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RESEARCH ARTICLE

On Graded Strongly J_{gr}^{Soc}-2-Absorbing Submodules

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ABSTRACT

Let $\mathfrak{B}$ be a graded commutative ring with unity, and let $\mathfrak{W}$ be a graded unital $\mathfrak{B}$-module. This study introduces and develops the concept of graded strongly J_{gr}^{Soc}-2-absorbing submodules, a natural extension of graded J_{gr}-2-absorbing submodules within the framework of graded module theory. The motivation for this generalization stems from the need to better capture the interplay between graded algebraic structures and the behaviors of certain radicals and socles under graded operations. A properly graded submodule N of $\mathfrak{W}$ is defined as a graded strongly J_{gr}^{Soc}-2-absorbing submodule if, for all $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$, the containment $buc \in N$ implies that at least one of the following conditions holds: $bc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$. Several fundamental properties of these submodules are established, along with characterizations that distinguish them from related graded structures. Moreover, the investigation reveals meaningful connections between these submodules and the graded socle and graded Jacobson radical of the module, offering new insights into their algebraic significance.

Keywords: Graded strongly J_{gr}^{Soc}-2-absorbing submodule, Graded J_{gr}-2-absorbing submodule, Graded J^{gr}-2-absorbing submodule, Graded 2-absorbing submodule, Graded prime submodules

Introduction

This article suggests that $\mathfrak{B}$ is a graded commutative ring with unity and $\mathfrak{W}$ is a graded unital $\mathfrak{B}$-module. First, there are some essential basics that must be clarified. A ring $\mathfrak{B}$ is defined that a graded ring if there exist $\{\mathfrak{B}_g\}_{g \in G}$ additive subgroups of $\mathfrak{B}$ such that $\mathfrak{B} = \oplus_{a \in G} \mathfrak{B}_a$ and $\mathfrak{B}_a \mathfrak{B}_b \subseteq \mathfrak{B}_{ab}$ for $a, b \in G$. If $x \in \mathfrak{B}_a$ then x is called a homogeneous element of degree a. The identity component of $\mathfrak{B}$, denoted by $\mathfrak{B}_e$ is a subring of $\mathfrak{B}$ and $1 \in \mathfrak{B}_e$. The set of all homogeneous elements of $\mathfrak{B}$, denoted by $h(\mathfrak{B}) = \cup_{a \in G} \mathfrak{B}_a$. An ideal E is called a graded ideal if $E = \oplus_{a \in G} E_a$, where E_a is called a-component of E.[1] Similarly, a module $\mathfrak{W}$ is defined that a graded module if there exist $\{\mathfrak{W}_g\}_{g \in G}$ additive subgroups of $\mathfrak{W}$, with $\mathfrak{W} = \oplus_{a \in G} \mathfrak{W}_a$ and $\mathfrak{B}_a \mathfrak{W}_b \subseteq \mathfrak{W}_{ab}$ for $a, b \in G$. The set of all homogeneous elements of $\mathfrak{W}$ denoted by $h(\mathfrak{W}) = \cup_{a \in G} \mathfrak{W}_a$. A submodule N of $\mathfrak{W}$ is defined that a graded submodule if $N = \oplus_{a \in G} N_a$ where N_a is a-component of N.[1] N is referred to as a graded submodule (respectively, a proper graded) submodule of $\mathfrak{W}$, by $N \leq_G^{sub} \mathfrak{W}$, (respectively, $N <_G^{sub} \mathfrak{W}$). In a similar way, E is referred to as a graded ideal (respectively, a proper graded ideal) of $\mathfrak{B}$ by $E \leq_G^{id} \mathfrak{B}$, (respectively, $E <_G^{id} \mathfrak{B}$). A nonzero graded submodule N of $\mathfrak{W}$ is defined as graded essential (briefly, Gr-essential) submodule if, for every nonzero graded submodule L of $\mathfrak{W}$, then $N \cap L \neq \{0\}$, see.[1] A submodule N is a graded maximal (Gr-maximal) submodule of $\mathfrak{W}$ if $N \neq \mathfrak{W}$ and $\exists D \leq_G^{sub} \mathfrak{W}$ such that $N \subseteq D \subseteq \mathfrak{W}$, then $D = \mathfrak{W}$ or $D = N$. For further details on graded rings and modules, readers may consult.[2–4]

The graded modules theory holds a prominent place, and its applications appear in several fields, for instance, in algebraic geometry, number theory, and homology. Also, it is used to account for some Hilbert functions.[5] The concept of graded 2-absorbing submodules over a graded commutative ring has recently

