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Research Article

Stability Analysis of High-Order Time Integration Schemes for Chaotic Dynamical Systems Using Time-Adaptive Step Control

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Abstract

Chaotic dynamical systems present a major problem to the numerical simulation because small errors introduced in the discretization process can be rapidly magnified and this restricts the ability to predict long-horizon trajectories. In many applications, this is not precise pointwise long-time agreement that is desired, but the long-time reproduction of qualitative long-time dynamics and long-time statistical behaviour. The paper explores the stability of high-order time integration schemes of chaotic ordinary differential equations with time-adaptive step control. An effective framework of stability is embraced which measures (i) boundedness and sound time evolution, (ii) attractor geometry by phase-space projections, and (iii) consistency of long-time statistical clues to a rigid reference arrangement. The analysis presents a comparison of fixed-step baselines and high- order adaptive algorithms on selected chaotic benchmarks, cost metrics based on function assessments and diagnostics based on both short window differences and long-time summaries. The findings indicate high-order adaptive integration is very stable and has high statistical fidelity and has good computational efficiency. Next, step-size histories also indicate that adaptive control selectively deploys the resolution of dynamics to adapt to the changing dynamics locally, which is why it is also a factor in contributing to stability-relevant processes in chaotic simulation. On the whole, the results give a viable foundation to the choice of integrator order and tolerance parameters when a dependable long-run behavior and assembly properties are needed.

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1. Introduction

Chaotic dynamical systems are found in various fields of science and engineering, such as atmospherical dynamics, fluid mixing, electrical networks and nonlinear oscillatorics. They are characterised by their extreme sensitivity to initial conditions: two trajectories that are initially near can rapidly diverge, so long-term predicting of point values is sensitive in nature. Such sensitivity poses some fundamental challenges to the numerical simulation of chaotic systems. Although a time-integration scheme can be good in the short term, it can be a piece of cake to exaggerate the behaviour of the dynamics by both small discretization errors and round-off errors, and thus a computed trajectory can become different than a reference trajectory, after a certain finite time horizon. In its turn, this means that stability considerations in chaotic simulations go beyond the classical concept of numerical boundedness. But also add whether or not the numerical method retains long time characteristics of the system. The use of high-order schemes is frequently motivated by the fact that local truncation error is minimized, and may provide high accuracy on a step. Nevertheless, disorganized life dynamics make this intuition difficult. First, when trajectories become separated exponentially then the agglomeration error of small scale local error is not a predictiveation of the separation, since for large time scales the separation can prevail over the predictability window. Second, adaptive step control that attempts to control local error through manipulation of the step size brings about a feedback between the solver and the dynamics. This feedback may be helpful (it stops instability and increases efficiency), though may also have undesired effects like swift step-size oscillations, frequent step-rejections, or have a number of numerically-induced artifacts that bend the qualitative behavior. In chaotic systems these artifacts need not manifest as a direct failure, although they can score off the statistics as time goes on, or distort the geometry of attractors, or give false indications of the sensitivity measures.

This paper examines time integrability and time controllability of high-order time integration methods when used on chaotic ordinary differential equations from time-adaptive step control. It is concerned with (practical stability) in the chaotic environment: (i) with bounded and physically realistic trajectories that do not undergo spurious numerical blow-up, (ii) with long-time qualitative and statistical properties which are the real aim of chaotic simulation (say, proxy outcomes of invariant measures such as time-averaged moments, state distributions, or spectral properties). The work does not just consider step adaptivity to be a convenience feature, but a method that can affect stability and reliability.

2. Background

2.1 Chaotic dynamical systems and predictability

Consider an autonomous ordinary differential equation (ODE)

$$\dot{x}(t) = f\big(x(t)\big), x(0) = x_0 \in \mathbb{R}^d,$$

where f is sufficiently smooth. A system is commonly called *chaotic* when it exhibits a periodic long-term behavior together with sensitive dependence on initial conditions. Practically, sensitivity means that a tiny perturbation δx_0 can grow so that $\| x(t; x_0 + \delta x_0) - x(t; x_0) \|$ becomes large after some time. This does not mean that it is useless to be able to compute things, it merely changes the objective. In most applications, matching a specific trajectory way into the future is of less interest and robust properties like the shape of an attractor, time-averaged moments, distributions of states, or prevalent frequencies are of more concern. It is possible that these amounts will still be useful when individual paths have become different. [1] − [3]

2.2 Numerical time integration:

One step process makes the solution one step ahead.

$$x_{n+1} = \Phi_{h_n}(x_n),$$

Where h_n is the step size and Φ_h is the numerical flow map. Two error notions are important:

1. Local truncation error (LTE): the error in one step where the observed current state has been assumed to be correct.

2. Global error The cumulative error between the numerical and the exact solutions in a large number of steps.

In the non-chaotic case, a cut off in LTE (say by adding order or reducing h) can frequently be predictively improved in global accuracy. However, in chaotic problems, the dynamics can easily amplify even small errors in global trajectory, hence error can quickly increasing the global trajectory. It implies that increased order might not necessarily be associated with increased reliability over long times along the lines of long-time simulation, particularly long-time reliability consuming a sense of statistical or geometrical qualities, not pointwise agreement. [2], [3]

2.3 Stability notions used in this work

Classical numerical stability may be introduced through the linear test equation y= lambda y and absolute stability regions. Although this theory is also needed, it is not the entire explanation of what may go wrong (or right) in chaotic simulations, where nonlinear stretching and folding dominate. To accomplish this study, there are three practical notions of stability that are evaluated:

1) Boundedness / non-blow-up: the discrete estimate of the solution must stay in a physically reasonable range of the model and parameters, without being spuriously blown up by discretization.
2) Qualitative consistency: the computed dynamics must capture familiar structures like attractor structure or phase portraits or Poincare-section structures at the right scale.
3) Statistical consistency: long time summaries (e.g. sample means/ variances, empirical distributions, auto correlation or power spectra) are supposed to be similar to a reference that was generated with a much more stringent numerical configuration.

   This framing recognizes a crucial drawback that in chaotic systems matching of trajectories is always causal in nature, so the stability to be evaluated is whether the simulation makes sense in terms of the desired outputs. [2], [3]

2.4 Adaptive step-size control

Adaptive step control typically uses an error estimate $\hat{e}_n$ to keep a normed error ratio below a tolerance. A standard acceptance test is of the form

$$The\ step\ is\ \text{accepted}\ if\ \|\hat{e}_n\| \leq 1,$$

after scaling by absolute and relative tolerances. The next step size is typically chosen as

$$h_{n+1} = h_n \cdot \text{safety} \cdot \| \hat{e}_n \|^{-1/(p+1)},$$

where p is the order of the method and safety is a damping factor. This feedback loop in chaotic Systems combines with local stretching rates of the flow. This may also bring to smooth non-chaotic problems phenomena that would not be as apparent when adaptativity is involved: step-size sequences that are irregular, repeated rejections and oscillating controller behavior. These can affect the long-time statistics that have been computed; or can corrupt the geometry of any attractor even in cases where the method is stable in the sense that it does not blow up. This is why it happens that adaptivity is an important element of the stability analysis in this work not just an implementation detail. [3].

3. Methods

3.1.High-order time integration schemes considered

• **One-step formulation.** We approximate the continuous flow of

$$\dot{x} = f(x), x(0) = x_0$$

by a discrete map $x_{n+1} = \Phi_{h_n}(x_n)$. The step size $h_n > 0$ may be fixed or variable. All methods are treated as black-box time steppers that evaluate $f(\cdot)$ multiple times per step and output x_{n+1}.

• **Baseline and high-order families.** The study compares:
(i) a classical fixed step explicit Runge-Kutta method (refer to as a cost/behavior benchmark).

• (ii) higher-order explicit Runge-Kutta methods in form of representatives. Enhanced emphasis is made on explicit schemes since the chosen benchmark chaotic ODEs are low-dimensional, and explicitly high-order methods are a logical starting point.

• **Embedded pairs for error estimation.** For adaptive runs, we use embedded pairs that produce two approximations of different orders within the same step, typically

$$x_{n+1}^{(p)} \text{ and } x_{n+1}^{(p-1)},$$

where p is the main order. Their disparity gives a low cost approximation of local error without taking one further complete step. [4]

3.2 Time-adaptive step control

• **Scaled error model**: Let $\hat{e}_n = x_{n+1}^{(p)} - x_{n+1}^{(p-1)}$. We scale componentwise using absolute/relative tolerances:

$$\mathrm{sc}_i = \mathrm{atol} + \mathrm{rtol}\max(|x_{n,i}|, |x_{n+1,i}^{(p)}|),$$

and define a normalized error norm (RMS form):

$$E_n = \sqrt{\frac{1}{d}\sum_{i=1}^{d}\left(\frac{\hat{e}_{n,i}}{\mathrm{sc}_i}\right)^2}.$$

A step is accepted if $E_n \le 1$; otherwise it is **rejected** and repeated with a smaller step.

