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A Structural Investigation of the Idempotent Graph Associated with the Ring Z_{pq}

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ABSTRACT

This study investigates the structural characteristics of the idempotent graph G_{Id} (R) associated with a commutative ring R. The graph is defined as a simple, undirected graph in which the vertices correspond to the elements of R, and two distinct vertices a and b are considered adjacent precisely when the condition $(a+b)^2 = a+b$ holds, meaning their sum is an idempotent element. The research specifically examines the idempotent graph constructed over the ring of integers modulo n, denoted Z_n, where $n = pq$ and p and q are distinct prime numbers satisfying $p < q$. Within this framework, the work provides a detailed analysis of several fundamental graph-theoretic properties of G_{Id} (R), including its radius, diameter, vertex degree, and chromatic number. The findings offer insight into how the algebraic structure of Z_{pq} influences the resulting graph, contributing to a deeper understanding of the interaction between ring theory and graph theory.

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1. INTRODUCTION

This interplay between graph theory and ring theory was first explored by Beck in 1988[1], when he introduced the concept of coloring elements of a commutative ring to construct a graph-theoretical model. This first idea was further developed by Anderson and Livingston, who examined graphs defined by the zero divisors of rings [2]. Since then, numerous researchers have extended and generalized these concepts in various directions [3,4,5,6,7,8,9]. Let R be a ring. An element $e \in R$ is called idempotent if it satisfies the condition$e^2 = e$. Building upon this notation, Pravesh Sharma and Sanghita Dutta introduced the concept of idempotent graph of a ring [10]. The idempotent graph of a ring R, denoted by G_{Id} (R), is a simple undirected graph where each vertex corresponds to an element of R. two distinct vertices $a, b \in R$ are adjacent if and only if $(a+b)^2 = a+b$. In this study, several fundamental concepts are used to describe the structural properties of the idempotent graph.

The eccentricity of a vertex in a graph is defined as the greatest distance from that vertex to any other vertex within the same graph. This concept helps in understanding how far a vertex is from the most distant point in the graph. The diameter of a graph is the maximum eccentricity among all its vertices. It represents the longest of all shortest-path distance between any pair of vertices, thereby indicating the overall spread of the graph. In contrast, the radius of a graph is the minimum eccentricity among its vertices, reflecting the distance from the most centrally located vertex to the farthest one. A connected graph is a type of graph in which there is at least one path between every pair of vertices. This implies that the graph consists of a single component, with no isolated subgraphs or vertices meanwhile; a non-planar graph is one that cannot be drawn in a two-dimensional plane without some of its edges crossing each other. This characteristic distinguishes it from planar graphs, which can be embedded in the plane without edge intersections, except at vertices.

In this paper, we investigate the idempotent graph of the ring Z_n, where $n = pq$, $with\, p$ and q being distinct prime numbers such that$p < q$. We analyze various structural properties of this graph, including the vertex set V (G), edge set E (G), order |V|, and the degree of vertex$deg(v)$, defined as the number of edges incident to$v$. Additionally, we consider path graphs of order n, denoted by P_n and cycle graph of order n, denoted by C_n . A path beginning at vertex a_1 and ending at vertex a_q is represented by $a_1 \to a_q$.

2. Idempotent graphs of the Ring Z_{pq} Where p and q Are Distinct Prime Numbers and $p < q$

In this section, we investigate the idempotent graphs associated with the commutative ring Z_{pq}, where p and q are distinct prime numbers satisfying $p < q$. The study focuses on examining the structural characteristics of these graphs, alongside exploring several of their fundamental properties. Specifically, we construct and analyze the idempotent graphs corresponding to the rings Z_{2q}, Z_{3q}, and Z_{5q}, highlighting patterns and properties unique to each case.

2.1. The Structure of the Idempotent Graph of the ring Z_{2q} where q is prime number and $q > 2$

The idempotent graph associated with the ring Z_{2q}, denoted by $G_{Id}(Z_{2q})$, exhibits a distinctive structure. In this context, for every element $a \in Z_{2q}$, there exists an element $b \in Z_{2q}$ such that$(a+b)^2 = a+b$, satisfying the idempotent condition.