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attracted many authors, and several generalizations emerged from it. The authors in,[6] introduced the concept of graded J_{gr}-2-absorbing and graded weakly J_{gr}-2-absorbing submodules as generalizations of graded 2-absorbing submodules. Let $N <_G^{sub} \mathfrak{W}$, then N is called a graded J_{gr}-2-absorbing (resp. graded weakly-J_{gr}-2-absorbing) submodule if, whenever $buc \in N$ (resp. $0 \neq buc \in N$) where $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$, then $bc \in N + J_{gr}(\mathfrak{W})$, $uc \in N + J_{gr}(\mathfrak{W})$, or $bu \in (N + J_{gr}(\mathfrak{W}):_{\mathfrak{B}}\mathfrak{W})$, where $J_{gr}(\mathfrak{W}) = \cap\ \{L : L$ is a Gr-maximal submodule of $\mathfrak{W}\}$ is the graded Jacobson radical of $\mathfrak{W}$. The symbols $Soc^{gr}(\mathfrak{W}) = \cap\ \{L : L$ is a Gr-essential submodule of $\mathfrak{W}\}$ is the graded Socle of $\mathfrak{W}$. This paper introduces and investigates the notion of graded strongly J_{gr}^{Soc}-2-absorbing submodules, proposed as a natural generalization of the concept of graded J_{gr}-2-absorbing submodules. The study establishes several results that elucidate the structural properties and intrinsic behavior of these submodules, with particular attention given to their homogeneous components.

Results and discussion

Definition 1: A proper graded submodule N of $\mathfrak{W}$ is called a graded strongly J_{gr}^{Soc}-2-absorbing (Gr^{St}-J_{gr}^{Soc}-2-absorbing) submodule of $\mathfrak{W}$ if, whenever $buc \in N$ where $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$, then $bc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. A graded ideal D is called a Gr^{St}-J_{gr}^{Soc}-2-absorbing the ideal of $\mathfrak{B}$ if it is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule of a graded $\mathfrak{B}$-module $\mathfrak{B}$.

Every Gr-2-absorbing submodule is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, but the converse does not have to be true in general, see the next example.

Example 1: *Let $\mathfrak{B} = \mathbb{Z}$ be a G-graded ring with $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{B}_1 = \langle 0\rangle$ where $G = \mathbb{Z}_2$. Let $\mathfrak{W} = \mathbb{Z}_{36}$ be a graded $\mathfrak{B}$-module with $\mathfrak{W}_0 = \mathbb{Z}_{36}$ and $\mathfrak{W}_1 = \langle\bar{0}\rangle$. Take $N = \langle\overline{12}\rangle$. From the definitions, $Soc^{gr}(\mathfrak{W}) = \mathbb{Z}_{36} \cap \langle\bar{6}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{2}\rangle = \langle\bar{6}\rangle$, $J_{gr}(\mathfrak{W}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$, and N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule. But, N is not Gr-2-absorbing submodule, since there exists $2, 3 \in h(\mathfrak{B})$, and $\bar{2} \in h(\mathfrak{W})$ with $2 \cdot 3 \cdot \bar{2} = \overline{12} \in \langle\overline{12}\rangle = N$, whilst $2 \cdot \bar{2} \notin \langle\overline{12}\rangle$, $3 \cdot \bar{2} = \bar{6} \notin \langle\overline{12}\rangle$ and $2 \cdot 3 \notin (\langle\overline{12}\rangle:_{\mathfrak{B}}\mathfrak{W}) = 12\mathbb{Z}$.*

Every Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule is a Gr-J_{gr}-2-absorbing submodule, but the converse is not always true. To elaborate, see the next example.

Example 2: *Let $\mathfrak{B} = \mathbb{Z}$ be a G-graded ring with $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{B}_1 = \langle 0\rangle$ where $G = \mathbb{Z}_2$. Let $\mathfrak{W} = \mathbb{Z}_{48}$ be a graded $\mathfrak{B}$-module with $\mathfrak{W}_0 = \mathfrak{W}$ and $\mathfrak{W}_1 = \langle\bar{0}\rangle$. Take $N = \langle\overline{24}\rangle$, then N is a Gr-$J_{gr}$-2-absorbing submodule of $\mathfrak{W}$ where $Soc^{gr}(\mathfrak{W}) = \mathfrak{W} \cap \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle = \langle\bar{8}\rangle$ and $J_{gr}(\mathfrak{W}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. But N is not Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, since there exists $2, 4 \in h(\mathfrak{B})$ and $\bar{3} \in h(\mathfrak{W})$ with $2 \cdot 4 \cdot \bar{3} = \overline{24} \in N$ whilst $2 \cdot \bar{3} = \bar{6} \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = N$, $4 \cdot \bar{3} = \overline{12} \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = N$, and $2 \cdot 4 = \bar{8} \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W}) = 24\mathbb{Z}$*

