3, 81^2, 82^4, 83$	$0^{88}, 1^{16}$	NC
	213			0^0	C
o_3^{67}	152	112	$68^2, 69^6, 70^7, 71^{15}, 72^{18}, 73^{16},$ $74^{31}, 75^{33}, 76^{19}, 77^{32}$ $, 78^{31}, 79^{19}, 80^{14}, 81^2, 82^3, 83^5, 84^1, 112^1$	$0^{88}, 1^{15}$	NC
	214			0^0	C
o_3^{68}	153	113	$69^5, 70^3, 71^{15}, 72^{13}, 73^{20}, 74^{25}, 75^{27}, 76^{31}, 77^{27}$ $, 78^{33}, 79^{21}, 80^{15}, 81^{10}, 82^3, 83^1, 84^5, 113^1$	$0^{88}, 1^{14}$	NC
	219			0^0	C
o_3^{69}	154	114	$69^3, 70^3, 71^6, 72^{17}, 73^{19}, 74^{16}, 75^{28}, 76^{31}, 77^{33}$ $, 78^{31}, 79^{20}, 80^{21}, 81^{15}, 82^4, 83^2, 84^3, 85^2, 114^1$	$0^{88}, 1^{13}$	NC
	219			0^0	C
o_3^{70}	155	115	$69^1, 70^2, 71^5, 72^{13}, 73^{18}, 74^{19}, 75^{16}, 76^{27}, 77^{40}$ $, 78^{33}, 79^{23}, 80^{23}, 81^{18}, 82^9, 83^2, 84^1, 85^3, 86^1, 115^1$	$0^{88}, 1^{12}$	NC
	221			0^0	C
o_3^{71}	156	116	$70^2, 71^4, 72^6, 73^{18}, 74^{17}, 75^{16}, 76^{21}, 77^{36}$ $, 78^{45}, 79^{24}, 80^{18}, 81^{22}, 82^{15}, 83^4, 84^1, 85^4, 86^1, 116$	$0^{88}, 1^{11}$	NC
	226			0^0	C
o_3^{72}	157	117	$70^2, 72^7, 73^{12}, 74^{20}, 75^{10}, 76^{13}, 77^{43}$	$0^{88}, 1^{10}$	NC

	227		$,78^{36},79^{32},80^{25},81^{20},82^{20},83^{6},84^{3},85^{3},86^{2},117$	0^{0}	C
o_3^{73}	158	118	$70^{1},71^{1},73^{14},74^{15},75^{17},76^{6},77^{20}$	$0^{88},1^{9}$	NC
	230		$,78^{58},79^{31},80^{24},81^{26},82^{17},83^{15},84^{2},85^{4},86^{3},11$	0^{0}	C
o_3^{74}	159	119	$71^{1},72^{1},73^{6},74^{16},75^{14},76^{11},77^{12}$	$0^{88},1^{8}$	NC
	231		$,78^{40},79^{49},80^{27},81^{26},82^{20},83^{14},84^{9},85^{4},86^{4},11$	0^{0}	C
o_3^{75}	160	120	$72^{1},73^{2},74^{11},75^{18},76^{12},77^{14}$	$0^{88},1^{7}$	NC
	233		$,78^{17},79^{42},80^{53},81^{26},82^{17},83^{18},84^{12},85^{6},86^{3},8$	0^{0}	C
o_3^{76}	161	121	$73^{2},74^{2},75^{15},76^{22},77^{11}$	$0^{88},1^{6}$	NC
	235		$,78^{14},79^{29},80^{52},81^{40},82^{14},83^{17},84^{22},85^{9},86^{2},8$	0^{0}	C
o_3^{77}	162	122	$74^{3},75^{4},76^{20},77^{22}$	$0^{88},1^{5}$	NC
	238		$,78^{10},79^{12},80^{56},81^{52},82^{15},83^{12},84^{20},85^{22},86^{2},8$	0^{0}	C
o_3^{78}	163	123	$75^{3},76^{14},77^{20}$	$0^{88},1^{4}$	NC
	241		$,78^{14},79^{12},80^{36},81^{56},82^{36},83^{11},84^{14},85^{20},86^{14},$	0^{0}	C
o_3^{79}	164	124	$76^{7},77^{22}$	$0^{88},1^{3}$	NC
	244		$,78^{18},79^{8},80^{16},81^{52},82^{60},83^{16},84^{7},85^{22},86^{18},8$	0^{0}	C
o_3^{80}	165	125	77^{15}	$0^{88},1^{2}$	NC
	245		$,78^{28},79^{12},81^{32},82^{72},83^{40},85^{15},86^{28},87^{12},125^{1}$	0^{0}	C
o_3^{81}	166	126	$78^{29},79^{26},82^{68},83^{76},86^{29},87^{26},126^{1}$	$0^{88},1^{1}$	NC
	248			0^{0}	C
o_3^{82}	167	127	$79^{55},83^{144},87^{55},127^{1}$	0^{88}	NC
	255			0^{0}	C

Conclusions

The idea of group action is a useful method to construct new arcs of higher degree and in the finite projective space of dimension higher than two. So, the lower bounds for $m_r(7,2)$ is founded for $3 \leq r \leq 126$.