The vertex set of $G_{Id}(Z_{2q})$ is partitioned into two disjoint subsets, V_1 and V_2, formed a bipartite graph these subsets are defined as follows:

- $V_1 = \{a_1, a_2, a_3, \dots, a_q\} \subseteq \{0,2,4,\dots,2q-2\}$
- $V_2 = \{b_1, b_2, b_3, \dots, b_q\} \subseteq \{1,3,5,\dots,2q-1\}$

Each vertex in V_1 ,except the end points a_1 and a_q , has degree 4, the vertices a_1 and a_q have degree 3. Similarly, all vertices in V_1 possess degree 4, with the exceptions of b_1 and b_q ,which each have degree 3. The internal structure of V_1 forms a linear path from a_1 to a_q, where consecutive vertices are adjacent. Additionally, each vertex $a_i \in V_1$ is connected to two vertices inV_2. Specifically, for$1 < i < q$, the vertex a_i is adjacent to b_{i-1} andb_{i+1}. In the boundary cases, a_1 is connected to b_1 and b_2, while a_q is connected to b_q and b_{q-1}. Likewise, the vertices in V_2 forms a linear path from b_1 to b_q, with each vertex b_i connected to its neighboring vertices. Furthermore, each $b_i \in V_2$ is adjacent to two vertices inV_1: for$1 < i < q$, it is connected to a_{i-1} anda_{i+1}, b_1 connects to a_1 and a_2, while b_q connects to a_q and a_{q-1}. The id arrangement results in several interwoven paths: the sequence$a_1 \rightarrow a_q$, the sequence$b_1 \rightarrow b_q$, and two additional paths formed by alternating between V_1 andV_2, namely $a_1 \rightarrow b_1$ and $a_q \rightarrow b_q$. Together, these four paths constitute a closed cycle of length$2q$, which forms the boundary of graph$G_{Id}(Z_{2q})$. The total number of vertices (i.e., the order of the graph) is $2q$, and the total number of edges (i.e., the size) is $4q-2$. A general schematic representation of the idempotent graph $G_{Id}(Z_{2q})$ is illustrated in Figure 1.

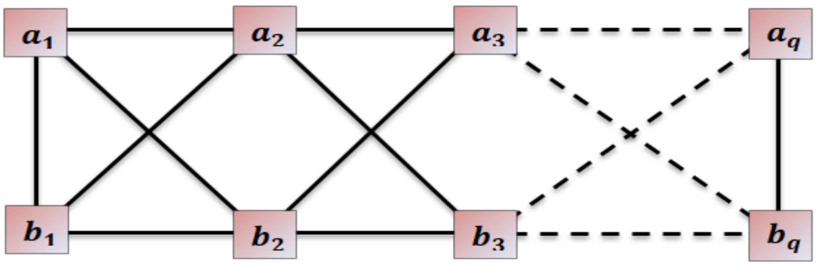


Figure 1. General form of the idempotent graph $G_{Id}(Z_{2q})$

Example 1 (Idempotent graph ofZ_{10}):

Let us consider the idempotent graph associated with the ring $Z_{10} \cong Z_{2.5}$, where $q = 5$. The vertex set of the graph $G_{Id}(Z_{10})$ can be partitioned into two disjoint subsets:

- $V_1 = \{0, 2, 4, 6, 8\}$, and
- $V_2 = \{1, 3, 5, 7, 9\}$

This classification is based on the structure induced by the ring's idempotent elements. The corresponding idempotent graph is illustrated in Figure 2.

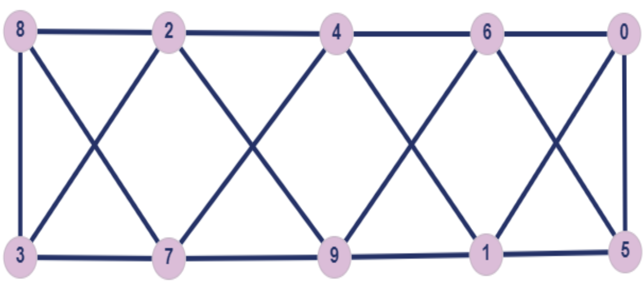


Figure 2. The idempotent graph $G_{Id}(Z_{10})$

2.2. The Idempotent Graph of the Ring Z_{3q}, Where $q > 3$ Is a Prime Number

The idempotent graph associated with the ringZ_{3q}, where q is a prime number greater than 3, possesses a distinctive structure. For every element a $\in Z_{3q}$, there exists an element $b \in Z_{3q}$ such that$(a + b)^2 = a + b$. This property gives rise to a well-defined graph structure based on idempotent behavior under addition and squaring.

The vertex set of the idempotent graph $G_{Id}(Z_{3q})$ can be partitioned into three disjoint subsets:

$V\big(G_{Id}(Z_{3q})\big) = V_1 \cup V_2 \cup V_3$,

Where each subset contains exactly q vertices, i.e., $|V_1| = |V_2| = |V_3|$.

These subsets are defined as

$V_1 = \{a_1, a_2, a_3, \ldots, a_q\} \subseteq \{0, 3, 6, \ldots, 3q - 3\}$,

$V_2 = \{b_1, b_2, b_3, \ldots, b_q\} \subseteq \{1, 4, 7, \ldots, 3q - 2\}$,

$V_3 = \{c_1, c_2, c_3, \ldots, c_q\} \subseteq \{2, 5, 8, \ldots, 3q - 1\}$,

Vertex Degrees

Each vertex in V_2 has degree 4.