The intersection of two *Gr^{St}-J_{gr}^{Soc}*-2-absorbing submodule is not always a *Gr^{St}-J_{gr}^{Soc}*-2-absorbing submodule. To elaborate, see the next example.

Example 3: *Let $\mathfrak{B} = \mathbb{Z}$ be a G-graded ring with $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{B}_1 = \langle 0\rangle$ where $G = \mathbb{Z}_2$. Let $\mathfrak{W} = \mathbb{Z}_{60}$ be a graded $\mathfrak{B}$-module with $\mathfrak{W}_0 = \mathfrak{W}$ and $\mathfrak{W}_1 = \langle\bar{0}\rangle$. Take $N = \langle\bar{5}\rangle$ and $K = \langle\bar{6}\rangle$, then N and K are Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, by definition. Nevertheless, $N \cap K = \langle\overline{30}\rangle$ is not Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule where $J_{gr}(\mathfrak{W}) = \mathfrak{W} \cap \langle\bar{2}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{5}\rangle = \langle\overline{30}\rangle$ and $J_{gr}(\mathfrak{W}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$, since there exists $2, 3 \in h(\mathfrak{B})$ and $\bar{5} \in h(\mathfrak{W})$ with $2 \cdot 3 \cdot \bar{5} = \overline{30} \in N$ whilst $2 \cdot \bar{5} = \overline{10} \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = \langle\overline{30}\rangle + \langle\overline{30}\rangle = \langle\overline{30}\rangle$, $3 \cdot \bar{5} = \overline{15} \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = \langle\overline{30}\rangle$, and $2 \cdot 3 \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W}) = 30\mathbb{Z}$.*

Theorem 1: *Let N and K be two Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule of $\mathfrak{W}$ with $N \subseteq (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ or $K \subseteq (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Then $N \cap K$ is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule of $\mathfrak{W}$.*

Proof: Obviously, $N \cap L <_G^{sub} \mathfrak{W}$, since $N \cap K \subseteq K \subsetneq \mathfrak{W}$. Now, let $buc \in N \cap K$ where $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$. If $N \subseteq (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, then $buc \in N$ as $N \cap K \subseteq N$. But N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, thus $bc \in$

$N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu\mathfrak{W} \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) = (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ since $N \subseteq (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Thus, $bc \in (N \cap K) + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uc \in (N \cap K) + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in ((N \cap K) + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$ since $(J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) \subseteq (N \cap K) + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Similarly, for $K \subseteq (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Thus $N \cap K$ is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule of $\mathfrak{W}$.

The following theorems give us some characterizations of Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule.

Theorem 2: *Let* $N <_G^{sub} \mathfrak{W}$. *Then* N *is a* Gr^{St}-J_{gr}^{Soc}-*2-absorbing submodule if and only if* $(N :_{\mathfrak{W}} bu) \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{W}} b)) \cup (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{W}} u)$ *for each* $b, u \in h(\mathfrak{B})$ *with* $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$.

Proof: (⇒) Let $c \in (N :_{\mathfrak{W}} bu) \cap h(\mathfrak{W})$ and $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$, then $buc \in N$, but N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, so $bc \in N + (J_{gr}(\mathfrak{W})) \cap Soc^{gr}(\mathfrak{W}))$ or $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Thus either, $c \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{W}} b)$ or $c \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{W}} u)$. That is, $c \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{W}} b) \cup (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{W}} u)$.

(⇐) Let $buc \in N$ where $b, u \in h(\mathfrak{B})$, $c \in h(\mathfrak{W})$ and $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$, then $c \in (N :_{\mathfrak{W}} bu)$. By hypothesis, $c \in (N :_{\mathfrak{W}} bu) \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}) :_{\mathfrak{W}} b)) \cup (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}) :_{\mathfrak{W}} u)$. It follows that $bc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ or $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Hence, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule.

