Baghdad Science Journal

Volume 23 | Issue 3 Article 21

3-25-2026

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How to Cite this Article

Jameel, Marwan S.; Allawee, Zeiad Yahya Ali; Al-Kawaz, Rana Z; and Hussein, Sura J (2026) "Using the Karush-Kuhn-Tucher Optimization Method on a Finite Fuzzy Markov Chain," *Baghdad Science Journal*: Vol. 23: Iss. 3, Article 21.
DOI: https://doi.org/10.21123/2411-7986.5247

BAGHDAD SCIENCE JOURNAL 2026;23(3):985–998

RESEARCH ARTICLE

Using the Karush-Kuhn-Tucher Optimization Method on a Finite Fuzzy Markov Chain

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ABSTRACT

It is highly interesting to explore the implications of Markov chain stability because its importance in engineering, dynamic systems, energy subject and decision-making issues. This paper deals with a method to assess the stability of Markov chains, finishing our notes in finite fuzzy Markov chains. The optimization of KKT was evaluated in two approaches: a. For one thing, optimized KKT methods substitute the random transition probability with a fuzzy value in the transition matrix. The problem illustrates the efficacy of method in proving the unique properties for power bounds of a 3 × 3 matrix depending where on this it lies. This transition matrix defines a fuzzy link within an essentially constrained state space. The second attempt was to model the problem of computing dependability vector by the proposed optimization technique. Additional components were successfully incorporated criteria into fuzzy 3 × 3 transition matrices via KKT optimization techniques. This enhanced the adaptability of the behavior. The model determined that ability constituted an optimization problem. This applicable feature involves the collection of data, namely the exchange rate of the Iraqi dinar against the US dollar as established by the Central Bank of Iraq for the period from October 1, 2023, to December 1, 2023. Data was evaluated, results were derived to fulfill the study objective, and suggestions were formulated based on the findings.

Keywords: Constrained optimization, Fuzzy number, Finite Markov chain, KKT-optimizer, Mathematical modelling

Introduction

Markov chains are a principal mathematical technique for handling dynamic systems. Currently, it has been applied in several fields to solve real-world problems. However, today's real-world problem is not crisp; it has vagueness or impreciseness. A new mathematical model called finite fuzzy Markov chain provides knowledge of the future that will happen with respect to impreciseness or uncertainty. [1] Optimization is a prominent tool used to improve the performance of many systems or industries. The purpose of optimization is to make the system as efficient as possible. A pragmatic approach to optimization was all that could solve an optimization problem. To solve such problems it is necessary to use a set of optimal solutions defined by a decision-maker and account this subjective information for quantitative model. Ultimately, it is sought to form an accurate representation of the efficient frontier, thus a variable concept of structure for the problem. [2]

Received 22 May 2024; revised 25 January 2025; accepted 27 January 2025.
Available online 25 March 2026

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https://doi.org/10.21123/2411-7986.5247

The KKT method is important technique for optimization algorithms, and has many applications in engineering and scientific problems. There are many attempts in the literature to develop approaches based on using the KKT conditions that are able to convert MOP into a trackability problem.[3] Gatti, Rocco and Sandholm pointed out that, in such cases, the KKT conditions impose further necessary but insufficient constraints. By representing the KKTV state in the Pareto optimal formulation, a unique solution is available for every Pareto point.[4] New developments, including those of Chalco have generalized the KKT optimality conditions to fuzzy optimization problems and permit handling uncertainty in the objectives that are given by means of fuzzy-valued functions more efficientl.[5] Stefanini and Jimenez (2005) further investigated the establishment of KKT conditions for interval and fuzzy optimization, which have led to robust optimization methods under uncertainty the applications of Fuzzy logic have also experienced substantial growth mostly in control, analysis, prediction of state of a system etc.[6] Relatively conventional fuzzy systems have difficulty in coping with random nature phenomena, while Markov models can be integrated inherently into the depiction of uncertainty through fuzzy logic to provide a viable remedy. The Markov model method, however computationally affordable, suffers from the limitation of sampling the system state space into those adequately approximated by transition probability matrices. To overcome these issues, researchers have proposed fuzzy Markov models and fuzzy dynamical programming, which combine the strengths of both methodologies.[7,8] For instance, network-based fuzzy inference systems (ANFIS) have been employed for predicting chaotic time series, and integrating Markov models allows for stochastic time series simulation.[9] Studies, such as those by Clempner et al,[10] have demonstrated the convergence properties of advanced optimization techniques applied to multi-objective Markov chains, further highlighting their potential. Hoang explored decision-making frameworks under uncertainty, focusing on chance-constrained optimization in game theory and Markov decision processes (MDPs).[11,12] Most interest studied by Socha in Genetic algorism utilized to calculate the values of fuzzy transition matrix powers, streamlining the analysis of reachability and stationary distribution for fuzzy Markov chains.[2]

The gap in the study of the inductivity or invariance of Markov chains, specifically fuzzy Markov chains, remains an open question because of the complexities of modern applied issues. Fuzzy logic is the primary consideration in modern decision-making applications. Consequently, summarizing the optimal instances of Markov chains, particularly fuzzy variants, becomes challenging; thus, it is preferable to seek optimization strategies capable of addressing the issue.[1,2] Fuzzy Markov chains have a beneficial intention—to aid in decision-making—but their stability is a major flaw, and optimization procedures are necessary to overcome this. This leads to the formation of the research pillars.

In this paper, we will study finite fuzzy Markov chains from two points of view. First, ambiguity concerning the transition probabilities in a transition matrix is taken into account by representing them as fuzzy numbers. Second, it solves the ergodic properties of fuzzy Markov chains based on KKT optimization. A fuzzy Markovian transition matrix is provided to demonstrate the representation of fuzzy relations over state space. The study validates the proposed approach through specific criteria, demonstrating its efficacy. Additionally, a detailed analysis of unique 3×3 fuzzy transition matrices explores how relaxed behavior can be achieved through KKT optimization. The optimization problem involving finite fuzzy Markov chain is formulated as a nonlinear programming problem. Then this problem is solved using the conventional optimization method. Finally, the experimental result is analyzed.