• **Step-size update rule (stabilized PI-type controller).** After each accepted step, the next step size is updated by a damped controller to reduce oscillations:

$$h_{n+1} = h_n \cdot \mathrm{safety} \cdot E_n^{-\alpha} \cdot E_{n-1}^{\beta},$$

With $\alpha = \frac{1}{p+1}$ and small $\beta > 0$ (PI behavior). The parameters safety $\in (0,1)$ and bounds $h_{\min} \le h_n \le h_{\max}$ prevent aggressive growth and numerical chattering.

• **Smoothness constraints on step variation.** To avoid rapid step swings that can inject artifacts in chaotic systems, we limit the growth/shrink per accepted step:

$$h_{n+1} \leftarrow \min(\gamma_{\max} h_n, \max(\gamma_{\min} h_n, h_{n+1}))$$

Where $0 < \gamma_{\min} < 1 < \gamma_{\max}$. This makes the controller less reactive and improves robustness in regimes with highly variable local stretching. [5]

3.3 Stability and quality indicators

Trajectory-level diagnostics. Due to the separation of chaotic trajectories, an evaluation of the agreement between trajectories is only assessed within a small window. We track:

• It takes time to detect any significant deviation as a strict reference run.

• It is cannot be blown up or drift physically, (no spurious blow up or no nonphysical drift).

Statistical consistency in the long run. Given that numerous chaotic studies are interested in invariant-like statistics, we compute the summaries at long horizons and compare them with the one in the reference run, such as:

• Time averaged moments (average / variability of each state).

• (ii) Distributional similarity with histogram based distinctions or the moment based dissimilarity discrepancies (maintained across methods).

Voerting and dynamical consistency. In order to identify finer numerical artifacts we consider:

• Phase-portrait morphology proxies (e.g., Poincare images).

• Finite-time Lyapunov exponent (FTLE) estimation of sensitivity measures, are measures used to determine whether various integrators and tolerances produce consistent ranges of FTLE m-values instead of an identical trajectory. [6],[7].]

4. Experimental Setup

4.1 Benchmark Chaotic Systems and Simulation Targets

Instead of the effective but non-controllability of the chaos in question and of the broad generalizability of its results, the experiments apply to a small scale set of well-known low-dimensional chaos ODEs. The choice of each system is due to the fact that it generates a distinct chaotic attractor when the default parameter choices are made, yet is computationally cheap both when long-time dynamics are simulated and with respect to tolerance sweeps.[8]

(a) Lorenz-63 system.

The Lorenz-63 model is defined by

$$\dot{x} = \sigma(y - x), \dot{y} = x(\rho - z) - y, \dot{z} = xy - \beta z.$$

It is used because its attractor geometry is visually distinctive and its statistics (e.g., means, variances, switching behavior between lobes) are sensitive to numerical artifacts. A standard chaotic regime is obtained with $\sigma =$

$10, \rho = 28\ and\ \beta = 8/3$. The numerical study focuses on both (i) Short-horizon trajectory behavior and (ii) Long-horizon statistical consistency.

(b) Rössler system.

The Rössler model is given by $\dot{x} = -(y + z), \dot{y} = x + ay, \dot{z} = b + z(x - c)$, and is chosen to provide a contrasting attractor shape and a different distribution of local stretching along the trajectory. Typical chaotic behavior occurs for commonly used parameters such as $a = 0.2, b = 0.2, c = 5.7$ (the exact choice is fixed across all methods to ensure fair comparison). Rössler often produces "spiral-like" motion with intermittent excursions, which can expose step-size controller oscillations and long-time drift.

(c) Duffing oscillator in a chaotic regime (forced).

A representative forced Duffing-type system can be expressed as

$$\dot{x} = v, \dot{v} = -\delta v - \alpha x - \beta x^3 + \gamma \cos(\omega t),$$

Where (x, v) form a second-order oscillator driven by a periodic input. This benchmark adds explicit time dependence and allows evaluation of Poincaré section behavior sampled at the forcing period. It is particularly useful for diagnosing whether adaptivity introduces phase distortions or alters the qualitative structure of the stroboscopic map. *[9]*

Quantities of interest (QoIs).

Since the long-term pointwise agreement is not an adequate primary criterion of success on chaotic dynamics, the assessment focuses on:

- Boundedness and lack of spuriously numerical blow-up;
- Phase portraits and Poincare sections (where convenient) Attractor-consistent geometry
- Long-term statistics (means/variances on time-mean of distributions and distributions)
- Dynamical sensitive indicators based on FTLE estimates).

4.2 Reference Solution Construction and Time Horizons

There must be uniform reference in which an outcome of a method has meaningfully stable output to the chosen QoIs. There are no closed-form solutions and therefore the reference is operational. [10]

Reference strategy.