Theorem 3: *Let* $N <_G^{sub} \mathfrak{W}$ *and* $L = \oplus_{g \in G} L_g \leq_G^{sub} \mathfrak{W}$. *Then* N *is a* Gr^{St}-J_{gr}^{Soc}-*2-absorbing submodule if and only if for each* $b, u \in h(\mathfrak{B})$, *and* $g \in G$ *with* $buL_g \subseteq N$, *then* $bL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ *or* $uL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, *or* $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$.

Proof: (⇒) Let $buL_g \subseteq N$ where $g \in G$ and $b, u \in h(\mathfrak{B})$. Suppose, on the contrary, $bL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$, then there exist nonzero elements $q, p \in L_g$ such that $bq \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $up \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$.

It is claimed that $uq \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $bp \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. To prove this claim, since $buL_g \subseteq N$. It follows that $buq \in N$. But N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, $bq \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$. Hence $uq \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. As a similar way, since $bup \in N$, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, $up \notin N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$. Thus $bp \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$.

Now, since $buL_g \subseteq N$ and $q + p \in L_g$, then $bu(q + p) \in N$. But N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule and $bu \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$, So either $b(q + p) \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ or $u(q + p) \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. If $b(q + p) \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. That is, $bq + bp \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, but $bp \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, thus $bq \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, as a contradiction. Similarly, for $u(q + p) \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})$, $up \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, as $uq \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, which is a contradiction. Therefore, $bL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$, as needed.

(⇐) Let $buc \in N$ where $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$, then $bu\langle c \rangle \subseteq N$. Here, take $L = \langle c \rangle = \mathfrak{B}c = \oplus_{g \in G} \mathfrak{B}_g c$ a graded submodule of $\mathfrak{W}$ generated by c. So, for each $g \in G$, $L_g = \mathfrak{B}_g c$. In particular, $L_e = \mathfrak{B}_e c$. Since $bu\mathfrak{B}_e c \subseteq bu\langle c \rangle \subseteq N$, by hypothesis, $b\mathfrak{B}_e c \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $u\mathfrak{B}_e c \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$. But $1 \in \mathfrak{B}_e$, then $bc = b1c \in N + (J_{gr}(\mathfrak{W})) \cap Soc^{gr}(\mathfrak{W}))$, $uc = u1c \in N + (J_{gr}(\mathfrak{W})) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in (N + (J_{gr}(\mathfrak{W})) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$. Therefore, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodul.

Theorem 4: *Let* $N <_G^{sub} \mathfrak{W}$, $L = \oplus_{g \in G} L_g \leq_G^{sub} \mathfrak{W}$, $I = \oplus_{h \in G} I_h \leq_G^{id} \mathfrak{B}$, *and* $J = \oplus_{t \in G} J_t \leq_G^{id} \mathfrak{B}$. *Then* N *is a* Gr^{St}-J_{gr}^{Soc}-*2-absorbing submodule if and only if for each* $g, h, t \in G$ *with* $I_h J_t L_g \subseteq N$, $I_h L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ *or* $J_t L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ *or* $I_h J_t \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$.

Proof: (⇒) Let $I_h J_t L_g \subseteq N$ where $h, t, g \in G$. Suppose, on the contrary, $I_h L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $J_t L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and $I_h J_t \nsubseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$, then there exist nonzero elements $i, i' \in I_h$ and $j, j' \in J_t$ such that $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $jL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and $i'j' \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}) :_{\mathfrak{B}} \mathfrak{W}))$.

Step (1): Since $I_hJ_tL_g \subseteq N$, $ijL_g \subseteq N$. But N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $jL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, then:
$ij \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, by Theorem 3.

Step (2): Since $I_hJ_tL_g \subseteq N$, $i'j'L_g \subseteq N$, but N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule and $i'j' \notin (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}):_{\mathfrak{B}}\mathfrak{W}))$. So either $i'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ or $j'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, by Theorem 3.

Case (i): If $i'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $j'L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Since $I_hJ_tL_g \subseteq N$:

On one hand, $ij'L_g \subseteq N$, but $j'L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, then $ij' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, by Theorem 3.

On the other hand, $(i + i')j'L_g \subseteq N$. But $(i + i')L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ (because $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})))$, $j'L_g \nsubseteq N + (J_{gr}((\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, so by Theorem 3, $(i + i')j' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. That is, $ij' + i'j' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, but $ij' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. Thus $i'j' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, as a contradiction.

Case (ii): If $i'L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $j'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Since $i'jL_g \subseteq N$ and $i'(j + j')L_g \subseteq N$. In a manner similar to case (i), $i'j' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, as a contradiction.