The given material is organized as follows: an introduction; Some Basic Concepts discusses more detail about Crisp Markov chains and finite fuzzy Markov chains; the following section discusses the application of the optimization method in finite fuzzy Markov chains and a quantitative result; then Outcomes and Analysis of the Practical Implementation. The last section presents a list of references used in the paper.

Some basic concepts

This article provides a novel methodology for the optimization of finite fuzzy stochastic models. The specific problem addressed is a recently revised mathematical modeling tool of finite fuzzy Markov chains and, more accurately, the optimization of the geometrical scenarios vector of finite fuzzy Markov chain transitions cardinalities so that the model's required type I and II fuzziness properties are satisfied within specified tolerated margins. Commencing with the concepts, elements, and variables pertinent to the research:

Definition 1 (A Markov chain): A Markov chain is a stochastic process that transitions between different states within a finite set of possible states. A Markov chain is a collection of distinct states and associated probabilities, where the future state or condition of a variable is heavily influenced by its most recent state.[13]

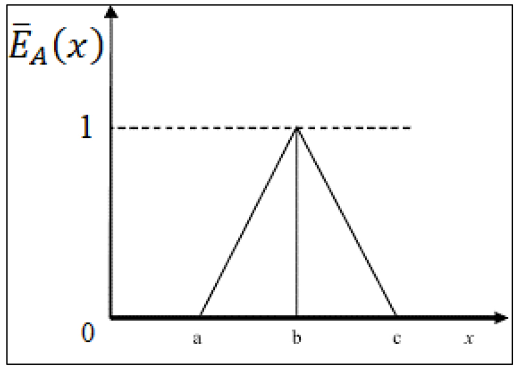

Fig. 1. Triangular fuzzy number graphic representation.

Definition 2 (A fuzzy set): A fuzzy set is defined as a math pair (U, m) where U represent set (non-empty) and m is a membership function that maps element U to the interval $[0, 1]$. The set U also known as Ω *or* X is referred to as universe of discourse. Each element $x \in U$ is assigned a value $m(x)$, which is referred to as the the grade of membership of x in (U, m). The membership function of the fuzzy set $A = (U, m)$ is denoted as $m = \mu_A$.[14]

Definition 3 (Fuzzy Markov chain): A fuzzy Markov chain $\bar{R} = (S,\ s_0, \bar{P})$ is a Markov chain where each p_{ij} from the transition matrix is substituted with a fuzzy probability $\bar{p}_{ij}$. These values now make up a fuzzy transition matrix $\bar{P} = \bar{p}_{ij}$, where there are such $p_{ij} \in \bar{p}_{ij}[1]$ is so that $P = p_{ij}$ is a valid transition matrix for a őnite Markov chain (the rows sum up to one).[15]

Definition 4 (A triangular fuzzy number): is a distinct sort of fuzzy number that has a triangle shape in mathematical terms $\bar{E} = (a|b|c)$ by three real numbers $a,\ b,\ c$ such that $a <\ b <\ c$ and the membership function $\bar{E} : \mathbb{R} \to [0, 1] \in \mathbb{R}$ satisfies the following conditions[16]:

1. $\bar{E}(b) = 1.$
2. The graph of $y = \bar{E}(x)$ forms represent by a linear segment from $(a,\ 0)\ to\ (b,\ 1)\ for\ x \in [a, b]$ and another linear segment from $(b,\ 0)\ to\ (c,\ 1)\ for\ x \in [b, c]$. In this graphical representation, the fuzzy number forms a triangle where the x-coordinate refers to the value under consideration, and the y-coordinate refers to its degree of membership, as seen in the illustration Fig. 1.
3. $\bar{E}(x) = 0\ for\ all\ x \in \mathbb{R}$ such that $x \leq a\ or\ x \geq c$.

Definition 5: Let $f(X)$ be a function where $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)$, then a point $X_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ is a maximum if $f(X_0 + \boldsymbol{h}) \leq f(X_0)$ for all $\boldsymbol{h} = (h_1, h_2, \ldots, h_n)$ where $|h_j|$ is sufficiently small for all j. In a similar manner X_0 is a minimum if $f(X_0 + \boldsymbol{h}) \geq f(X_0)$. A function $f(X)$ has an extreme point that might be a maximum or a minimum for the function.[17]

Consider the problem[17]

$$Maximize\ z = \ f(X) \tag{1}$$

Subject to

$g(X) \geq 0$

By introducing nonnegative slack variables, it is possible to transform the inequality constraints into equations. Let $S_i^2 \geq 0$ be the slack quantity added to the i^{th}constraint $g_i(X) \leq 0$ and Define

$$S = (S_1.\ S_2.\ \ldots.\ S_m)^T,\ S^2\ =\ \left(S_1^2.S_2^2.\ldots.S_m^2\right)^T$$

m is the overall count of inequality restrictions.

KKT optimization technique

Considering the optimization problem 1 may be non-linear programming when the objective or the constraint is non-linear; in that case, the KKT condition will conduct the Lagrange function (LF) to solve the issue. LF is thus given by

$$\mathrm{L}\left((\mathrm{X},\ \mathrm{S},\ \lambda)\right) = \ f\,(X) - \ \lambda\left[g\,(\mathrm{x}) + S^2\right] \tag{2}$$

Considering the constraints18. For optimality to be achieved, it is essential that λ is either nonnegative for maximization problems or nonpositive for minimization issues. This outcome is substantiated by observing the vector.

The symbol **λ** represents the rate at which the function f changes in relation to the function g

$$\lambda = \frac{\partial f}{\partial g}$$

In the context of maximizing, as the value of the constraint $(\boldsymbol{X}) < 0$ grows from 0 to the vector ∂g, the range of possible solutions becomes less restricted. Consequently, the objective function f cannot decrease, indicating that **λ** is greater than or equal to zero. Similarly, in the case of minimization, as the values on the right-hand side of the constraints rise, the objective function f cannot increase. This means that the Lagrange multiplier **λ** is less than or equal to zero ($\boldsymbol{\lambda} < \mathbf{0}$). If the constraints are expressed as equalities, meaning that $(\boldsymbol{X}) = 0$, then **λ** is allowed to have any sign without restrictions.[18]

The limitations on **λ** are fulfilled as a constituent of the Karush-Kuhn-Tucker (KKT) necessary requirements. Now, proceed to construct the remaining necessary conditions.