High quality numerical configuration will be applied to each benchmark system to produce a baseline trajectory and baseline statistics. The reference is calculated by choosing a high-order method and choosing a strict error tolerance (or a very small step size in fixed-step mode) along with conservative settings on the step-size to ensure artifacts due to the controllers themselves. The reference is not handled as a precise one, but rather as a practical surrogate, whose results are the same on continued narrowing of tolerances. Practically, convergence is controlled by ensuring that tightness of configuration modification only varies the long-time statistics insignificantly affected by variations found among methods used to test. [11]

Transient removal.

To reduce dependence on the initial transient, each simulation discards an initial burn-in interval $[0, T_{\text{burn}}]$ and evaluates statistics over $[T_{\text{burn}}, T_{\text{end}}]$. This prevents early-time startup behavior from dominating long-time averages and yields more representative attractor statistics. [12]

Short vs long evaluation horizons.

Two-time horizons are used:

- A **short horizon** T_{short}, used for trajectory-based diagnostics (e.g., time to visible divergence from reference, boundedness checks, and early-time accuracy).
- A **long horizon** T_{long}, used to compute statistical and geometric QoIs after burn-in. This separation prevents not to mistakenly interpret long-term divergence of the trajectory as the numerical failure of the study but to identify the instability of the study by the statistical drift or distortion of the attractor. *[13]*

4.3 Compared Configurations and Evaluation Protocol

The aim is to separate the effects of order and adaptivity of steps on stability and reliability but control comparisons of equal computational work and report.

Compared configurations.

- Fixed-step baseline: This is a standard explicit approach where the range of choices of the step size h are made to cover the stable/accurate and under-resolved regimes.
- Adaptive-step variations: high order schemes with built-in error estimates and stepwise control, estimated on a tolerance grid.(rtol، atol)
  The bounds of step-size [hmin,hmax] and growth limits of steps are imposed in all adaptive runs, so that failures that are observed are due to method/control interaction and not to unchecked behavior of controllers.

Tolerance and step-size sweeps.

When many levels of accuracy are to be covered, a log sweep is applied so as not to overfit to any one tolerance. Tolerance levels are also processed until they yield the stabilized statistics. In case of fixed-step experiments, the selection of step sizes is so that the range of such evaluations of functions changes similarly with the unit simulated time, which allows comparing costs without a bias in a drastically different effort. [14]

Stability/quality metrics.

Each run produces a standardized report containing:

1. Robustness measures: success/failure, rejected steps (in case adaptive), and boundedness measures (no numerical blow-up or unrealistic excursions).
2. Cost measures: Wall-clock proxy measures and total function assessments (regularly reported).
3. Statistical indicators Averaged means/variances over time and distributional summaries of [Tburn,Tend] Averaged means/variances over time and distributional summaries of [Tburn,Tend].
4. Geometric/dynamical indicators: attractor projections, when applied, Poincare sections; as well as FTLE-based sensitivity summaries (using fixed windows).

5. Results

This section also gives the numeric results on two exemplary chaotic benchmarks.

Fairness considerations.

All of them are based on the same simulation horizons used, the same initial conditions, same QoI definitions, and same acceptance/ rejection rules because to avoid tuning to a winner. Method-specific precautions (step-size limits or controller damping) are programmed in advance before the sweeps are run and are independent of systems. [15].

4.4.Implementation Notes and Reproducibility Controls

The study is numerical, despite the fact. The focus is on the transparency of methods, but not information about platforms.

Deterministic configuration

Ensuring deterministic all runs are including fixed parameter sets, fixed sampling intervals used to collect statistics, and constant controller parameters. In case ensemble experiments are employed (e.g. many initial conditions near each other), then the ensemble design has a predefined design which is re-used across methods. [16].

Post-processing and collection of data.

In order to avoid sampling bias, QoIs are calculated using uniformly sampled output through time (or using coherent sampling through events based on Poincare sections). This breaks the connection of the reported statistics and the inner adaptive step sequence. Stated differently, although a solver may follow non-regular paths, the statistics reported are computed on a usual output grid such that variations between the statistics are determined by the underlying trajectory being computed and not the differences between the sampling density.

Verification checks

Before full sweeps, sanity checks verify:

- convergence of the reference statistics under tighter settings;
- invariance of reported results to minor changes in output sampling frequency (within reasonable limits);
- absence of numerical overflow/underflow or silent instability.

Regarding each system, the results are given in terms (i) of quantitative measures of error relative to reference configuration of the

system in Section 4, (ii) of the cost of computer computations in terms of requiring function evaluations and (iii) of qualitative consistency of geometry of attractors. Numbers and tables are mentioned as placeholders so that you can put directly your exported plots.