Case (iii): If $i'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $j'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. Since $j'L_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $jL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $(j + j')L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$.

Since $I_hJ_tL_g \subseteq N$, then $i(j + j')L_g \subseteq N$, but N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule of $\mathfrak{W}$, $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, and $(j + j')L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, so by Theorem 3, $i(j + j') \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. That is, $ij + ij' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W}))$. But $ij \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$ by (*), thus $ij' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$.

Also, since $I_hJ_tL_g \subseteq N$, then $(i + i')jL_g \subseteq N$. But N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule and $(i + i')L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ (because $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})))$ and $jL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, thus $(i + i')j \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, by Theorem 3. Since $ij \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, then $i'j \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$.

Finally, since $I_hJ_tL_g \subseteq N$ then $(i + i')(j + j')L_g \subseteq N$. But $(i + i')L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ and $(j + j')L_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ as $iL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $jL_g \nsubseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. So, by Theorem 3, $(i + i')(j + j') \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. That is, $ij + ij' + i'j + i'j' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. Since $ij, ij', i'j \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, thus $i'j' \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, as a contradiction.

Therefore, from the three cases if for each $g, h, t \in G$, with $I_hJ_tL_g \subseteq N$, then $I_hL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $J_tL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $I_hJ_t \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$.

(⇐) Let $buc \in N$ where $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$. Then $bu\langle c\rangle \subseteq N$. Here, let $I = \langle b\rangle = \mathfrak{B}b = \oplus_{h\in G}\mathfrak{B}_hb$, $J = \langle u\rangle = \mathfrak{B}u = \oplus_{t\in G}\mathfrak{B}_tb$ be two graded ideals of $\mathfrak{B}$ generated by b, u respectively and $L = \langle c\rangle = \mathfrak{B}c = \oplus_{g\in G}\mathfrak{B}_gc$ be a graded submodule of $\mathfrak{W}$ generated by c. So, for each $h, t, g \in G$, $I_h = \mathfrak{B}_hb$, $J_t = \mathfrak{B}_tu$, and $L_g = \mathfrak{B}_gc$. In particular, $I_e = \mathfrak{B}_eb$, $J_e = \mathfrak{B}_eu$, and $L_e = \mathfrak{B}_ec$. Since $\mathfrak{B}_eb\mathfrak{B}_eu\mathfrak{B}_ec \subseteq \langle b\rangle\langle u\rangle\langle c\rangle \subseteq N$, by hypothesis $\mathfrak{B}_eb\mathfrak{B}_ec \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $\mathfrak{B}_eu\mathfrak{B}_ec \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $\mathfrak{B}_eb\mathfrak{B}_eu \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. In particular, $bc \in N + (J_{gr}(\mathfrak{W})) \cap Soc^{gr}(\mathfrak{W})$, $uc \in N + (J_{gr}(\mathfrak{W})) \cap Soc^{gr}(\mathfrak{W})$, or $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$ as $1 \in \mathfrak{B}_e$. Therefore, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule.

Corollary 1: *Let* $N <_G^{sub} \mathfrak{W}$. *Then* N *is a* Gr^{St}-J_{gr}^{Soc}-2-*absorbing submodule if and only if for each* $c \in h(\mathfrak{W})$ *and* $\oplus_{h\in G}I_h = I \leq_G^{id} \mathfrak{B}$, $\oplus_{t\in G}J_t = J \leq_G^{id} \mathfrak{B}$, *with* $I_hJ_tc \subseteq N$, $I_hc \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ *or* $J_tc \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, *or* $I_hJ_t \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$.

Proof: (⇒) Let $I_hJ_tc \subseteq N$, where $c \in h(\mathfrak{W})$, then $I_hJ_t\langle c\rangle \subseteq N$. Here, take $L = \langle c\rangle = \mathfrak{B}c = \oplus_{g\in G}\mathfrak{B}_gc \leq_G^{sub} \mathfrak{W}$. So for each $g \in G$, $L_g = \mathfrak{B}_gc$. Particularly, $L_e = \mathfrak{B}_ec$. Since $I_hJ_t\mathfrak{B}_ec \subseteq I_hJ_t\langle c\rangle \subseteq N$ and N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, by Theorem 4, $I_h\mathfrak{B}_ec \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ or $J_t\mathfrak{B}_ec \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $I_hJ_t \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. In particular, $I_hc \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$ or $J_tc \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $I_hJ_t \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$, since $1 \in \mathfrak{B}_e$.