By calculating the partial derivatives of L with respect to $\boldsymbol{X}$, $\boldsymbol{S}$, and **λ**, then the derive will given

$$\frac{\partial L}{\partial X} = \ A\ f\,(X) - \lambda\ A\ g\,(X) = 0$$

$$\frac{\partial L}{\partial S_i} = -2\lambda_i S_i = 0. \quad i = 1.2.\ldots.m$$

$$\frac{\partial L}{\partial \lambda_i} = \ -\lambda(g\,(X) + S^2 = 0$$

Where A indicates to denote a matrix utilized for scaling or transforming the function (X) and gradients of $g(X)$. The precise function of A is based on the particular optimization problem being addressed. The subsequent set of equations yields the following outcomes:

1. If $\lambda_i \neq 0$, then $S_i^2 = 0$, this implies that the associated resource is plentiful and hence fully utilized (equality constraint).
2. If $S_i^2 > 0$, then $\lambda_i = 0$. This implies that resource i is abundant and, as a result, it does not impact the value of f (i.e., $\lambda_i = \frac{\partial f_i}{\partial g_i}$).

From the second and third sets of equations, obtaining $(\boldsymbol{X}) = 0$, $i = 1{,}2, \ldots, m$
This new condition essentially repeats the foregoing argument, because if $\lambda_i > 0$,
$(\boldsymbol{X}) = 0 \mathrm{or} S_i{}^2 = 0$; $\mathrm{and if} g(\boldsymbol{X}) < 0, S_i{}^2 > 0,\ \mathrm{and}\ \ \lambda_i = 0.$
The KKT necessary conditions for maximization problem are summarized as:

$$\boldsymbol{\lambda}\ \geq\ \mathbf{0}$$

$$A\,(\boldsymbol{X})\ -\boldsymbol{\lambda}\mathrm{A}g\,(\boldsymbol{X})\ =\ 0$$

$$(\boldsymbol{X}) = 0,\ i\ =\ 1, 2,\ \ldots,\ m$$

$$g\,(\boldsymbol{X})\ \leq\ 0$$

In minimization, the same restrictions apply, except that λ must be non-negative. Both maximization and minimization issues do not restrict the sign of the Lagrange multipliers linked to equality criteria.

KKT on a finite fuzzy Markov chain

The methodology employs restricted fuzzy matrix multiplication to determine the maximum and minimum of $(f_{ij})^{\,n}$ on $Dom[\alpha]$, identifying the endpoints of the $\alpha - cuts$ of $(\bar{p}_{ij})^n$ utilized constrained fuzzy arithmetic. The objective of this problem is then to model it as a fuzzy optimization problem, which the KKT approach being the central strategy.

We have discussed the basic theory of fuzzy Markov chains and how to utilize the Karush-Kuhn-Tucker (KKT) optimization method. Specifically, the section now clarifies the process of constructing a fuzzy transition matrix, wherein each entry is replaced with a fuzzy number to account for inherent uncertainties. The concept of α-cuts, which represent bounds on fuzzy values, has been introduced to describe the uncertainty intervals effectively:

Uncertainty Representation: if P is a $r \times r$ crisp transition matrix for a regular Markov chain then $\lim_{n\to\in fty} p^n = \pi$ where each row in π *is* $w = (w_1.w_2. \ \ldots \ w_n)$, $wi > 0$ and $\sum_{i=1}^{r} w_i = 1$,
Here w is the solution of $wP \ = \ w$ satisfying $w_i \ > \ 0$ and $\sum_{i=1}^{r} w_i = 1$.

If $Q = [q_{ij}]$ is a $r \times r$ crisp transition matrix for a regular Markov chain, then consider $\bar{p} = [\bar{p}_{ij}]$ where $\bar{p}_{ij}$ gives the uncertainty (if any) in q_{ij}.

α-Cuts for Fuzzy Values: If $(p_{11}, \ldots, p_{rr}) \in Dom[\alpha]$, then $p = [p_{ij}]$ is also transition matrix for a regular Markov chain.

Let $\bar{p} \to \bar{\pi}$ where each row in $\overline{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1. \ \bar{\pi}_2. \ \ldots \ \bar{\pi}_r)$.

Also let $\bar{\pi}_j \ [\alpha] = [\pi_{j1} \ (\alpha), \pi_{j2} \ (\alpha)], \ j = 1, \ldots, r$.

Then, show how to compute the α −cuts of $\bar{\pi}_j$.

For each $(p_{11}, \ldots, p_{rr}) \in Dom[\alpha]$, set $p = [p_{ij}]$ and get $p^n = \pi$. Let

$$\Gamma(\alpha) \ = \ \{w|w \ a \ row \ in \ \Pi, \ (p_{11}, \ \ldots, \ p_{rr}) \ \in \ Dom\,[\alpha]\}.$$

Then $\pi_{j1}(\alpha) = \min\{w_j \mid w \in \Gamma(\alpha)\}$

$$\pi_{j2}(\alpha) = max\{\, W_J|w \ \in \ \Gamma(\alpha)\}$$

where w_j is the i^{th} component in the vector w.[19]

Stationary Distribution with Fuzzy Values: the following example shows how to find the limit of 3×3 regular fuzzy Markov chains using the restricted fuzzy matrix multiplication, then using Krush-Kanukar Method.

