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Saad Naji Al-Azzawi
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq, Saadna_math@csw.uobaghdad.edu.iq

Mahmood A. Shamran
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq, mahmoodas_math@csw.uobaghdad.edu.iq

Aseel H. Ali
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq, aseel.h@csw.uobaghdad.edu.iq

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RESEARCH ARTICLE

New Statistical Distribution SMART1

Saad Naji Al-Azzawi, Mahmood A. Shamran *, Aseel H. Ali

Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq

ABSTRACT

This paper introduces a new nonclassical statistical distribution, SMART1, derived from a nonpolynomial function. This function is used in ecology to model population growth rates, in medicine to classify the relationship between activity and tumor volume in cancer, and in economics to analyze the relationship between supply and demand. The proposed distribution differs from the Gompertz distribution, which is fundamentally based on the exponential function. By contrast, the SMART1 distribution is constructed on principles of mathematical analysis, specifically through the identification of local maximum endpoints of the Gompertz growth function and the subsequent verification that the resulting function satisfies the criteria of a probability density function. This distribution (SMART1) is flexible and can model various real-world phenomena on a bounded interval $(0, \beta)$, especially in reliability analysis or survival modeling, where an upper lifetime limit exists. All statistical concepts are expressed in terms of the distribution's scale parameter β defines the upper bound (or support limit) of the distribution. The random variable X can only take values in the interval $(0, \beta)$. In other words, β is a "lifetime limit" or "maximum capacity"; and the shape parameter α determines how the risk or likelihood is distributed over time: Low α : high initial risk that decreases (e.g., early failures). High α : low initial risk that increases with time (e.g., aging or wear-out). These include the probability density function, the cumulative distribution function, the reliability function, the hazard function, order statistics, moments, and key measures such as the mode and the median.

Keywords: Cumulative distribution function (*C.d.f*), Gompertz growth function, Hazard function *H(x)*, Median, Mode, Moments function *M(x)*, Probability density function (*pdf*), Reliability function *R(x)*

Introduction

Most statistical distributions use an exponential-type kernel, which decays to zero as $x \to \infty$. Given the finite nature of real-world time intervals, we focus on functions $f\,(x)$ that satisfy $f\,(0) = 0$ and $f\,(\beta) = 0$ while remaining positive on the open interval $(0,\beta)$. Examples include the logistic family and the Gompertz growth function. Rządzkowski et al. applied the Gompertz equation in time-series analysis.[1] Lavrenčič et al. used the Gompertz equation to study the degradation of dry matter in forages harvested at different times, finding that the Gompertz model accurately represented the degradation data and outperformed other models in their analysis.[2] Akın et al. generalized and compared dynamic logistic and Gompertz equations to describe the growth patterns of bacteria and tumors, studying two types of Gompertz equations.[3]

Asadi et al. proposed a generalized Gompertz model to study real data on the growth of infection numbers during the COVID-19 pandemic. Their approach involved using stochastic processes, such as birth and death processes, which are time-inhomogeneous.[4] Shamran introduced a modified weighted Pareto distribution Type I. He calculated the essential statistical functions for this distribution, which represents a new form of Pareto distribution Type I with three parameters.[5] Sánchez et al. studied the effect of COVID-19 vaccine coverage on

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* Corresponding author.
E-mail addresses: saadna_math@csw.uobaghdad.edu.iq (S. N. Al-Azzawi), mahmoodas_math@csw.uobaghdad.edu.iq (M. A. Shamran), aseel.h@csw.uobaghdad.edu.iq (A. H. Ali).

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the spread of the monkeypox virus using both a Gompertz model and a logistic model.[6] Jabbar and Al-Saedi examined the growth of two types of tumor cells using the Gompertz equation.[7] Wang and Guo investigated the application of the Gompertz equation to microbial growth and biogas production, confirming the model's suitability for analyzing microbial communities throughout the growing season.[8]

Hussain, A. S. et al. used the Gompertz distribution to develop a new approach for enhancing the rate of occurrence in non-homogeneous Poisson processes, studying it through both traditional methods and artificial intelligence.[9] Hussain A. S. et al. proposed a method to improve the parameter estimation for Gompertz–Makeham processes by developing a maximum likelihood approach that incorporates artificial intelligence and Bayesian statistical techniques.[10]

Hansen CW and Strulik H. studied the increase in mortality rate with advancing age, considering that this increase follows an exponential pattern. They posit that aging itself is not the primary cause of death; rather, they linked the frailty index to the mortality rate through the self-production of health deficits, which serve as characteristic indicators of frailty.[11]

Materials and methods

1- Probability density function (p.d.f)

Let x be a random variable (r.v), the function satisfies the conditions.

1- $f(x) \geq 0$
2- $\int_{-\infty}^{\infty} f(x) = 1$

called probability density function ($p.d.f$).

2- Cumulative distribution function ($C.d.f$)

Let X be a non-negative random variable (r.v), the cumulative distribution function *F(x)* is defined as $F(x) = \int_{-\infty}^{x} f(t)dt$

3- Reliability Function *R(x)*

Let X be a non-negative random variable (r.v), the Reliability Function R(x) is defined as $R(x) = 1 - F(x)$.

4- Hazard Function H(x)

Let X be a non-negative random variable (r.v), the Hazard Function *H(x)* is defined as $H(x) = \frac{p.d.f}{R(x)} = \frac{f(x)}{R(x)}$.

5- Function under Consideration.