Each vertex in V_1 and V_3 also has degree 4, except for the endpoints $a_1, a_q \in V_1$ and $c_1, c_q \in V_3$, which have degree 3.

Adjacency Relations

The vertices in V_1 and V_3 form paths: $(a_1, a_2, \ldots, a_q)$ and $(c_1, c_2, \ldots, c_q)$, respectively, such that each vertex is adjacent to its immediate predecessor and successor in the sequence (when applicable).

For each$a_i \in V_1$, adjacency with vertices in V_2 is defined as follows:

If$1 < i < q$, then a_i is adjacent to b_{i-1} and b_{i+1}

If$i = 1$, then a_1 is adjacent to b_1 and b_2

If$i = q$, then a_q is adjacent to b_{q-1} and b_q .

Each $b_i \in V_2$ connects to two vertices in V_1 and two inV_3:

If$1 < i < q$, then b_i is adjacent to a_{i-1} , a_{i+1} and c_{i-1} , c_{i+1}.

If$i = 1$, then b_1 is adjacent to a_1 , a_2 and c_1 , c_2

If$i = q$, then b_q is adjacent to a_{q-1} , a_q and c_{q-1} , c_q .

Structural Properties

These adjacency rules create several notable paths and cycles in the graph:

A path from a_1 to a_q withinV_1.

A path from c_1 to c_q withinV_3.

A vertical path connecting$a_1 \to b_1 \to c_1$, and similarly$a_q \to b_q \to c_q$.

These four paths—$(a_1 \to a_q)$,$(c_1 \to c_q)$, $(a_1 \to c_1)$ and $(a_q \to c_q)$—together form a cycle of length $2q + 2$, which outlines the boundary of the graph $G_{Id}(Z_{3q})$.

Graph Parameters

The **order** of the graph (number of vertices) is$3q$.

The **size** of the graph (number of edges) is$6q - 2$.

An illustrative example of this graph structure is provided in Figure 3 (not included here), which visually represents the general form of$G_{Id}(Z_{3q})$.

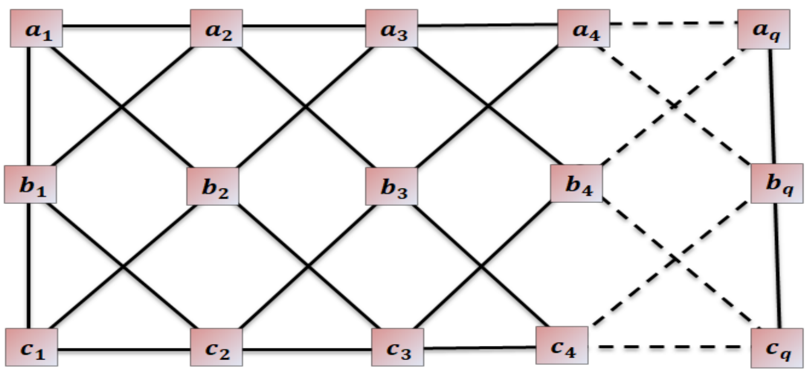


Figure 3. General form of the idempotent graph $G_{Id}(Z_{3q})$

Example 2: The Idempotent Graph of the Ring $Z_{15} \cong Z_{3.5}$, with $q = 5$
Consider the ringZ_{15}, where $q = 5$ is a prime number satisfying $q > 3$. The corresponding idempotent graph $G_{Id}(Z_{15})$. Can be constructed based on the partitioning of its elements into three disjoint subsets:
$V_1 = \{0, 3, 6, 9, 12\}$
$V_2 = \{1, 4, 7, 10, 13\}$
$V_3 = \{2, 5, 8, 11, 14\}$

Each subset corresponds to a congruence class modulo 3 withinZ_{15}, and each contains exactly 5 elements, consistent with the general structure defined for$G_{Id}(Z_{3q})$. The vertices inV_1, V_2 and V_3 follow the adjacency and degree patterns described in the general case: internal vertices typically have degree 4, while the endpoints of the paths in V_1 and V_3 have degree 3. The complete structure of the graph $G_{Id}(Z_{15})$ is illustrated in Figure 4