Let $Q = [q_{ij}]$ be a 3×3 transition matrix of a regular Markov chain, then consider

$$\bar{p} = \begin{vmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ \bar{p}_{31} & \bar{p}_{23} & \bar{p}_{33} \end{vmatrix}$$

Where $\bar{p}_{ij}$ gives the uncertainty (if any) in q_{ij} for $i\,.j = 1.2$.

$$if \ \ (\,p_{11}.p_{21}.p_{31}.p_{12}.p_{22}.p_{23}.p_{13}.p_{23}.p_{33}) \ \in \ \mathrm{Dom}\,[\alpha]\,, \ \ \text{then}$$

$$= \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{23} & p_{33} \end{vmatrix}$$

is a regular transition matrix and so P^n is convergent.
Optimization with KKT Conditions:

$$\text{solving} \ \ [W_1 \ \ W_2 \ \ W_3\ ] \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{23} & p_{33} \end{vmatrix} = [W_1 \ \ W_2 \ \ W_3\ ] \ \text{where} \ W_1 \ \ W_2 \ \ W_3 \ > 0$$

And $W_1 + W_2 + W_3 = 1$

Where

$$W_3 = \frac{p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32}}{p_{22}p_{11} - p_{12}p_{21}} > 0$$

$$W_2 = -\frac{((p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(p_{32}p_{11} - p_{12}p_{31}))}{(p_{22}p_{11} - p_{12}p_{21})^2}$$

$$W_1 = -\frac{((p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(-p_{31}p_{22} + p_{21}p_{32}))}{(p_{22}p_{11} - p_{12}p_{21})^2}$$

Now

$$\frac{\partial W_1}{\partial p_{11}} = \frac{((p_{33}p_{22} - p_{23}p_{32})(-p_{31}p_{22} + p_{21}p_{32}))}{(1 - 2p_{12} - p_{23} + p_{12}p_{21} + p_{23}p_{12} - p_{13} - p_{13}p_{21} + p_{23}p_{13})^2} > 0$$

$$\frac{\partial W_2}{\partial p_{21}} = \frac{(-p_{33}p_{12} + p_{13}p_{32})(p_{32}p_{11} - p_{12}p_{31})}{(p_{22}p_{11} - p_{12}p_{21})^2} < 0$$

$$-\frac{(2(p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(p_{31}p_{11} - p_{12}p_{31})p_{12})}{(p_{22}p_{11} - p_{12}p_{21})^3} < 0$$

$$\frac{\partial W_3}{\partial p_{31}} = \frac{-p_{13}p_{22} + p_{23}p_{12}}{p_{22}p_{11} - p_{12}p_{21}} < 0$$

$$\frac{\partial W_1}{\partial p_{12}} = \frac{(-p_{33}p_{21} + p_{23}p_{31})(-p_{31}p_{22} + p_{21}p_{32})}{(p_{22}p_{11} - p_{12}p_{21})^2}$$

$$+\frac{(2(p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(-p_{31}p_{22} + p_{21}p_{32})p_{21}))}{(p_{22}p_{11} - p_{12}p_{21})^3} < 0$$

$$\frac{\partial W_2}{\partial p_{22}} = -\frac{(p_{33}p_{11} + p_{13}p_{31})(p_{32}p_{11} - p_{12}p_{31})}{(p_{22}p_{11} - p_{12}p_{21})^2}$$

$$+\frac{(2(p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(p_{32}p_{11} + p_{12}p_{31})p_{11}))}{(p_{22}p_{11} - p_{12}p_{21})^3} < 0$$

$$\frac{\partial W_3}{\partial p_{32}} = \frac{-p_{23}p_{11} + p_{13}p_{21}}{p_{22}p_{11} - p_{12}p_{21}} < 0$$

$$\frac{\partial W_1}{\partial p_{13}} = \frac{(-p_{31}p_{22} + p_{21}p_{32})^2}{(p_{22}p_{11} - p_{12}p_{21})^2} < 0$$

$$\frac{\partial W_2}{\partial p_{23}} = -\frac{(-p_{32}p_{11} + p_{12}p_{31})(p_{32}p_{11} - p_{12}p_{31})}{(p_{22}p_{11} - p_{12}p_{21})^2} < 0$$

$$\frac{\partial W_3}{\partial p_{33}} = 1 > 0$$

$\bar{\mathrm{p}}_{11}[\alpha] = [\mathrm{p}_{111}(\alpha), \mathrm{p}_{112}(\alpha)]$ and $\bar{\mathrm{p}}_{12}[\alpha] = [\mathrm{p}_{121}(\alpha), \mathrm{p}_{122}(\alpha)]$ …….... and it is possible to take all the elements of the matrix $\bar{P}$ in the same manner.
by restricted matrix multiplication $P^n \to \Pi$ where each row in Π is $\pi = (\pi_1, \pi_2, \pi_3)$

$\bar{\pi}_1\ [\alpha] = [\bar{\pi}_{11}(\alpha), \bar{\pi}_{12}(\alpha)]$, $\bar{\pi}_2\ [\alpha] = [\bar{\pi}_{21}(\alpha), \bar{\pi}_{22}(\alpha)]$ and $\bar{\pi}_3\ [\alpha] = [\bar{\pi}_{31}(\alpha), \bar{\pi}_{22}(\alpha)]$.

$\pi_{11}(\alpha) = \min\left\{w_1 \middle| \left(\textit{all the elements the matrix } \bar{P}\right) \in \textit{Dom}[\alpha]\ \right\} = \bar{w}_{11}(\alpha)$.

$$\pi_{12}(\alpha) = \max\left\{w_1 | \left(all\ the\ elements\ the\ matrix\ \bar{P}\right) \in Dom[\alpha]\ \right\} = \ \bar{w}_{12}(\alpha).$$

$$\pi_{21}(\alpha) = \min\left\{w_2 | \left(all\ the\ elements\ the\ matrix\ \bar{P}\right) \in Dom[\alpha]\ \right\} = \ \bar{w}_{21}(\alpha).$$

$$\pi_{22}(\alpha) = \max\left\{w_2 | \left(all\ the\ elements\ the\ matrix\ \bar{P}\right) \in Dom[\alpha]\ \right\} = \ \bar{w}_{22}(\alpha).$$

$$\pi_{31}(\alpha) = \min\left\{w_3 | \left(all\ the\ elements\ the\ matrix\ \bar{P}\right) \in Dom[\alpha]\ \right\} = \ \bar{w}_{31}(\alpha).$$

$$\pi_{32}(\alpha) = \max\left\{w_3 | \left(all\ the\ elements\ the\ matrix\ \bar{P}\right) \in Dom[\alpha]\ \right\} = \ \bar{w}_{32}(\alpha).$$