The considered function has the form.

$$g(x; \alpha, \beta) = x^{\alpha} ln\left(\frac{\beta}{x}\right), \ \alpha, \ \beta > 1$$

Properties of $\boldsymbol{g(x; \alpha, \beta)}$

1 : $\boldsymbol{g(x = \beta; \alpha, \beta)} = 0$ become $g(\beta; \alpha, \beta) = \beta^{\alpha} \ ln\left(\frac{\beta}{\beta}\right)$

2 : $\boldsymbol{g(x; \alpha, \beta)}$ is defined at zero, and its value is zero

$$\lim_{x \to 0} g(x; \alpha, \beta) = \lim_{x \to 0} x^{\alpha} ln\left(\frac{\beta}{x}\right) = 0$$

Since $x < \ \beta \ then \ ln\left(\frac{\beta}{x}\right) > 0$, therefore $\lim_{x \to 0} x^{\alpha} ln\left(\frac{\beta}{x}\right) = 0$

3 : $\boldsymbol{g(x; \alpha, \beta)}$ is positive in $(0, \beta)$

Proof:

$$\frac{d}{dx} g(x; \alpha, \beta) = \alpha x^{\alpha-1} ln\left(\frac{\beta}{x}\right) + x^{\alpha} \ \frac{-\frac{\beta}{x^2}}{\frac{\beta}{x}}$$

$$= \alpha x^{\alpha-1} ln\left(\frac{\beta}{x}\right) - \ x^{\alpha-1}$$

$$= x^{\alpha-1}\left(\alpha \ ln\left(\frac{\beta}{x}\right) - 1\right)$$

$$\frac{d}{dx}g(x;\alpha,\beta)=0 \ then \ ln\left(\frac{\beta}{x}\right)=\frac{1}{\alpha},\ \text{then}\frac{\beta}{x}=e^{\frac{1}{\alpha}},\text{so } x=\beta e^{-\frac{1}{\alpha}}$$

$$\frac{d^2}{dx^2}g(x;\alpha,\beta)=(\alpha-1)x^{\alpha-2}\left(\alpha\ ln\left(\frac{\beta}{x}\right)-1\right)+\alpha x^{\alpha-1}\frac{-\frac{\beta}{x^2}}{\frac{\beta}{x}}\alpha x^{\alpha-2}$$

at $x=\beta e^{-\frac{1}{\alpha}}$

$$\frac{d^2}{dx^2}g(x;\alpha,\beta)=\left(\beta e^{-\frac{1}{\alpha}}\right)^{\alpha-2}\left[(\alpha-1)\left(\alpha\ ln\left(e^{\frac{1}{\alpha}}\right)-1\right)-\alpha\right]$$

$$=(\beta e^{-\frac{1}{\alpha}})^{\alpha-2}(-\alpha)=-\alpha\,(\beta e^{-\frac{1}{\alpha}})^{\alpha-2}<0$$

Then, the point $(\beta e^{-\frac{1}{\alpha}},\ g(\beta e^{-\frac{1}{\alpha}}))$ *is a* maximum point and therefore >0 in $(0,\ \beta)$.

Results and discussion

To construct the probability density function for the function $g(x;\alpha,\beta)$ *is evaluated* as follows:

1. SMART1 probability density function (p.d.f)

To derive the probability density function for the SMART1 distribution finding the integral of $g(x;\alpha,\beta)$ as follows:

$\int_0^\beta g(x;\alpha,\beta)\,dx=\int_0^\beta x^\alpha ln(\frac{\beta}{x})dx$ by using the integration by parts method

$$\int_0^\beta x^\alpha ln\left(\frac{\beta}{x}\right)dx=\frac{x^{\alpha+1}}{\alpha+1}ln\left(\frac{\beta}{x}\right)|_0^\beta-\int_0^\beta\frac{x^{\alpha+1}}{\alpha+1}\left(\frac{-\frac{\beta}{x^2}}{\frac{\beta}{x}}\right)\text{dx}$$

$$=\frac{1}{\alpha+1}\int_0^\beta x^\alpha dx=\frac{x^{\alpha+1}}{(\alpha+1)^2}|_0^\beta=\frac{\beta^{\alpha+1}}{(\alpha+1)^2}$$

Therefore, multiplying the function $g(x;\alpha,\beta)$ *by* $\frac{(\alpha+1)^2}{\beta^{\alpha+1}}$ to get the function $f(x;\alpha,\beta)$ as in the form

$$f(x;\alpha,\beta)=\begin{cases}\frac{(\alpha+1)^2}{\beta^{\alpha+1}}x^\alpha ln\left(\frac{\beta}{x}\right), & 0\le x\le\beta,\ \alpha,\ \beta>1\\ 0 & otherwise\end{cases}$$

$f(x;\alpha,\beta)$ is *a p.d.f.*

since $0\le f(x)\le 1$ therefore we need $\frac{(\alpha+1)^2}{\alpha\beta e}\le 1$ then $\beta\ge\frac{(\alpha+1)^2}{\alpha e}$

$$f(x;\alpha,\beta)=\begin{cases}\frac{(\alpha+1)^2}{\beta^{\alpha+1}}x^\alpha ln\left(\frac{\beta}{x}\right), & 0<x<\beta,\ \beta\ge\frac{(\alpha+1)^2}{\alpha e}\\ 0 & otherwise\end{cases}$$

is *a p.d.f.*

the carve of (p.d.f) in Fig. 1.