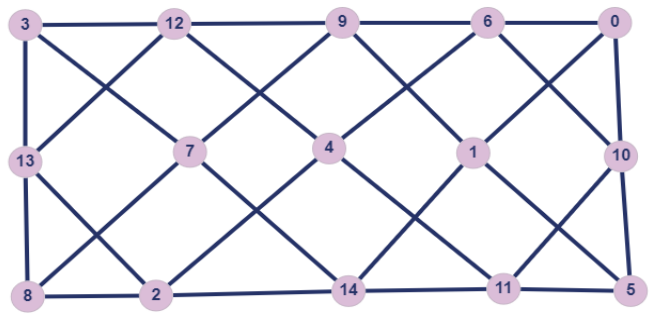


Figure 4. The idempotent graph $G_{Id}(Z_{15})$

2.3. The Idempotent Graph of the Ring Z_{5q}, Where $q > 5$ Is a Prime Number

Let q be a prime number greater than 5. Consider the ringZ_{5q}, the ring of integers modulo$5q$. The idempotent graph associated with this ring, denoted$G_{Id}(Z_{5q})$, is defined using the property that for every element $a \in Z_{5q}$, there exists another element $b \in Z_{5q}$ such that $(a + b)^2 = a + b$. This ensures that the graph has a specific structure influenced by the algebraic properties of the ring.
The vertex set of $G_{Id}(Z_{5q})$ is partitioned into five disjoint subsets:

$$V\big(G_{Id}(Z_{5q})\big) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5,$$

Where each partite set V_i contains exactly q vertices. These sets are defined based on congruence classes modulo 5:
$V_1 = \{a_1, a_2, a_3, \dots, a_q\} \subseteq \{0, 5, 10, \dots, 5q - 5\},$
$V_2 = \{b_1, b_2, b_3, \dots, b_q\} \subseteq \{1, 6, 11, \dots, 5q - 4\},$
$V_3 = \{c_1, c_2, c_3, \dots, c_q\} \subseteq \{2, 7, 12, \dots, 5q - 3\},$
$V_4 = \{d_1, d_2, d_3, \dots, d_q\} \subseteq \{3, 8, 13, \dots, 5q - 2\},$
$V_5 = \{e_1, e_2, e_3, \dots, e_q\} \subseteq \{4, 9, 14, \dots, 5q - 1\},$

The connections between the vertices in the graph are governed by a specific adjacency pattern:

- Vertices in V_1 form a simple path, i.e., each a_i is connected to a_{i+1} for$1 < i < q$. Additionally, each a_i is adjacent to b_{i-1} and b_{i+1} in V_2, where applicable. For boundary cases, a_1 connect to b_1 andb_2, and a_q is adjacent to b_q andb_{q-1}.
- Each vertex $b_i \in V_2$ is adjacent to two vertices from V_1 (namely a_{i-1} anda_{i+1}) and two from V_3 (namely c_{i-1} andc_{i+1}), with appropriate adjustments for the endpoints ($i = 1$ and $i = q$).
- The same adjacency rule applies recursively for the sets V_3 andV_4, i.e., each vertex in V_3 is connected to two vertices in V_2 and two inV_4, and each vertex in V_4 is adjacent to two vertices in V_3 and two in V_5.
- Finally, the vertices in V_5 also form a simple path from e_1 toe_q, analogous to the path formed inV_1. This adjacency structure naturally creates a sequence of connections such as $a_1 \to b_1 \to c_1 \to d_1 \to e_1$, forming a path from a_1 in V_1 to e_1 in V_5. A similar path exists from a_q to e_q. Along with the paths within V_1 and V_5, these connections create a cycle of length $2q + 6$, which forms the boundary of the graph $G_{Id}(Z_{5q})$.

In terms of degrees:
Most vertices in the graph have degree 4.
However, the vertices at the ends of the paths in V_1 and V_5, specifically a_1, a_q, e_1, and e_q , have degree 3 due to their positions in the graph structure.
The **order** (total number of vertices) of $G_{Id}(Z_{5q})$ is $5q$, while the **size** (total number of edges) is $10q - 2$.
A general representation of the idempotent graph $G_{Id}(Z_{5q})$ for $q > 5$ is illustrated in Figure 5

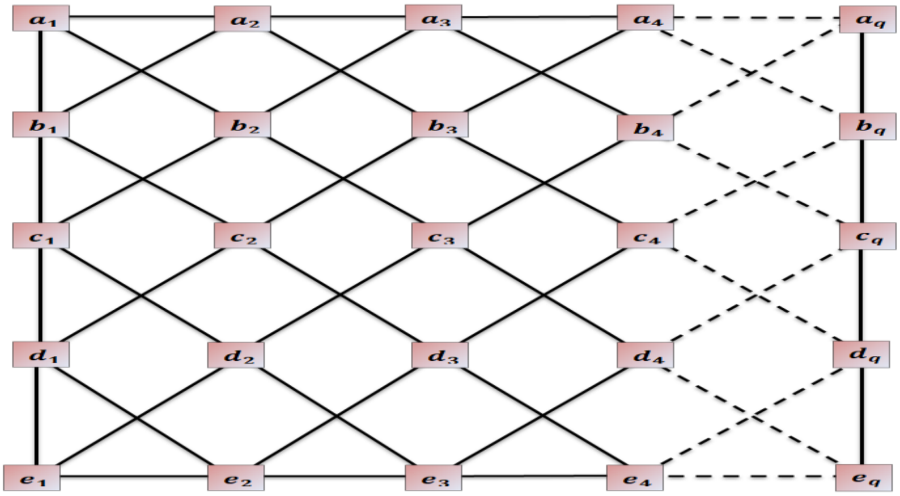


Figure 5. General form of the idempotent graph $G_{Id}(Z_{5q})$

Example 3. Let us examine the idempotent graph associated with the ring $Z_{35} = Z_{5.7}$ where $q = 7$. The vertex set of the graph $G_{Id}(Z_{35})$ is partitioned into five disjoint subsets as follows:

$V_1 = \{0, 5, 10, 15, 20, 25, 30\}$

$V_2 = \{1, 6, 11, 16, 21, 26, 31\}$

$V_3 = \{2, 7, 12, 17, 22, 27, 32\}$

$V_4 = \{3, 8, 13, 18, 23, 28, 33\}$

$V_5 = \{4, 9, 14, 19, 24, 29, 34\}$

Each subset V_i represents a distinct equivalence class of vertices under the defined graph structure. The corresponding idempotent graph $G_{Id}(Z_{35})$ is illustrated in Figure 6.

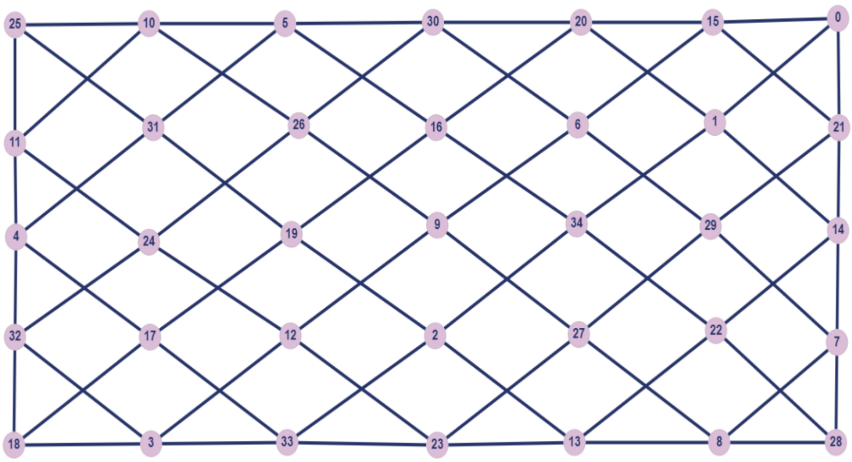


Figure 6. The idempotent graph $G_{Id}(Z_{35})$

Based on the observations and analyses in Sections 2.1, 2.2, and 2.3, we can derive a general structure for the idempotent graph corresponding to the ring Z_{pq}, where p and q are distinct prime numbers with $p < q$. The vertex set of the idempotent graph $G_{Id}(Z_{pq})$ can be partitioned into p distinct partite sets, denoted as follows:

$V_1 = \{a_i\} \subseteq \{0, 0+p, 0+2p, 0+3p, \dots, pq-p\}$

$V_2 = \{b_i\} \subseteq \{1, 1+p, 1+2p, 1+3p, \dots, pq-(p-1)\}$

$V_3 = \{c_i\} \subseteq \{2, 2+p, 2+2p, 2+3p, \dots, pq-(p-2)\}$

$\vdots$

$V_p = \{x_i\} \subseteq \{p-1, (p-1)+p, (p-1)+2p, (p-1)+3p, \dots, pq-1\}$

For $i = 1, 2, 3, \dots, q$. Collectively, the full vertex set of the graph is given by:

$$V\big(G_{Id}(Z_{pq})\big) = V_1 \cup V_2 \cup \dots \cup V_p$$

with each partite set containing exactly q vertices, i.e., $|V_1| = |V_2| = |V_3| = \dots = |V_P| = q$.
In this graph structure, each vertex in V_k is adjacent to two vertices in V_{k+1} for $1 \le k < p$. More specifically, if $e_i \in V_k$, then it is connected to h_{i-1} and $h_{i+1} \in V_{k+1}$. For boundary cases, if $i = 1$, then $e_1 \in V_k$ is adjacent to h_1 and h_2; and if $i = q$, then e_q is adjacent to h_q and h_{q-1}. This adjacency rule applies to all vertices e_i, $h_i \in V(Z_{pq})$.