Therefore

$$\bar{\pi}_1\ [\alpha] = [\bar{w}_{11}(\alpha), \bar{w}_{12}(\alpha)],\ \bar{\pi}_2\ [\alpha] = [\bar{w}_{21}(\alpha), \bar{w}_{22}(\alpha)]\ \ and\ \bar{\pi}_3\ [\alpha] = [\bar{w}_{31}(\alpha), \bar{w}_{22}(\alpha)]$$

For all $0 < \alpha < 1$.
hence show how $\pi_{11}(\alpha)$ can be derived using KKT conditions then all others:
hence show howthat what want to minimize

$$f\left(All\ P_{ij}\right) = -\frac{((p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(-p_{31}p_{22} + p_{21}p_{32}))}{(p_{22}p_{11} - p_{12}p_{21})^2}$$

Subject to

$$g_i\left(all\ P_{ij}\ \right) \leq 0,\ \ 1 \leq i \leq 18$$

As follows

$$g_1(P_{11}, \ldots\ldots, P_{33}) = P_{11} - P_{112}(\alpha) \leq 0$$

$$g_2(P_{11}, \ldots\ldots, P_{33}) = P_{111}(\alpha) - P_{11} \leq 0$$

$$g_3(P_{11}, \ldots\ldots, P_{33}) = P_{12} - P_{122}(\alpha) \leq 0$$

$$g_4(P_{11}, \ldots\ldots, P_{33}) = P_{112}(\alpha) - P_{12} \leq 0$$

$$g_5(P_{11}, \ldots\ldots, P_{33}) = P_{13} - P_{131}(\alpha) \leq 0$$

$$g_6(P_{11}, \ldots\ldots, P_{33}) = P_{131}(\alpha) - P_{13} \leq 0$$

$$g_7(P_{11}, \ldots\ldots, P_{33}) = P_{21} - P_{212}(\alpha) \leq 0$$

$$g_8(P_{11}, \ldots\ldots, P_{33}) = P_{211}(\alpha) - P_{21} \leq 0$$

$$g_9(P_{11}, \ldots\ldots, P_{33}) = P_{22} - P_{222}(\alpha) \leq 0$$

$$g_{10}(P_{11}, \ldots\ldots, P_{33}) = P_{221}(\alpha) - P_{22} \leq 0$$

$$g_{11}(P_{11}, \ldots\ldots, P_{33}) = P_{23} - P_{232}(\alpha) \leq 0$$

$$g_{12}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{231}\left(\alpha\right)-P_{23}\leq 0$$

$$g_{13}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{31}-P_{312}\left(\alpha\right)\leq 0$$

$$g_{14}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{311}\left(\alpha\right)-P_{31}\leq 0$$

$$g_{15}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{32}-P_{322}\left(\alpha\right)\leq 0$$

$$g_{16}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{321}\left(\alpha\right)-P_{32}\leq 0$$

$$g_{17}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{33}-P_{332}\left(\alpha\right)\leq 0$$

$$g_{18}\left(P_{11},\ldots\ldots.,P_{33}\right)=P_{331}\left(\alpha\right)-P_{33}\leq 0$$

The KKT conditions will be

$$\lambda=\left(\lambda_1,\ldots\ldots\ldots\ldots,\lambda_{18}\right)\leq 0$$

$$\nabla f\left(P_{11},\ldots\ldots.,P_{33}\right)-\lambda\nabla g\left(P_{11},\ldots\ldots.,P_{33}\right)=0$$

$$\lambda_i g_i\left(P_{11},\ldots\ldots.,P_{33}\right)=0,\ \ i=1,2,3,4,5,6,7,8,9$$

$$g\left(P_{11},\ldots\ldots.,P_{33}\ \right)=\begin{pmatrix} g_1\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \\ g_2\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \\ \vdots \\ g_9\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \\ g_{10}\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \\ g_{11}\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \\ \vdots \\ g_{17}\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \\ g_{18}\ (\ P_{11},\ldots\ldots.,P_{33}\ ) \end{pmatrix}\leq 0$$

Or

$$\lambda=\left(\lambda_1,\ldots\ldots\ldots\ldots,\lambda_{18}\right)\leq 0$$

$$\left(\frac{\partial f}{\partial P_{11}},\frac{\partial f}{\partial P_{12}},\ldots\ldots.,\frac{\partial f}{\partial P_{33}}\right)$$

$$-\left(\lambda_1,\ldots,\lambda_{18}\right)\begin{pmatrix} \frac{\partial g_1}{\partial P_{11}} & \frac{\partial g_1}{\partial P_{12}} & \cdots & \frac{\partial g_1}{\partial P_{32}} & \frac{\partial g_1}{\partial P_{33}} \\ \frac{\partial g_2}{\partial P_{11}} & \frac{\partial g_2}{\partial P_{12}} & & \frac{\partial g_2}{\partial P_{32}} & \frac{\partial g_2}{\partial P_{33}} \\ \vdots & & \ddots & & \vdots \\ \frac{\partial g_{17}}{\partial P_{11}} & \frac{\partial g_{17}}{\partial P_{12}} & \cdots & \frac{\partial g_{17}}{\partial P_{32}} & \frac{\partial g_{17}}{\partial P_{33}} \\ \frac{\partial g_{18}}{\partial P_{11}} & \frac{\partial g_{18}}{\partial P_{12}} & & \frac{\partial g_{18}}{\partial P_{32}} & \frac{\partial g_{18}}{\partial P_{33}} \end{pmatrix}=O$$

$$\Rightarrow \frac{\partial f}{\partial_{P_{11}}} - \lambda_1 + \lambda_2 = 0,\ \frac{\partial f}{\partial_{P_{12}}} - \lambda_3 + \lambda_4 = 0,\ \frac{\partial f}{\partial_{P_{13}}} - \lambda_5 + \lambda_6 = 0$$

$$\frac{\partial f}{\partial_{P_{21}}} - \lambda_7 + \lambda_8 = 0,\ \frac{\partial f}{\partial_{P_{22}}} - \lambda_9 + \lambda_{10} = 0,\ \frac{\partial f}{\partial_{P_{23}}} - \lambda_{11} + \lambda_{12} = 0$$