2. SMART1 Cumulative distribution function (*C.d.f*).

Lemma 1: *Let X be a continuous non-negative random variable (r.v), the cumulative distribution function (C.d.f), $F(x;\alpha,\beta)$ is in the form,*

$$F(x;\alpha,\beta)=\frac{x^{\alpha+1}}{\beta^{\alpha+1}}\left((\alpha+1)\ ln\left(\frac{\beta}{x}\right)+1\right)$$

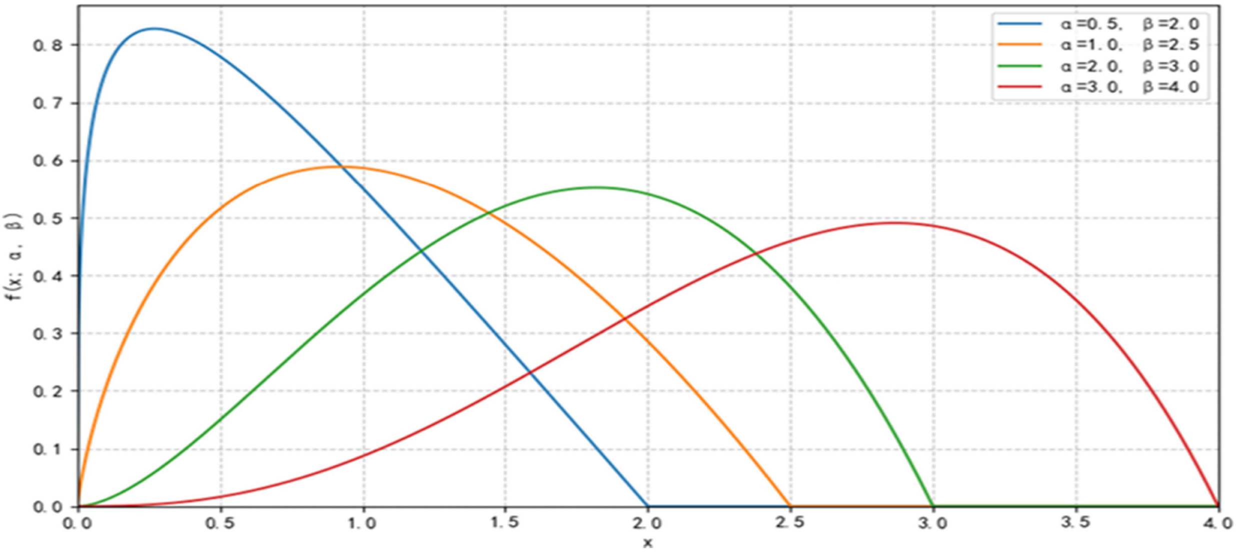

Fig. 1. SMART1 probability density function (pdf) for different values of α (shape) and β (scale).

Proof:

$$F\,(x;\alpha,\beta)=\int_0^x f\,(t;\alpha,\beta)\,dt$$

$$=\int_0^x \frac{(\alpha+1)^2}{\beta^{\alpha+1}}\,t^{\alpha} ln\left(\frac{\beta}{t}\right)dt$$

$$=\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\int_0^x t^{\alpha} ln\left(\frac{\beta}{t}\right)dt$$

By the integral by parts method

$$=\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\left[\frac{t^{\alpha+1}}{\alpha+1} ln\left(\frac{\beta}{t}\right)\Bigg|_0^x-\int_0^x \frac{t^{\alpha+1}}{\alpha+1}t^{-1}dt\right]$$

$$=\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\left[\frac{x^{\alpha+1}}{\alpha+1} ln\left(\frac{\beta}{x}\right)+\frac{t^{\alpha+1}}{(\alpha+1)^2}\Bigg|_0^x\right]$$

$$=\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\left[\frac{x^{\alpha+1}}{\alpha+1} ln\left(\frac{\beta}{x}\right)+\frac{x^{\alpha+1}}{(\alpha+1)^2}\right]$$

$$=\frac{(\alpha+1)}{\beta^{\alpha+1}}x^{\alpha+1}\,ln\left(\frac{\beta}{x}\right)+\frac{x^{\alpha+1}}{\beta^{\alpha+1}}$$

Then, $F(x;\alpha,\beta)=\dfrac{x^{\alpha+1}}{\beta^{\alpha+1}}\,((\alpha+1)ln(\dfrac{\beta}{x})+1)$

The Cumulative distribution function curve is shown in Fig. 2.

3. SMART1 Reliability function $R(x;\alpha,\beta)$.

Lemma 2: *Let X be a continuous non-negative r.v having a SMART1 distribution, the reliability function of x as in the form:*

$$R\,(x;\alpha,\beta)=1-\left[\frac{\alpha+1}{\beta^{\alpha+1}}\,x^{\alpha+1} ln\left(\frac{\beta}{x}\right)+\frac{x^{\alpha+1}}{\beta^{\alpha+1}}\right]$$