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References:

[1] J. W. P. Hirschfield, "*Finite projective spaces of three dimensions*," Oxford University Press, 1985.

[2] J. W. P. Hirschfeld, "*Projective geometries over finite fields*,"2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.

[3] S. Ball and J. W. P. Hirschfeld, "Bounds on $(n;r)$-arcs and their application to linear codes," *Finite Fields and Their Applications*, vol. 11, no. 3, pp. 326-336, Aug. 2005.

[4] R. Daskalov, "New Large (n,r)-arcs in $PG(2,q)$," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 17, no. 1, pp. 125–133, Apr. 2022.

[5] R. Daskalov and E. Metodieva, "Three new large (n,r) arcs in $PG(2,31)$," in 2020 Algebraic and Combinatorial Coding Theory (ACCT), *IEEE*, pp. 1- 4, Oct. 2020.

[6] R. N. Daskalov, E. Metodieva, and R. Daskalov, “Four new large (n, r)-arcs in $PG(2,31)$,” 2018. [Online]. Available: https://www.researchgate.net/publication/332963540.

[7] M. Braun, “New lower bounds on the size of (n, r)-arcs in $PG(2, q)$,” *Journal of Combinatorial Designs*, vol. 27, no. 11, pp. 682–687, Nov. 2019.

[8] M. Braun, “Update: Some new results on lower bounds on $(n; r)$-arcs in $PG(2, q)$ for $q < 31$,” Jun. 2021, [Online]. Available: http://arxiv.org/abs/2106.05908.

[9] F. N. Abdullah, N. Yassen, and K. Yahya, “Bounds on minimum distance for linear codes over $GF(q)$,” *Italian Journal of Pure and Applied Mathematics*, no. 45, pp. 894-903, 2021.

[10] A. E. Mustafa Sulaimaan and N. Y. Kasm Yahya, “The reverse construction of complete (k, n)-arcs in three-dimensional projective space $PG(3,4)$,” *J. Phys. Conf. Ser*, vol. 1591, no. 1, p. 012078, Jul. 2020.

[11] N. A. M. Al-Seraji, A. J. Al-Rikabi, and E. B. Al-Zangana, “Represent the space $PG(3, 8)$ by subspaces and sub-geometries,” *AIP Conference Proceedings*, vol. 2414, no. 1, p. 040037, 2023.

[12] N. A. Al-seraji, A. J. Al-Rikabi, and E. B. Al-Zangana, “New arcs in $PG(3,8)$ by singer group,” *Al-Mustansiriyah Journal of Science*, vol. 33, no. 2, pp. 70-76, Jun. 2022.

[13] J. S. Radhi and E. B. Al-Zangana, “Complete (k, r)-caps from orbits in $PG(3,11)$,” *Iraqi Journal of Science*, pp. 347-353, Jan. 2023.

[14] J. S. Radhi and E. B. Al-Zangana, “Construction of complete $(k; r)$-arcs from orbits in $PG(3,11)$,” *Al-Mustansiriyah Journal of Science*, vol. 33, no. 3, pp. 48-53, Sep. 2022.

[15] S. M. Attook and E. B. Al-Zangana, “Use algebra of group action to find special types of caps in $PG(3,13)$,” *Iraqi Journal of Science*, pp. 4092-4101, Aug. 2023.

[16] S. M. Attook and E. B. Al-Zangana, “Special arcs in $PG(3,13)$,” *Al-Mustansiriyah Journal of Science*, vol. 33, no. 4, pp. 86-91, Dec. 2022.

[17] E. B. Al-Zangana, “On non-singular plane cubic curves over F_q, $q = 2,3,5,7$," *Journal of College of Education, Mustansiriyah University*, no.1, pp. 149-160, 2013.

[18] Z. Shehab Hamed, “On the geometric pictures of the non-equivalent smallest images of $(k; n)$-arcs for $n = 2,3$ in $PG(2,5)$,” *J. Phys Conf Ser*, vol. 1294, no. 3, p. 032021, Sep. 2019.

[19] N. A. M. Al-Seraji , “Classification of the projective line over Galois field of order 31,” *Iraqi Journal of Science*, pp. 1881–1901, Apr. 2023.

[20] H. R. Shakir, and S. A. Mehdi, "A new method for color image encryption using chaotic system and DNA encoding ," *Al-Mustansiriyah Journal of pure and Applied Sciences*, vol. 1, no. 1, pp. 68-79, 2023.

[21] A. A. Salih, and A. S. Mahmood, "Enhace key stage generation for developing kasumi encryption algorithm," *Al-Mustansiriyah Journal of pure and Applied Sciences*, vol. 2, no. 4, pp. 104-112, 2024.

[22] The GAP Group, GAP Reference Manual. Available online: http://www.GAP-system.org/, (accessed Oct. 30, 2022).