$$\frac{\partial f}{\partial_{P_{31}}} - \lambda_{13} + \lambda_{14} = 0,\ \frac{\partial f}{\partial_{P_{32}}} - \lambda_{15} + \lambda_{16} = 0,\ \frac{\partial f}{\partial_{P_{33}}} - \lambda_{17} + \lambda_{18} = 0$$

$$\lambda_1 (P_{11} - P_{112}(\alpha)) = 0, \quad \lambda_2 (P_{111}(\alpha) - P_{11}) = 0, \quad \lambda_3 (P_{12} - P_{122}(\alpha)) = 0$$

$$\lambda_4 (P_{112}(\alpha) - P_{12}) = 0, \quad \lambda_5 (P_{13} - P_{131}(\alpha)) = 0, \quad \lambda_6 (P_{131}(\alpha) - P_{13}) = 0$$

$$\lambda_7 (P_{21} - P_{212}(\alpha)) = 0, \quad \lambda_8 (P_{211}(\alpha) - P_{21}) = 0, \quad \lambda_9 (P_{22} - P_{222}(\alpha)) = 0$$

$$\lambda_{10} (P_{221}(\alpha) - P_{22}) = 0, \quad \lambda_{11} (P_{23} - P_{232}(\alpha)) = 0, \quad \lambda_{12} (P_{231}(\alpha) - P_{23}) = 0$$

$$\lambda_{13} (P_{31} - P_{312}(\alpha)) = 0, \quad \lambda_{14} (P_{311}(\alpha) - P_{31}) = 0, \quad \lambda_{15} (P_{32} - P_{322}(\alpha)) = 0$$

$$\lambda_{16} (P_{321}(\alpha) - P_{32}) = 0, \quad \lambda_{17} (P_{33} - P_{332}(\alpha)) = 0, \quad \lambda_{18} (P_{331}(\alpha) - P_{33}) = 0$$

Here analyzing the possible cases that lambda (λ) values well:

If $\lambda_2 = 0$ then $\frac{\partial f}{\partial_{P_{11}}} = \lambda_1 \le 0$ which is not possible since $\frac{\partial f}{\partial_{P_{11}}} =, \ldots,$ so $P_{11} = P_{112}(\alpha)$, while $\lambda_1 = 0$ then $\lambda_2 = -\frac{\partial f}{\partial_{P_{11}}}$.

If $\lambda_3 = 0$ then $\frac{\partial f}{\partial p_{12}} = -\lambda_4 \le 0$ which is not possible since $\frac{\partial f}{\partial p_{12}} =, \ldots,$ so $P_{12} = P_{122}(\alpha)$, while $\lambda_4 = 0$ then $\lambda_3 = \frac{\partial f}{\partial_{P_{11}}}$.

If $\lambda_6 = 0$ then $\frac{\partial f}{\partial_{P_{13}}} = \lambda_5 \le 0$ which is not possible since $\frac{\partial f}{\partial_{P_{13}}} =, \ldots,$ so $P_{13} = P_{132}(\alpha)$, while $\lambda_5 = 0$ then $\lambda_6 = -\frac{\partial f}{\partial_{P_{13}}}$.

If $\lambda_7 = 0$ then $\frac{\partial f}{\partial p_{21}} = -\lambda_8 \le 0$ which is not possible since $\frac{\partial f}{\partial p_{21}} =, \ldots,$ so $P_{21} = P_{212}(\alpha)$, while $\lambda_8 = 0$ then $\lambda_7 = \frac{\partial f}{\partial_{P_{21}}}$.

If $\lambda_{10} = 0$ then $\frac{\partial f}{\partial_{P_{22}}} = \lambda_9 \le 0$ which is not possible since $\frac{\partial f}{\partial_{P_{22}}} =, \ldots,$ so $P_{22} = P_{222}(\alpha)$, while $\lambda_9 = 0$ then $\lambda_{10} = -\frac{\partial f}{\partial_{P_{22}}}$.

If $\lambda_{11} = 0$ then $\frac{\partial f}{\partial p_{23}} = -\lambda_{12} \le 0$ which is not possible since $\frac{\partial f}{\partial p_{23}} =, \ldots,$ so $P_{23} = P_{232}(\alpha)$, while $\lambda_{12} = 0$ then $\lambda_{11} = \frac{\partial f}{\partial_{P_{23}}}$.

If $\lambda_{14} = 0$ then $\frac{\partial f}{\partial_{P_{31}}} = \lambda_{13} \le 0$ which is not possible since $\frac{\partial f}{\partial_{P_{31}}} =, \ldots,$ so $P_{31} = P_{312}(\alpha)$, while $\lambda_{13} = 0$ then $\lambda_{14} = -\frac{\partial f}{\partial_{P_{31}}}$.

If $\lambda_{15} = 0$ then $\frac{\partial f}{\partial p_{32}} = -\lambda_{16} \le 0$ which is not possible since $\frac{\partial f}{\partial p_{32}} =, \ldots,$ so $P_{32} = P_{322}(\alpha)$, while $\lambda_{16} = 0$ then $\lambda_{15} = \frac{\partial f}{\partial_{P_{32}}}$.

If $\lambda_{18} = 0$ then $\frac{\partial f}{\partial_{P_{33}}} = \lambda_{17} \le 0$ which is not possible since $\frac{\partial f}{\partial_{P_{33}}} =, \ldots,$ so $P_{33} = P_{332}(\alpha)$, while $\lambda_{17} = 0$ then $\lambda_{18} = -\frac{\partial f}{\partial_{P_{33}}}$.