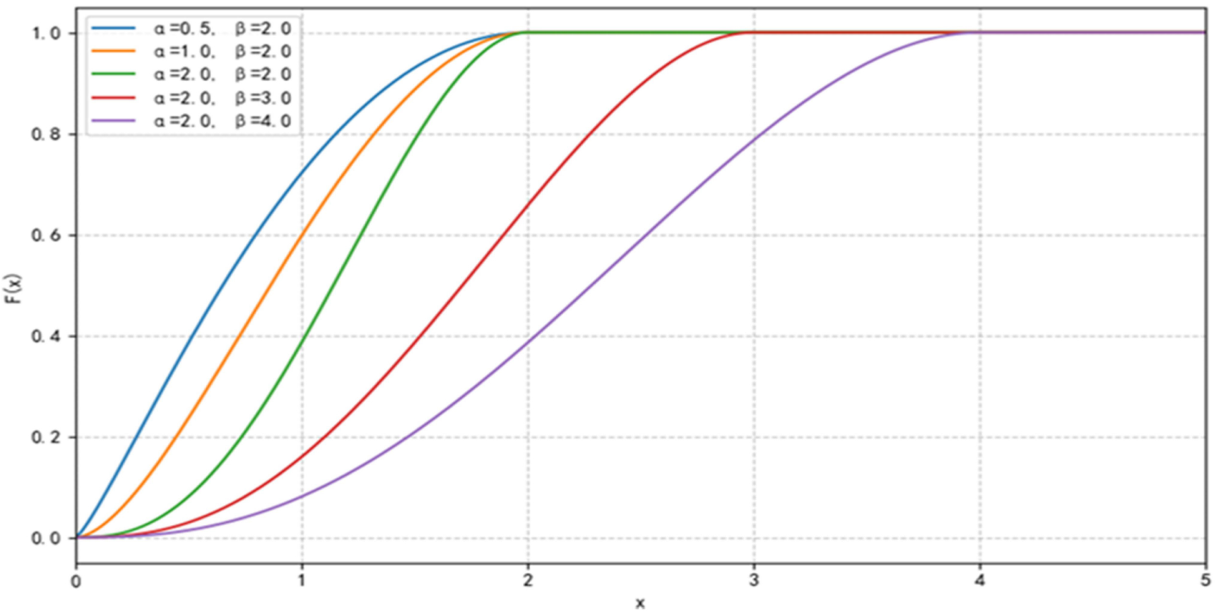

Fig. 2. SMART1 cumulative distribution function (*C.d.f*).

Proof:

$$R\,(x;\alpha,\beta) = 1 - F\,(x;\alpha,\beta)$$

$$= 1 - \left[\frac{x^{\alpha+1}}{\beta^{\alpha+1}}\left((\alpha+1)\,ln\left(\frac{\beta}{x}\right)+1\right)\right]$$

$$= 1 - \left[\frac{\alpha+1}{\beta^{\alpha+1}}\,x^{\alpha+1} ln\left(\frac{\beta}{x}\right)+\frac{x^{\alpha+1}}{\beta^{\alpha+1}}\right]$$

The SMART1 Reliability function curve is shown in Fig. 3.

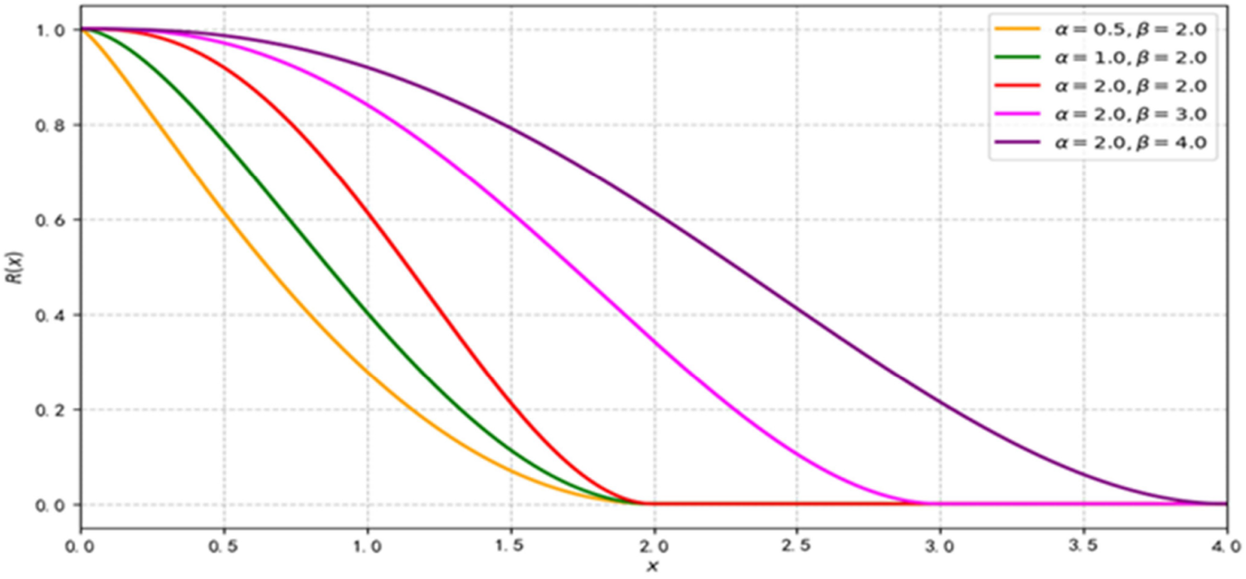

Fig. 3. SMART1 reliability function $R(x;\alpha,\beta)$.