Additionally, the vertices in V_1 form a linear path from a_1 to a_q, and similarly, the vertices in V_p form a path from x_1 to x_q. The vertices $\{a_1, b_1, \dots, x_1\}$ form a path from a_1 to x_1, while the sequence $\{a_q, b_q, \dots, x_q\}$ forms another path from a_q to x_q. These four paths collectively form a cycle of length $pq-(p-2)(q-2)$, which constitutes the boundary of the graph $G_{Id}(Z_{pq})$.
With respect to vertex degrees, every vertex in the graph has degree 4, except for the boundary vertices a_1, $a_q \in V_1$ and x_1, $x_q \in V_p$, each of which has degree 3. Notably, the elements $2a_1$, $2a_q$, $2x_1$, and $2x_q$ are idempotent in Z_{pq}.
In summary, the idempotent graph $G_{Id}(Z_{pq})$ has an order of pq and a total of $2pq-2$ edges. The general structural form of this graph is illustrated in figure 7

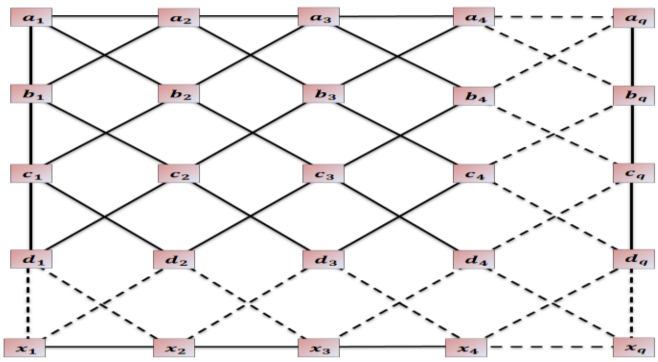


Figure 7. General form of the idempotent graph $G_{Id}(Z_{pq})$

Remark. Since the ring Z_{pq} is commutative, the identity $(a+b)^2 = a+b$ implies that $(b+a)^2 = b+a$ as well. Consequently, if a vertex a is adjacent to a vertex b, then b is also adjacent to a. Given that the graph $G_{Id}(Z_{pq})$ is simple, each pair of adjacent vertices is represented by a single undirected edge, ensuring no multiple edges exist between any two vertices $a, b \in Z_{pq}$.
However, if the graph were defined as a non-simple graph using the same adjacency condition, the commutativity of the ring Z_n would result in two directed edges between each adjacent pair of vertices—one in each direction. In such a representation, the general structure of the idempotent graph $G_{Id}(Z_{pq})$ would appear as depicted in figure 8.

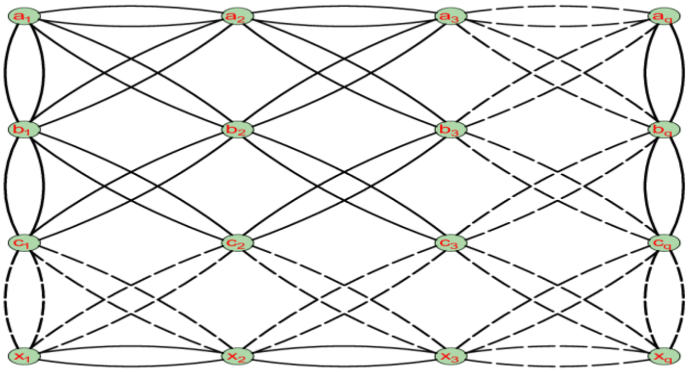


Figure 8. The idempotent graph

To characterize the idempotent graph of the ring Z_{pq}, where p and q are prime number with $p < q$, it is essential to introduce the concept of the direct product of two graphs. Direct Product:[11] The direct product of two graphs H and G, denoted by $H \times G$, id defined as a graph whose vertex set is the cartesian product $V\ (H) \times V\ (G)$. Two vertices (a, b) and $(a`, b`)$ in $H \times G$ are adjacent if and only if a is adjacent to $a`$ in $H$ and $b$ is adjacent to $b`$ in G. Formally,

$$V\ (H \times G) = \{(a, b) |\ a \in V(H), b \in V(G)\}$$

And

$$E\ (H \times G) = \{((a, b), (a`, b`)) |\ (a, a`) \in E(H),\ (\ b, b`) \in E(G)\}.$$

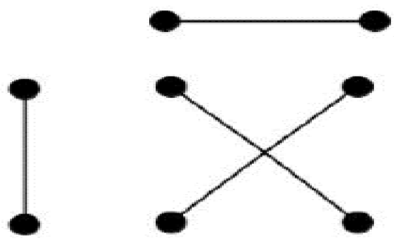
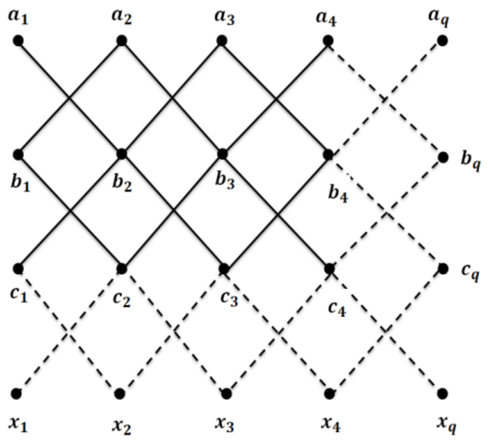