Since all the conditions have been met, the objective is given as follows:

$$min\left[-\frac{((p_{33}p_{11}p_{22} - p_{33}p_{12}p_{21} - p_{13}p_{31}p_{22} - p_{23}p_{11}p_{32} + p_{23}p_{12}p_{31} + p_{13}p_{21}p_{32})(-p_{31}p_{22} + p_{21}p_{32}))}{(p_{22}p_{11} - p_{12}p_{21})^2}\right]$$

is equal to their min $f = \left[-\frac{(p_{33(\alpha)}p_{11(\alpha)}p_{22(\alpha)} - p_{33(\alpha)}p_{12(\alpha)}p_{21(\alpha)} - p_{13(\alpha)}p_{31(\alpha)}p_{22(\alpha)} - p_{23(\alpha)}p_{11(\alpha)}p_{32(\alpha)} + p_{23}p_{12}(\alpha)p_{31} + p_{13}(\alpha)p_{21}(\alpha)p_{32})(-p_{31(\alpha)}p_{22(\alpha)} + p_{21(\alpha)}p_{32(\alpha)})}{(p_{22(\alpha)}p_{11(\alpha)} - p_{12(\alpha)}p_{21}(\alpha))^2} \right]$

Where

$$P_{11} \in [P_{111}(\alpha), P_{112}(\alpha)], \quad P_{12} \in [P_{121}(\alpha), P_{122}(\alpha)], \quad P_{13} \in [P_{131}(\alpha), P_{132}(\alpha)]$$

$$P_{21} \in [P_{211}(\alpha), P_{212}(\alpha)], \quad P_{22} \in [P_{221}(\alpha), P_{222}(\alpha)], \quad P_{23} \in [P_{231}(\alpha), P_{232}(\alpha)]$$

$$P_{31} \in [P_{311}(\alpha), P_{312}(\alpha)], \quad P_{32} \in [P_{321}(\alpha), P_{322}(\alpha)], \quad P_{33} \in [P_{331}(\alpha), P_{332}(\alpha)]$$

Similarly for all cases of $\pi_{12}(\alpha)$, $\pi_{21}(\alpha)$, $\pi_{22}(\alpha)$, $\pi_{31}(\alpha)$ and $\pi_{32}(\alpha)$.

Key Insight: The general KKT framework which ensures that the optimization takes into account both fuzzy uncertainties, i.e., their corresponding intervals and constraints on Markov chains. The approach can readily account for the fuzziness, so that system steady-state behaviour can be described in presence of imprecisiness.

Results and discussion of the practical application

In this study, we use Markov Chain to define and analysis the time series in exchange rate of Iraqi dinar. In particular, the study considers three regimes in which the exchange rate can vary: a decline (1), stability (2) and increase (3). To describe the switches between these discrete states, a Markov transition matrix is derived using historical data and uncertainties in the system are processed by fuzzy logic to include impreciseness in both of the system model and data. A Markov Chain as a system' that transforms from one state to another and which is observed over time. It depends only on the current state and nothing else, none of the chain of events leading to that state. Those are three states in this situation are:

1. Collapse (1): A fall in the value of currency.
2. Stability (2): The exchange rate is steady.
3. Rise (3): Appreciation of the exchange rate.

The Markov matrix is constructed by studying the historical exchange rate data to ascertain the likelihood of going from one state to another. For instance, if the exchange rate was in a state of decline (1) on one day, the model calculates the probability that the rate will either stay in decline, move to a stable state, or rise on the next day. The transition matrix is constructed as follows:

$$U = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

be a crisp transition matrix. In this matrix, the values represent the probabilities of transitioning from one state to another. For example:

- The probability of staying in state 1 (decline) is 0.5.
- The probability of moving from state 1 (decline) to state 2 (stability) is 0.25.
- The probability of moving from state 1 (decline) to state 3 (rise) is 0.25.

The sum of the values in each row is 1, as the system must transition to one of the possible states. As mentioned above, have uncertainties in all the entries, so the model uncertainties by fuzzy numbers between 0 and 1. Since fuzzy number is a mathematical representation that allows for vagueness or imprecision in a value. Instead of assigning a fixed probability (e.g., 0.25), a fuzzy number assigns a range of possible values. For example, instead of saying that the probability of moving from state 1 to state 2 is exactly 0.25, we say it could be between 0.15 and 0.35, with a central tendency around 0.25.

The transition matrix is therefore modified to include fuzzy numbers, as shown below for the transition probabilities between states:

$$\bar{P}_{11} = (0.4\,|0.5|\,0.6)\,,\bar{P}_{12} = (0.15\,|0.25|\,0.35)\,,\bar{P}_{13} = (0.15\,|0.25|\,0.35)\,,$$

$$\bar{P}_{21} = (0.15\,|0.25|\,0.35)\,,\bar{P}_{22} = (0.4\,|0.5|\,0.6)\,,\bar{P}_{23} = (0.15\,|0.25|\,0.35)\,,$$

$$\bar{P}_{31} = (0.15\,|0.25|\,0.35)\,,\bar{P}_{32} = (0.15\,|0.25|\,0.35)\,,\bar{P}_{33} = (0.4\,|0.5|\,0.6)\,,$$

$\bar{P}_{11}$ means the probability of staying in state 1 (decline) is between 0.4 and 0.6, with a central value of 0.5. $\bar{P}_{12}$ means the probability of transitioning from state 1 (decline) to state 2 (stability) is between 0.15 and 0.35. The same type of fuzzy numbers is applied to other transitions as well.

Thus, instead of a crisp matrix, we now have a fuzzy matrix where each entry is a fuzzy number representing the uncertainty about the transition probabilities. Now the $\alpha - cut$ Fuzzy representation giver by,

$$\bar{P}_{11}\,[\alpha] = (0.4 + 0.1\alpha, 0.6 - 0.1\alpha)\,,\bar{P}_{12} = (0.15 + 0.1\alpha, 0.35 - 0.1\alpha)\,,$$

$$\bar{P}_{13} = (0.15 + 0.1\alpha, 0.35 - 0.1\alpha)\,,$$

$$\bar{P}_{21}\,[\alpha] = (0.15 + 0.1\alpha, 0.35 - 0.1\alpha)\,,\bar{P}_{22} = (0.4 + 0.1\alpha, 0.6 - 0.1\alpha)\,,$$

$$\bar{P}_{23} = (0.15 + 0.1\alpha, 0.35 - 0.1\alpha)\,,$$

$$\bar{P}_{31}\,[\alpha] = (0.15 + 0.1\alpha, 0.35 - 0.1\alpha)\,,\bar{P}_{32} = (0.15 + 0.1\alpha, 0.35 - 0.1\alpha)\,,$$