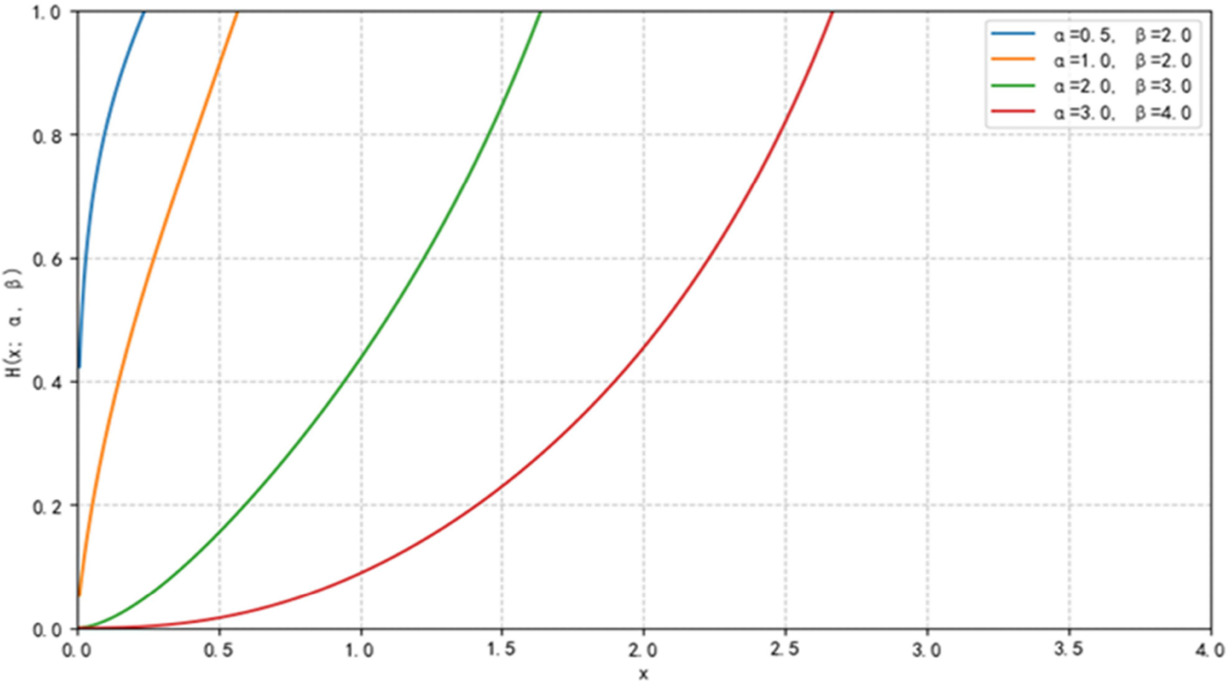

Fig. 4. SMART1 hazard function *H(x)*.

4. SMART1 hazard function $H(x; \alpha, \beta)$.

Lemma 3: *Let X be a continuous positive r.v. distributed has SMART1 distribution the hazard function for x as follows:*

$$H(x; \alpha, \beta) = \frac{\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\, x^{\alpha} \ln\left(\frac{\beta}{x}\right)}{1 - \left[\frac{x^{\alpha+1}(\alpha+1)}{\beta^{\alpha+1}} \ln\left(\frac{\beta}{x}\right) + \frac{x^{\alpha+1}}{\beta^{\alpha+1}}\right]}$$

Proof:

$$H(x; \alpha, \beta) = \frac{f(x; \alpha, \beta)}{R(x; \alpha, \beta)}$$

$$= \frac{\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\, x^{\alpha} \ln\left(\frac{\beta}{x}\right)}{1 - \left[\frac{x^{\alpha+1}}{\beta^{\alpha+1}} \left((\alpha+1) \ln\left(\frac{\beta}{x}\right) + 1\right)\right]}$$

$$= \frac{\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\, x^{\alpha} \ln\left(\frac{\beta}{x}\right)}{1 - \left[\frac{x^{\alpha+1}(\alpha+1)}{\beta^{\alpha+1}} \ln\left(\frac{\beta}{x}\right) + \frac{x^{\alpha+1}}{\beta^{\alpha+1}}\right]}$$

SMART1 hazard function *H(x)* curve as in Fig. 4.

5. SMART1 Moments function $M(x; \alpha, \beta)$.

Lemma 4: *X be a positive r.v. distributed as SMART1 distribution, the moments function for x as follows:*

$$M(x; \alpha, \beta) = E\left(x^r\right) = \frac{(\alpha+1)^2 \beta^r}{(\alpha+r+1)^2}$$

Proof:

$$M(x;\alpha,\beta)=E\left(x^{r}\right)=\int_{0}^{\beta}x^{r}f\left(\frac{\beta}{x}\right)\,dx$$

$$=\int_{0}^{\beta}\frac{(\alpha+1)^{2}}{\beta^{\alpha+1}}\,x^{r}x^{\alpha}\,ln\left(\frac{\beta}{x}\right)dx$$

$$=\frac{(\alpha+1)^{2}}{\beta^{\alpha+1}}\int_{0}^{\beta}x^{\alpha+r}\,ln\left(\frac{\beta}{x}\right)dx$$

By integration by parts:

$$=\frac{(\alpha+1)^{2}}{\beta^{\alpha+1}}\left[\frac{x^{\alpha+r+1}}{\alpha+r+1}ln\left(\frac{\beta}{x}\right)\right]\Bigg|_{0}^{\beta}-\int_{0}^{\beta}\left(\frac{x^{\alpha+r+1}}{\alpha+r+1}\cdot\frac{-\frac{\beta}{x^{2}}}{\frac{\beta}{x}}\right)dx$$

$$=\frac{(\alpha+1)^{2}}{\beta^{\alpha+1}}\cdot\frac{x^{\alpha+r+1}}{(\alpha+r+1)^{2}}\Bigg|_{0}^{\beta}$$

$$=\frac{(\alpha+1)^{2}}{\beta^{\alpha+1}}\cdot\frac{\beta^{\alpha+r+1}}{(\alpha+r+1)^{2}}$$

$$=\frac{(\alpha+1)^{2}\beta^{r}}{(\alpha+r+1)^{2}},\ r=1,2,3,\ldots$$

The mean and variance for the SMART1 distribution can be obtained by Lemma 4. If we substitute $r = 1{,}2$ in the moments function gate

$$M(1)=\text{ Mean }(x)=E(x)=\frac{(\alpha+1)^{2}\beta}{(\alpha+2)^{2}}$$

And, $M(2)=E(x^{2})=\frac{(\alpha+1)^{2}\beta^{2}}{(\alpha+3)^{2}}$

Since, $\text{var}(x)=E(X^{2})\ -\ (E(x))^{2}$

$$var(x)=\frac{(\alpha+1)^{2}\beta^{2}}{(\alpha+3)^{2}}-\left(\frac{(\alpha+1)^{2}\beta}{(\alpha+2)^{2}}\right)^{2}$$

$$var(x)=\frac{(\alpha+1)^{2}\beta^{2}}{(\alpha+3)^{2}}-\frac{(\alpha+1)^{4}\beta^{2}}{(\alpha+2)^{4}}$$