Figure 9. $P_2 \times P_2$

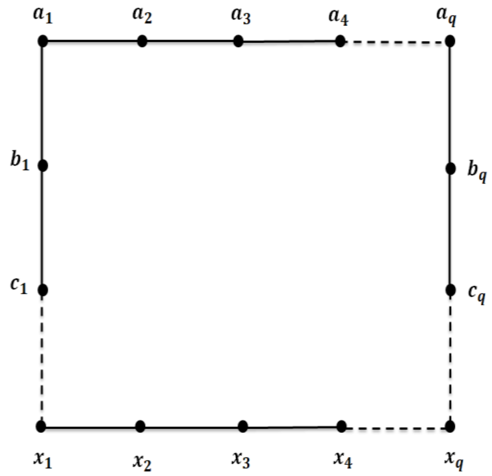
3. Some Fundamental Properties of the Idempotent Graph of the Ring Z_{pq} (Where p and q are Prime Numbers and $p < q$)

In this section, we examine several fundamental characteristics of the idempotent graph associated with the ring Z_{pq}, where p and q are distinct prime numbers satisfying p < q.

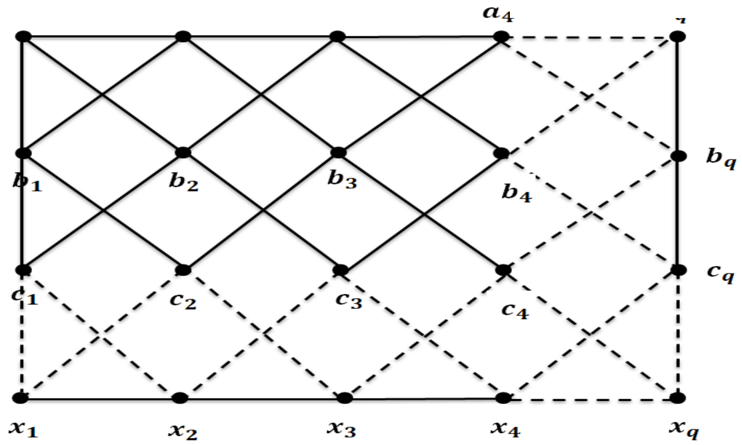
Theorem 1:
The idempotent graph of the ring Z_{pq}, denoted by $G_{Id}(Z_{pq})$, is a novel graph structure that can be described as $G_{Id}(Z_{pq}) \cong P_q \times P_p \cup C_n$, Where $n = pq - (p-2)(q-2)$.
Proof:
The vertex set of $G_{Id}(Z_{pq})$ is partitioned into p subsets, each containing exactly q vertices. Within this construction, each vertex in the subset V_k is adjacent to two vertices in the next subset V_{k+1}, thereby forming a direct product of two paths, specifically $P_q \times P_p$, as illustrated in figure 10. Furthermore, the graph incorporates a boundary structure in the form of a cycle of order $pq - (p-2)(q-2)$. The union of this cycle C_n with the Cartesian product $P_q \times P_p$ completes the construction of the idempotent graph $G_{Id}(Z_{pq})$, as depicted in figure 10.



$P_q \times P_p$ Direct product of two paths P_q and P_p in the graph $G_{Id}(Z_{pq})$



Border of the graph $G_{Id}(Z_{pq})$ on the shape cycle $C_{pq-(p-2)(q-2)}$



$P_q \times P_p \cup C_{pq-(p-2)(q-2)}$

Figure 10. The idempotent graph

Where $\{a_1, a_2, a_3, \dots, a_q\} \subseteq V_1$, $\{b_1, b_2, b_3, \dots, b_q\} \subseteq V_2$, $\{c_1, c_2, c_3, \dots, c_q\} \subseteq V_3$, …. , $\{x_1, x_2, x_3, \dots, x_q\} \subseteq V_p$

Corollary 2:
The idempotent graph of the commutative ring Z_{pq} (where p and q are prime numbers with $p < q$) is connected.
Proof:
In the idempotent graph $G_{Id}(Z_{pq})$, each vertex belonging to the subset V_k is adjacent to two vertices in the subsequent subset V_{k+1}. This structure ensures that for any pair of vertices in the graph, there exists at least one path connecting them. Consequently, the graph $G_{Id}(Z_{pq})$ is connected.
Corollary 3:
The idempotent graph $G_{Id}(Z_{pq})$ is non-planar.
Proof: According to Theorem 1, the idempotent graph can be expressed as
$G_{Id}(Z_{pq}) \cong P_q \times P_p \cup C_n$,
where $P_q \times P_p$ represents the direct product of two paths and C_n is a cycle of order n. It is well-known that the Cartesian product of two paths of order greater than or equal to 3 is non-planar. Hence, $G_{Id}(Z_{pq})$ is non-planar.