$$\bar{P}_{33} = (0.4 + 0.1\alpha, 0.6 - 0.1\alpha)\,,$$

Results After Several Iterations: To analyze how the system evolves over time, the fuzzy transition matrix is used to calculate the probability distribution of the system after several iterations. This distribution is represented by the vector $\pi(p_i)$, which provides the probability of being in each state after a certain number of transitions. Using constrained matrix $P^n \to \pi$ where each row in π is $\pi = (\pi_1, \pi_2, \pi_3)$, getting

$$\bar{\pi}_1\ [\alpha] = [0.233 + 0.1\alpha, 0.433 - 0.1\alpha]\,,$$

$$\bar{\pi}_2\ [\alpha] = [0.233 + 0.1\alpha, 0.433 - 0.1\alpha]\ \ and$$

$$\bar{\pi}_3\ [\alpha] = [0.233 + 0.1\alpha, 0.433 - 0.1\alpha]\,,$$

- The probabilities for all α −cuts of states are given as intervals $0 < \alpha < 1$, reflecting the vagueness due to the uncertainty on the system's evolution. For instance: when $\alpha = 0$, the values are $\bar{\pi}_1\ [0]\pi 1$=[0.233,0.433], and "for being in state 1 (decline)" shows the probability interval.
- For α =1, we have $\bar{\pi}_1\ [1] = \pi 1 =$ [0.333, 0.333], indicating that the system converges to an equilibrium state.

Also, the value of $\bar{\pi}_2$ *and* $\bar{\pi}_2$ will be equivalent to the same value(res) which is a triangular fuzzy number. And so $\bar{\pi}_1 = (0.233, 0.333, 0.433)$ is a triangle fuzzy number. These outcomes are described by triangular fuzzy numbers, and also used to express the interval of the values for probability in each state.

On the applications side, the work suggests that institutions success is highly contingent on optimal control of co-operations, optimal achievement and regular monitoring of plan implementation. After 2022, the Central Bank of Iraq set the dollar exchange rate and regulations to ensure its stability. Such policy measures require the study of exchange rate predictability and the computation of possible scenarios, highlighting the applicability of optimization techniques in economic models.

Conclusion

This paper has showed that finite Markov chains with fuzzy logic consider uncertainties and imprecision on the transition probabilities. Based on the Karush-Kuhn-Tucker (KKT) optimization method, optimum levels of a fuzzy transition matrix are found to satisfy the power and steady-state requirements. This concept also gave a sound way of modeling and reasoning under the condition of uncertainty for systems. Incorporating fuzzy values in the transition matrix enabled an improved description of internal links in the state space, and uncertainty and variability in dynamics of system were made explicit. The α-cuts made it possible to characterize the boundaries of uncertainty and have a better understanding of the steady state behavior. KKT framework preserved the respect of the optimization to both Markov chain constraints and interval uncertainties in the fuzzy entries. This double agreement emphasizes the generality and accuracy of the method for complex systems. In addition, the application to an applied example like exchange rate dynamics demonstrated its practical importance and applicability under real-world circumstances. In deduction, this work contributes to further develop theoretical aspects of fuzzy Markov chains and optimization that can be applied in a good framework for analyzing systems with uncertainty. In the future the approach can be extended to larger state spaces or additional optimization objectives to expand its applicability.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Mosul.

Authors' contribution statement

M.S.J. and Z.Y.A. designed the study. M.S.J.and R.Z.A expressed KKT Optimization Technique. M.S.J., Z.Y.A, and R performed KKT on a Finite Fuzzy Markov Chain. Z.Y.A and S.J.H analyzed the data. Results and Discussion of the Practical Application by all authors. S.J.H, and R.Z.A wrote the paper with input from all authors.

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استخدام طريقة Karush-Kuhn-Tucher الامثلية على سلسلة ماركوف المضبضبة المنتهية

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الخلاصة

تعتبر دراسة استقرارية سلاسل ماركوف مسألة مهمة حسب تطبيقها في الهندسة والأنظمة الديناميكية ومسائل الطاقة بالإضافة إلى مسائل اتخاذ القرار. تناول البحث طريقة الوصول إلى استقرار سلاسل ماركوف بشكل عام والسلاسل المضبضبة المنتهية بشكل خاص. تم استخدام طريقة KKT الامثلية لهذا الغرض، من خلال منظورين: حيث يتم تمييزه من خلال طريقة KKT الامثلية، أولاً يتم استبدال عدم اليقين المرتبط باحتمالية الانتقال الصريحة في مصفوفة الانتقال بقيم غامضة. في هذه الفرضية، نجحنا في إثبات الطبيعة الفريدة لحدود القوى لمصفوفة 3 × 3 وفقًا لمعايير محددة. بالإضافة إلى ذلك، تعمل مصفوفة الانتقال بمثابة تمثيل لعلاقة ضبابية داخل مساحة الحالة التي تكون محدودة بطبيعتها .تضمن الجهد الثاني نمذجة قضية تحديد متجه الاعتمادية باستخدام استراتيجية الامثلية المقترحة. تم دمج المكونات الإضافية بنجاح في معايير مصفوفات الانتقال الضبابية 3 × 3 عبر تقنيات تحسين KKT. أدى هذا إلى تعزيز قابلية التكيف للسلوك الخاص بسلاسل الماركوفية المضبضبة، حيث حدد النموذج أن القدرة تشكل مشكلة الامثلية. تمكنا من إضافة بعض الشروط إلى مصفوفات الانتقال المضبضبة 3 × 3 للحصول على سلوك واضح باستخدام طريقة تحسين KKT. في الجانب التطبيقي جمع البيانات والتي تتمثل بسعر صرف الدينار العراقي مقابل الدولار الأمريكي الصادر عن البنك المركزي العراقي للفترة من 1 تشرين الأول 2023 ولغاية 1 كانون الأول 2023. وبعد تحليل البيانات للمثال التطبيقي استنتجنا أن قدرة النموذج كانت على شكل مسالة امثلية مقيدة، حيث حلها هي الحصول على النتائج لتحقيق هدف البحث، وتم تقديم التوصيات في ضوء الاستنتاجات التي توصل إليها هذا البحث.

الكلمات المفتاحية: الامثلية المقيدة، العدد الضبابي، سلسلة ماركوف المنتهية، طريقة KKT الامثلية، النمذجة الرياضية.