(⇐) Let $buc \in N$, where $b, u \in h(\mathfrak{B})$ and $c \in h(\mathfrak{W})$. Then $\langle b\rangle\langle u\rangle c \subseteq N$. Take $I = \langle b\rangle = \mathfrak{B}b = \oplus_{h\in G}\mathfrak{B}_hb$, $J = \langle u\rangle = \mathfrak{B}u = \oplus_{t\in G}\mathfrak{B}_tu$ are two graded ideals of $\mathfrak{B}$ generated by b, u respectively. So, for each $h, t \in G$, $I_h = \mathfrak{B}_hb$, $J_t = \mathfrak{B}_tu$. In particular, $I_e = \mathfrak{B}_eb$, $J_e = \mathfrak{B}_eu$. Since $\mathfrak{B}_eb\mathfrak{B}_euc \subseteq \langle b\rangle\langle u\rangle c \subseteq N$ thus by hypothesis, $\mathfrak{B}_ebc \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $\mathfrak{B}_euc \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $\mathfrak{B}_eb\mathfrak{B}_eu \subseteq (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}}\mathfrak{W})$. In particular, $bc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $bu \in (N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})) :_{\mathfrak{B}} \mathfrak{W})$ as $1 \in \mathfrak{B}_e$. Therefore, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule.

The next example explains that the residual of Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule is not always a Gr^{St}-J_{gr}^{Soc}-2-absorbing ideal. It is followed by the theorem that explains why the statement is true under specific conditions.

Example 4: *Let $\mathfrak{B} = \mathbb{Z}$ be a G-graded ring with $\mathfrak{B}_0 = \mathbb{Z}$ and $\mathfrak{B}_1 = \langle 0 \rangle$ where $G = \mathbb{Z}_2$. Let $\mathfrak{W} = \mathbb{Z}_{72}$ with $\mathfrak{W}_0 = \mathbb{Z}_{72}$ and $\mathfrak{W}_1 = \langle \bar{0} \rangle$. Take $N = \langle \overline{36} \rangle = \{\bar{0}, \overline{36}\}$. By definition, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule. Nevertheless, $(N:_{\mathfrak{B}}\mathfrak{W})$ is not Gr^{St}-J_{gr}^{Soc}-2-absorbing ideal, since there exists $3, 4 \in h(\mathfrak{B})$ with $3 \cdot 3 \cdot 4 = 36 \in (N:_{\mathfrak{B}}\mathfrak{W}) = 36\mathbb{Z}$, whilst $3 \cdot 4 \notin (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathbb{Z}) \cap Soc^{gr}(\mathbb{Z})) = 36\mathbb{Z} + \{0\} = 36\mathbb{Z}$, and $3 \cdot 3 = 9 \notin (36\mathbb{Z} + (J_{gr}(\mathbb{Z}) \cap Soc^{gr}(\mathbb{Z})) :_{\mathfrak{B}} \mathfrak{B}) = (36\mathbb{Z} :_{\mathfrak{B}} \mathbb{Z}) = 36\mathbb{Z}$.*

A graded $\mathfrak{B}$-module $\mathfrak{W}$ is called a graded finitely generated (Gr-finitely generated), if, $\mathfrak{W} = \mathfrak{B}c_{g_1} + \mathfrak{B}c_{g_2} + \cdots + \mathfrak{B}c_{g_n}$, for some $c_{g_1}, c_{g_2}, \ldots, c_{g_n} \in h(\mathfrak{W})$ see.[7] A graded $\mathfrak{B}$-module $\mathfrak{W}$ is called a graded faithful (Gr-faithful) if, whenever $r\mathfrak{W} = \{0\}$ where $r \in h(\mathfrak{B})$, then $r = 0$. In,[2] a graded multiplication (Gr-multiplication) $\mathfrak{B}$-module $\mathfrak{W}$ is defined as for any $N \leq_G^{sub} \mathfrak{W}$ there exists $E \leq_G^{id} \mathfrak{B}$ such that $N = I\mathfrak{W}$. It is obvious that, $N = (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W}$ for every $N \leq_G^{sub} \mathfrak{W}$ if, $\mathfrak{W}$ is a Gr-multiplication. Also, in,[8] a graded $\mathfrak{B}$-module $\mathfrak{W}$ is called a graded cancellation (Gr-cancellation) if, whenever $D\mathfrak{W} \subseteq E\mathfrak{W}$ where $D, E \leq_G^{id} \mathfrak{B}$, then $D \subseteq E$. For further details modules, readers may consult.[9]