Theorem 4:
The chromatic number of the idempotent graph $G_{Id}(Z_{pq})$ is 3, i.e. $\mathcal{X}\left(G_{Id}(Z_{pq})\right) = 3$.
Proof:
Consider the vertex partition V_k for $k = 1, 3, 5, \dots, p$. These subsets can be colored using two alternating colors such that adjacent vertices are assigned distinct colors. For the remaining subsets V_k, where $k = 2, 4, 6, \dots, p-1$, a third color is used. This ensures a proper coloring of all vertices without any adjacent vertices sharing the same color. Thus, three colors suffice to color the entire graph $G_{Id}(Z_{pq})$, which implies $\mathcal{X}\left(G_{Id}(Z_{pq})\right) = 3$, as illustrated in Figure 11.

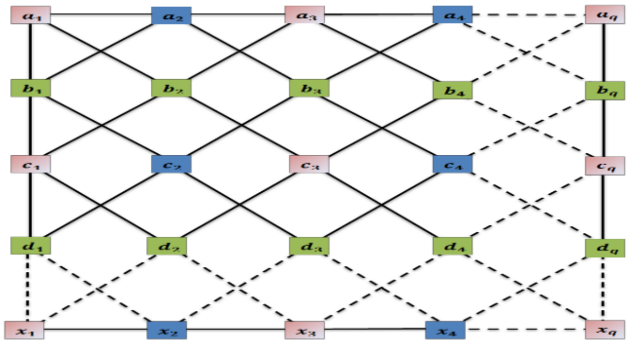


Figure 11. Chromatic number of the graph $G_{Id}(Z_{pq})$

Proposition 5: Diameter of idempotent graph $G_{Id}(Z_{pq})$ is equal to $q-1$, $dim\left(G_{Id}(Z_{pq})\right) = q-1$. The Radius of idempotent graph $G_{Id}(Z_{pq})$ is equal to p or equal to $\frac{q-1}{2}$, $rad\left(G_{Id}(Z_{pq})\right) = p$ or $rad\left(G_{Id}(Z_{pq})\right) = \frac{q-1}{2}$.

Proof: To find diameter and radius of the graph $G_{Id}(Z_{pq})$ we have two cases:
Case1: If $p = 2$ and $q > 2$ in the graph $G_{Id}(Z_{pq})$ the eccentricity of the vertices in $G_{Id}(Z_{2q})$ are equal to $q-1, q-2, q-3, \dots, \frac{q-1}{2}$ see Fig.1, q is prime number usually $q-1 > q-2 > q-3 > \dots > \frac{q-1}{2}$. Therefore, $q-1$ is the maximal eccentricity and $\frac{q-1}{2}$ is minimal eccentricity $G_{Id}(Z_{2q})$. Hence $dim\left(G_{Id}(Z_{2q})\right) = q-1$ and $rad\left(G_{Id}(Z_{2q})\right) = \frac{q-1}{2}$.
Case2: If p and q are prime number, $p \geq 3$ and $q > p$ in the graph $G_{Id}(Z_{pq})$, the eccentricity of vertex a is equal to $q-1$ or equal to p for all $a \in V(G_{Id}(Z_{pq}))$ see Fig.7. If p and q are prime number and $p < q$ then $p < q-1$ but if $p = 2$ and $q = 3$ then $p = q-1$ this is going to be the first case, so we are going to ignore it in this case, we are just going to get the part $p < q-1$ and in the graph $G_{Id}(Z_{pq})$ p and q are prime number and $p < q$, therefore the maximal eccentricity in graph $G_{Id}(Z_{pq})$ equal to $q-1$ and minimal eccentricity equal to p. Hence $dim\left(G_{Id}(Z_{pq})\right) = q-1$, $rad\left(G_{Id}(Z_{pq})\right) = p$.

Theorem 6

Let $G_{Id}(Z_{pq})$ denote the idempotent graph associated with the ring Z_{pq}, where p and q are distinct prime numbers such that $p < q$. Then, for every element $a \in Z_{pq}$, the degree of the corresponding vertex satisfies $deg(a) = 3$ or $deg(a) = 4$.

Proof.
This result is verified through the graphical representation illustrated in Figure7 (General form of the idempotent graph), which demonstrates the degree values of each vertex in the idempotent graph of Z_{pq}

4. Conclusion

This research introduces a new structural representation of the idempotent graph corresponding to the ring Z_{pq}, where p and q are distinct prime numbers. Building on this formulation, the study establishes several key graph-theoretic properties, including the diameter, radius, planarity, and vertex degree of the graph. The result enhances our understanding of how algebraic characteristics of rings relate to associated graph structures. Moreover, the framework developed here opens opportunities for future investigations, allowing scholars to extend this approach to a wider class of rings.

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