The standard deviation $S.D(x)=\ \sqrt{var(x)}$

$$S.\,D(x)=\sqrt{\frac{(\alpha+1)^{2}\beta^{2}}{(\alpha+3)^{2}}-\frac{(\alpha+1)^{4}\beta^{2}}{(\alpha+2)^{4}}}$$

$$S.D(x)=\frac{\beta(\alpha+1)\ \sqrt{2\alpha^{2}+8\alpha+7}}{(\alpha+2)^{2}(\alpha+3)}$$

6. SMART1 Moment generating function $M_x(t)$.

Lemma 5: *Let X be a continuous non-negative r.v distributed as the SMART1 distribution, the moments generating function of x, $M_x(t)$ is in the form:*

$$M_{x}(t)=\frac{(\alpha+1)^{2}}{\beta^{\alpha+1}}\sum_{i=0}^{\infty}\frac{t^{i}\beta^{\alpha+i+1}}{i!\,(\alpha+i+1)^{2}}$$

Proof:

$$M_x(t) = E\left(e^{tx}\right) = \int_o^\beta e^{tx}\, f(x)\, dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \int_o^\beta e^{tx}\, x^\alpha \ln\left(\frac{\beta}{x}\right) dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{t^i}{i!} \int_o^\beta x^{\alpha+i} \ln\left(\frac{\beta}{x}\right) dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{t^i}{i!} \left(\mathrm{Ln}\left(\frac{\beta}{x}\right) \frac{x^{\alpha+i+1}}{\alpha+i+1} \Bigg|_0^\beta - \int_0^\beta \frac{x^{\alpha+i+1}}{\alpha+i+1} \cdot \frac{-\frac{\beta}{x^2}}{\frac{\beta}{x}} dx \right)$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{t^i}{i!} \int_0^\beta \frac{x^{\alpha+i}}{\alpha+i+1}\, dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{t^i}{i!} \frac{x^{\alpha+i+1}}{(\alpha+i+1)^2} \Bigg|_0^\beta$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{t^i\, \beta^{\alpha+i+1}}{i!\,(\alpha+i+1)^2}$$

7. SMART1 Median

Lemma 6: *Let X be a continuous non-negative r.v distributed as the SMART1 distribution, then the median of x is in the form:*

$$M_{n+1} = m_n - \frac{(\alpha+1)\ {m_n}^{\alpha+1} \ln\left(\frac{\beta}{m_n}\right) + {m_n}^{\alpha+1} - \frac{\beta^{\alpha+1}}{2}}{(\alpha+1)^2\ \ln\left(\frac{\beta}{m_n}\right) + {m_n}^{\alpha}},\ n = 0, 1, 2, \ldots$$

Proof:

$$\int_o^m f(x)\, dx = \frac{1}{2}$$

$$\int_o^m \frac{(\alpha+1)^2}{\beta^{\alpha+1}} x^\alpha \ln\left(\frac{\beta}{x}\right) dx = \frac{1}{2}$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \left[\frac{m^{\alpha+1}}{\alpha+1} \ln\left(\frac{\beta}{m}\right) + \frac{m^{\alpha+1}}{(\alpha+1)^2}\right] = \frac{1}{2}$$

$$(\alpha+1)\, m^{\alpha+1}\ ln\left(\frac{\beta}{m}\right) + m^{\alpha+1} = \frac{\beta^{\alpha+1}}{2}$$

Let $\mathrm{h(m)} = (\alpha+1)\mathrm{m}^{\alpha+1} \ln\left(\frac{\beta}{\mathrm{m}}\right) + \mathrm{m}^{\alpha+1} - \frac{\beta^{\alpha+1}}{2} = 0$

by newton Raphson method:

$$m_{n+1} = m_n - \frac{h(m_n)}{h'(m_n)},\ n = 0, 1, 2, \ldots$$

$$m_{n+1} = m_n - \frac{(\alpha+1)\ m^{\alpha+1} \ln\left(\frac{\beta}{m_n}\right) + m^{\alpha+1} - \frac{\beta^{\alpha+1}}{2}}{(\alpha+1)^2\ \ln\left(\frac{\beta}{m_n}\right) {m_n}^{\alpha}},\ n = 0, 1, 2, \ldots$$

Must find the approximate value for m that satisfies the equation above.