Theorem 5: *Let $\mathfrak{W}$ be a finitely generated faithful Gr-multiplication $\mathfrak{B}$-module. Then N be a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule of $\mathfrak{W}$ if and only if $(N:_{\mathfrak{B}}\mathfrak{W})$ is a Gr^{St}-J_{gr}^{Soc}-2-absorbing the ideal of $\mathfrak{B}$.*

Proof: ($\Rightarrow$) Obviously, $(N:_{\mathfrak{B}}\mathfrak{W}) <_G^{id} \mathfrak{B}$, since if $(N:_{\mathfrak{B}}\mathfrak{W}) = \mathfrak{B}$, then $1 \in (N:_{\mathfrak{B}}\mathfrak{W})$ thus $\mathfrak{W} \subseteq N$, a contradiction. Now, let $I_hJ_tD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W})$ where $I = \oplus_{h\in G} I_h$, $J = \oplus_{t\in G} J_t$, and $D = \oplus_{g\in G} D_g$ are graded ideals of $\mathfrak{B}$, then $I_hJ_tD_g\mathfrak{W} \subseteq N$. Take $L_g = D_g\mathfrak{W}$, then $L = \oplus_{g\in G} L_g = \oplus_{g\in G} D_g\mathfrak{W} \leq_G^{sub} \mathfrak{W}$. Thus, $I_hJ_tL_g \subseteq N$, since N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, by Theorem 4, $I_hL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $J_tL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $I_hJ_t\mathfrak{W} \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$. But $J_{gr}(\mathfrak{W}) = J_{gr}(\mathfrak{B})\mathfrak{W}$, $Soc^{gr}(\mathfrak{W}) = Soc^{gr}(\mathfrak{B})\mathfrak{W}$ and $N = (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W}$, since $\mathfrak{W}$ is a finitely generated faithful Gr-multiplication. Thus $I_hD_g\mathfrak{W} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W} + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))\mathfrak{W}$, $J_tD_g\mathfrak{W} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W} + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))\mathfrak{W}$, or $I_hJ_t\mathfrak{W} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W} + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))\mathfrak{W}$. Since $\mathfrak{W}$ is a finitely generated faithful Gr-multiplication, by Theorem 2.180 in,[8] $\mathfrak{W}$ is a Gr-cancellation. So, $I_hD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$, $J_tD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$, or $I_hJ_t \subseteq N + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$. Hence $I_hD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$, $J_tD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$, or $I_hJ_t\mathfrak{B} \subseteq N + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$. Therefore, $(N:_{\mathfrak{B}}\mathfrak{W})$ is a Gr^{St}-J_{gr}^{Soc}-2-absorbing the ideal of $\mathfrak{B}$.

($\Leftarrow$) Obviously, $N <_G^{sub} \mathfrak{W}$, since if $N = \mathfrak{W}$, then $\mathfrak{B}\mathfrak{W} \subseteq N$. That is, $\mathfrak{B} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})$, a contradiction. Now, let $I_hJ_tL_g \subseteq N$ where $\oplus_{h\in G} I_h = I$, $\oplus_{t\in G} J_t = J \leq_G^{id} \mathfrak{B}$ and $\oplus_{g\in G} L_g = L \leq_G^{sub} \mathfrak{W}$. Since $\mathfrak{W}$ is a Gr-multiplication, then $\exists D \leq_G^{id} \mathfrak{B}$ such that $L = D\mathfrak{W}$. In particular, $L_g = D_g\mathfrak{W}$, for $g \in G$. Thus $I_hJ_tD_g\mathfrak{W} \subseteq N$, so $I_hJ_tD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W})$. Since $(N:_{\mathfrak{B}}\mathfrak{W})$ is a Gr^{St}-J_{gr}^{Soc}-2-absorbing ideal, by Theorem 4, $I_hD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$, $J_tD_g \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$, or $I_gJ_t\mathfrak{B} \subseteq (N:_{\mathfrak{B}}\mathfrak{W}) + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))$. It follows that $I_hD_g\mathfrak{W} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W} + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))\mathfrak{W} = N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $J_tD_g\mathfrak{W} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W} + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))\mathfrak{W} = N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $I_hJ_t\mathfrak{W} \subseteq (N:_{\mathfrak{B}}\mathfrak{W})\mathfrak{W} + (J_{gr}(\mathfrak{B}) \cap Soc^{gr}(\mathfrak{B}))\mathfrak{W} = N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, as $\mathfrak{W}$ is a finitely generated faithful Gr-multiplication $\mathfrak{B}$-module. Hence $I_hL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, $J_tL_g \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, or $I_hJ_t\mathfrak{W} \subseteq N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$, since $L_g = D_g\mathfrak{W}$. Therefore, N is a Gr^{St}-J_{gr}^{Soc}-2-absorbing submodule, by Theorem 4.