8. SMART1 Characteristic function *G(t)*.

Lemma 7: *Let X be a continuous positive r.v. that has a SMART1 distribution on (0, β) with probability density function, then the Characteristic function G(t) of x is in the form:*

$$G(t) = \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{n=0}^{\infty} \frac{t^n i^n \beta^{\alpha+n+1}}{n!\,(\alpha+n+1)^2}$$

Proof:

$$G(t) = E\left(e^{itx}\right) = \int_o^{\beta} e^{itx}\, f(x)\, dx$$

$$= \int_o^{\beta} e^{itx}\, \frac{(\alpha+1)^2}{\beta^{\alpha+1}} x^{\alpha} \ln\left(\frac{\beta}{x}\right) dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{n=0}^{\infty} \frac{t^n i^n}{n!} \int_o^{\beta} x^{\alpha+n} \ln\left(\frac{\beta}{x}\right) dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{n=0}^{\infty} \frac{t^n i^n}{n!} \left(\ln\left(\frac{\beta}{x}\right) \frac{x^{\alpha+n+1}}{\alpha+n+1} \Bigg|_0^{\beta} - \int_0^{\beta} \frac{x^{\alpha+n}}{\alpha+n+1}\, dx \right)$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{n=0}^{\infty} \frac{t^n i^n}{n!} \frac{x^{\alpha+n+1}}{(\alpha+n+1)^2} \Bigg|_0^{\beta}$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{n=0}^{\infty} \frac{t^n i^n}{n!} \frac{\beta^{\alpha+n+1}}{(\alpha+n+1)^2}$$

9. Ordered statistics of SMART1

Lemma 8: *Let $x_1, x_2, \ldots, x_n$ be continuous r.vs have SMART1 distribution, then the ordered statistics function $f_{x_j}(x)$ is in the form:*

$$f_{x_j}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_x(x)\, [F_x(x)]^{j-1}\, [1 - F_x(x)]^{n-j}$$

then $f_{x_j}(x) = \frac{n!}{(j-1)!(n-j)!} (\frac{(\alpha+1)^2}{\beta^{\alpha+1}} x^{\alpha} \ln(\frac{\beta}{x}))\, (\frac{x^{\alpha+1}(\alpha+1)}{\beta^{\alpha+1}} \ln(\frac{\beta}{x}) + \frac{x^{\alpha+1})}{\beta^{\alpha+1}})^{j-1}$

$$\times \left(1 - \left[\frac{x^{\alpha+1}(\alpha+1)}{\beta^{\alpha+1}} \ln\left(\frac{\beta}{x}\right) + \frac{x^{\alpha+1})}{\beta^{\alpha+1}}\right]\right)^{n-j}$$

10. Mode for SMART1

Lemma 9: *Let X be a continuous r.v. that has a SMART1 distribution, then the mode of x is in the form:*

$$x = e^{\frac{-1}{\alpha}} \beta$$

Proof:

$$f'(x;\ \alpha, \beta) = \frac{d}{dx}\left(\frac{(\alpha+1)^2}{\beta^{\alpha+1}} x^{\alpha} \ln\left(\frac{\beta}{x}\right)\right) = 0$$

$$\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\left(\frac{d}{dx}x^{\alpha}\ln\left(\frac{\beta}{x}\right)\right)=0$$

$$\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\left(\alpha\ln\left(\frac{\beta}{x}\right)x^{\alpha-1}-x^{\alpha-1}\right)=0$$

$$\alpha(\alpha+1)^2\beta^{-\alpha-1}\ln\left(\frac{\beta}{x}\right)x^{\alpha-1}-(\alpha+1)^2\beta^{-\alpha-1}\,x^{\alpha-1}=0$$

$$(\alpha+1)^2\beta^{-\alpha-1}\left(\alpha\ln\left(\frac{\beta}{x}\right)-1\right)x^{\alpha-1}=0$$

Then $\ln\left(\frac{\beta}{\mathrm{x}}\right)=\frac{1}{\alpha}$

$\left(\frac{\beta}{x}\right)=e^{\frac{1}{\alpha}}$, then $x=\beta e^{\frac{-1}{\alpha}}$

11. Mean time to failure (*MTTF*).

Lemma 10: *Let X be a continuous r.v. that has a SMART1 distribution, then the mean time to failure (MTTF) is in the form:*

$$MTTF\,(t)=\frac{\alpha^2+1}{(\alpha+2)^2}\beta$$

Proof:

$$MTTF\,(t)=\int_0^{\beta}R\,(t)\,dx$$

$$=\int_0^{\beta}\left[1-\left(\beta^{-\alpha-1}\left((\alpha+1)\ln\left(\frac{\beta}{t}\right)+1\right)t^{\alpha+1}\right)\right]dt$$

$$=\int_0^{\beta}dt-\beta^{-\alpha-1}\int_0^{\beta}t^{\alpha+1}\,dt-\ (\alpha+1)\,\beta^{-\alpha-1}\int_0^{\beta}\frac{t^{\alpha+1}}{\alpha+2}dt$$

$$=\beta-\ \frac{\beta}{\alpha+2}-\frac{\beta}{(\alpha+2)^2}$$

$$MTTF\,(t)=\frac{\alpha^2+1}{(\alpha+2)^2}\beta$$

12. SMART1 Factorial moments generating function *L(t)*.

Lemma 11: *Let X be a non-negative continuous r.v. that has a SMART1 distribution, then the Factorial moments generating function of X is in the form:*

$$L\,(t)=\frac{(\alpha+1)^2}{\beta^{\alpha+1}}\sum_{i=0}^{\infty}\frac{(lnt)^i\beta^{\alpha+i+1}}{(\alpha+i+1)^2}$$

Proof:

$$L\,(t)=E\,(t^x)=E\left(e^{lnt^x}\right)=E\left(e^{xlnt}\right)=\ \mu_x\,(\ln t)$$

$$\mu_x\,(\ln t)=\int_0^{\beta}e^{xlnt}\ f\,(x)\,dx$$

$$=\frac{(\alpha+1)^2}{\beta^{\alpha+1}}=\int_0^{\beta}e^{xlnt}\ x^{\alpha}\ln\left(\frac{\beta}{x}\right)\ dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{(lnt)^i}{i!} \int_0^{\beta} x^{\alpha+i} \ln\left(\frac{\beta}{x}\right) dx$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{(lnt)^i}{\substack{i!\\ z}} \left(\ln\left(\frac{\beta}{x}\right) \frac{x^{\alpha+i+1}}{\alpha+i+1} \Bigg|_0^{\beta} - \int_0^{\beta} \frac{x^{\alpha+i}}{\alpha+i+1} dx \right)$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{(lnt)^i \, x^{\alpha+i+1}}{i!(\alpha+i+1)^2} \Bigg|_0^{\beta}$$

$$= \frac{(\alpha+1)^2}{\beta^{\alpha+1}} \sum_{i=0}^{\infty} \frac{(lnt)^i \, \beta^{\alpha+i+1}}{i! \, (\alpha+i+1)^2}$$

Conclusion

This study introduces the SMART1 distribution, a novel, mathematically rigorous probability model derived from the Gompertz growth equation. By moving beyond the conventional exponential assumptions of classical growth-based distributions, SMART1 leverages the intrinsic analytical properties of the Gompertz function to provide a more refined and flexible representation of bounded stochastic processes. The full suite of its statistical machinery, including the probability density function (PDF), cumulative distribution function (CDF), survival and hazard functions, moment-generating function (MGF), mean, variance, and mean time to failure (MTTF), has been rigorously derived, confirming its validity as a proper probability distribution. With its two interpretable parameters β (scale), which defines the finite upper bound of the support (i.e., maximum lifetime or capacity), and α (shape), which governs the temporal evolution of risk (from early-failure to wear-out patterns), SMART1 is uniquely suited for modeling real-world phenomena constrained to a finite interval. Its applicability spans multiple disciplines, including reliability engineering, survival analysis, ecology, oncology, and economics. Altogether, SMART1 bridges deterministic growth modeling and probabilistic inference, offering a robust, adaptable, and theoretically grounded framework for both theoretical exploration and practical data analysis in contexts where an upper limit on outcomes is inherent.

Author's declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Furthermore, any Figures and images that are not ours have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' contribution statement

S.N. proposed the research idea. S.N. and M.A.S derived the necessary equations for the research. M.A.S reviewed and verified the derivations. A.H.A and M.A.S plotted the appropriate functions and included the required tables. The researchers participated equally in writing the research paper.

Data availability

The datasets generated during and analyzed during the current study are available from the corresponding author on reasonable request.

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توزيع إحصائي جديد SMART1

سعد ناجي العزاوي، محمود عريبي شمران، اسيل حسين علي

قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق.

الخلاصة

تقدّم هذه الدراسة توزيعًا إحصائيًا جديدًا غير تقليدي يُسمّى SMART1، مشتقًّا من دالة غير متعددة الحدود. تُستخدم هذه الدالة في علم البيئة لنمذجة معدلات نمو الجماعات الحيوية، وفي الطب لتصنيف العلاقة بين النشاط وحجم الورم في السرطان، وفي الاقتصاد لتحليل العلاقة بين العرض والطلب. يختلف التوزيع المقترح عن توزيع غومبرتز (Gompertz) الذي يعتمد أساسًا على الدالة الأسّية. وعلى النقيض من ذلك، يُبنى توزيع SMART1 على أسس التحليل الرياضي، وبالتحديد من خلال تحديد نقاط القيم العظمى المحلية لدالة نمو غومبرتز، ثم التحقق من أن الدالة الناتجة تستوفي شروط دالة الكثافة الاحتمالية. يتميّز توزيع SMART1 بالمرونة، إذ يمكنه تمثيل العديد من الظواهر الواقعية ضمن مجال محدود $(\beta,0)$، ولا سيما في تحليل الاعتمادية أو نماذج البقاء حيث يوجد حدّ أعلى للعمر. وتُصاغ جميع المفاهيم الإحصائية بدلالة معلمة القياس β التي تمثل الحد الأعلى (أو مجال الدعم) للتوزيع، بحيث لا يمكن للمتغير العشوائي X أن يأخذ إلا قيماً ضمن الفترة $(\beta,0)$. وبعبارة أخرى، تمثل β «حدّ العمر» أو «السعة القصوى» التي لا يمكن لأي مشاهدة أن تتجاوزها. أما معلمة الشكل α فتحدد كيفية توزيع المخاطر أو الاحتمالات عبر الزمن، وذلك على النحو الآتي: عندما تكون α منخفضة: يكون الخطر الابتدائي مرتفعًا ثم يتناقص (مثل حالات الأعطال المبكرة). عندما تكون α مرتفعة: يكون الخطر الابتدائي منخفضًا ثم يزداد مع الزمن (مثل حالات التقادم أو التآكل). ويتضمن هذا التوزيع اشتقاق كل من: دالة الكثافة الاحتمالية (pdf)، ودالة التوزيع التراكمي (CDF)، ودالة الاعتمادية (Reliability Function)، ودالة المخاطرة (Hazard Function)، وإحصاءات الرتب (Order Statistics)، والعزوم (Moments)، إضافة إلى مقاييس مهمة مثل المنوال والوسيط.

الكلمات المفتاحية: دالة التوزيع التراكمي (CDF)، دالة نمو غومبرتز، دالة المخاطرة $H(x)$، الوسيط، المنوال، دالة العزوم $M(x)$، دالة الكثافة الاحتمالية (pdf)، دالة الاعتمادية $R(x)$.