Conclusion

This work presents a detailed study of a new class of graded submodules, termed graded strongly $\mathbf{J_{gr}^{Soc}}$-2-absorbing submodules, as a natural extension of previously introduced graded $\mathbf{J_{gr}}$-2-absorbing submodules. By establishing several foundational characterizations and theorems, the paper clarifies how this concept interacts with existing graded structures such as the graded Jacobson radical and the graded socle. Through carefully constructed examples, it has been demonstrated that this class exhibits a richer behavior, properly containing the class of graded 2-absorbing submodules but not necessarily being closed under common operations such as intersection. Additionally, the conditions under which the residual of a submodule inherits the same structure were explored, with a complete characterization achieved in the setting of finitely generated faithful graded multiplication modules. These results deepen the understanding of the internal structure of graded modules and provide a useful framework for further generalizations. Future investigations may focus on exploring categorical properties, homological consequences, and potential applications in the study of graded homological algebra or algebraic geometry. In graded rings, does there exist a strongly J_{gr}^{Soc}-2-absorbing submodule which is not graded strongly J_{gr}^{Soc}-2-absorbing? At present, the answer is not known, and therefore this is left as an open problem for future investigation.

Authors' declaration

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- No animal studies are present in the manuscript.
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Author's contribution statement

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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في الحالة المتدرجة J_{gr}^{Soc} -بقوة2حول الزمر الجزئية الممتصة

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الملخص

لتكن $\mathfrak{B}$ حلقة تبادلية مُدرَّجة ذات عنصر وحدة، وليكن $\mathfrak{W}$ وحدة مُدرَّجة أحادية على $\mathfrak{B}$.تُقدِّم هذه الدراسة وتطوِّر مفهوم الوحدات الجزئية الممتصّة بقوة من النمط 2-J_{gr}^{Soc} في السياق المُدرَّج، وذلك بوصفه تعميمًا طبيعيًا للوحدات الجزئية الممتصّة 2-J_{gr} ضمن إطار نظرية الوحدات المُدرَّجة. وينبع الدافع وراء هذا التعميم من الحاجة إلى فهمٍ أعمق للتفاعل بين البُنى الجبرية المُدرَّجة وسلوك بعض الجذور والجذور الاجتماعية (السوسل) تحت العمليات المُدرَّجة. تُعرَّف الوحدة الجزئية المُدرَّجة الحقيقية Nمن $\mathfrak{W}$ بأنها وحدة جزئية مُدرَّجة ممتصّة بقوة من النمط 2-J_{gr}^{Soc} إذا كان لكل $b,u \in h(\mathfrak{B})$ ولكل $c \in h(\mathfrak{W})$، فإن الاحتواء $buc \in N$ يقتضي تحقق أحد الشروط الآتية على الأقل: $uc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$، $bc \in N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W}))$، $bu \in \left(N + (J_{gr}(\mathfrak{W}) \cap Soc^{gr}(\mathfrak{W})):_{\mathfrak{B}} \mathfrak{W}\right)$.
تم إثبات عدد من الخصائص الأساسية لهذه الوحدات الجزئية، إضافةً إلى تقديم توصيفات تميّزها عن البُنى المُدرَّجة ذات الصلة. كما تكشف الدراسة عن روابط ذات دلالة بين هذه الوحدات الجزئية وكلٍّ من السوسل المُدرَّج والجذر الجاكوبسوني المُدرَّج للوحدة، مما يقدّم رؤى جديدة حول أهميتها الجبرية.

الكلمات المفتاحية: الوحدة الجزئية المُدرَّجة الممتصّة بقوة 2-J_{gr}^{Soc}، الوحدة الجزئية المُدرَّجة الممتصّة 2-J_{gr}، الوحدة الجزئية المُدرَّجة الممتصّة 2-Soc^{gr}، الوحدة الجزئية المُدرَّجة الممتصّة-2، الوحدات الجزئية الأولية المُدرَّجة.