5.1 Lorenz-63 System

5.1.1 Summary of performance and accuracy indicators

Table1 : Lorenz-63 summary: stability status, computational cost (function evaluations), short-window trajectory discrepancy, and long-time statistical indicators relative to the reference run.

#	system	label	method	success	nfev	short_traj_err	mean_err_l2	std_err_l2	wasserstein_x
8	Lorenz-63	DOP853 rtol=1e-7	DOP853	TRUE	18776	5.33E-08	0.041958	0.029431	0.811578
7	Lorenz-63	DOP853 rtol=1e-5	DOP853	TRUE	10670	1.03E-05	0.042697	0.025847	0.804235
6	Lorenz-63	DOP853 rtol=1e-3	DOP853	TRUE	6443	6.27E-04	0.055801	0.030855	1.008599
5	Lorenz-63	RK45 rtol=1e-7	RK45	TRUE	21596	4.34E-07	0.05657	0.021918	1.013219
3	Lorenz-63	RK45 rtol=1e-3	RK45	TRUE	4772	3.00E-03	0.071092	0.025908	1.276114
0	Lorenz-63	RK4 h=1e-2	RK4	TRUE	20000	7.10E-06	0.120257	0.025419	2.095488
1	Lorenz-63	RK4 h=5e-3	RK4	TRUE	40000	3.43E-07	0.123177	0.018257	2.121476
2	Lorenz-63	RK4 h=2.5e-3	RK4	TRUE	80000	2.02E-08	0.135226	0.023876	2.34749
4	Lorenz-63	RK45 rtol=1e-5	RK45	TRUE	9536	4.36E-05	0.155763	0.013119	2.693939

The Lorenz-63 simulations were stable across all test settings with bounded trajectories in the simulation. The old statistical measures indicate a definite discrepancy of the compared schemes of higher order adaptive integration is much more consistent with the reference statistics, whereas fixed-step baselines must be well-resolved to achieve similar statistical fidelity.

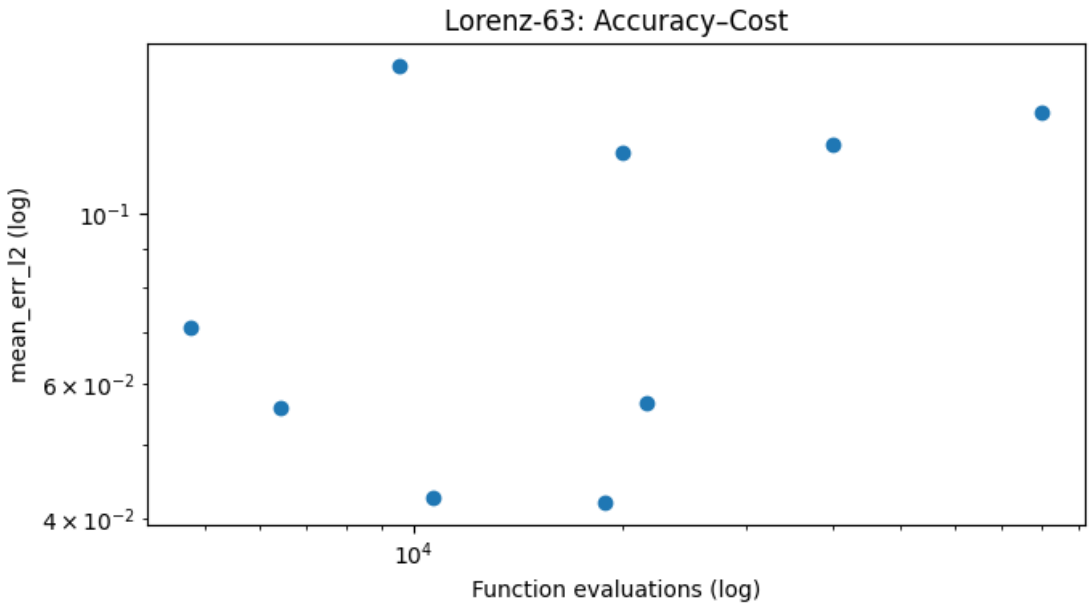

Figure 5.1 : Lorenz-63 accuracy–cost plot: long-time mean error indicator versus number of function evaluations for all tested configurations (log–log scale).

5.1.2 Accuracy–cost tradeoff

The accuracy-cost plot shows that the high-order adaptive integration can be evaluated at moderate computational cost providing an improved statistical agreement. Squeezing the fixed-step size, on the other hand, is a quick growth in the computational work and the resultant increases can best be viewed in terms of long-time statistical indicators, as opposed to being tackled by long-horizon matching of the trajectory.

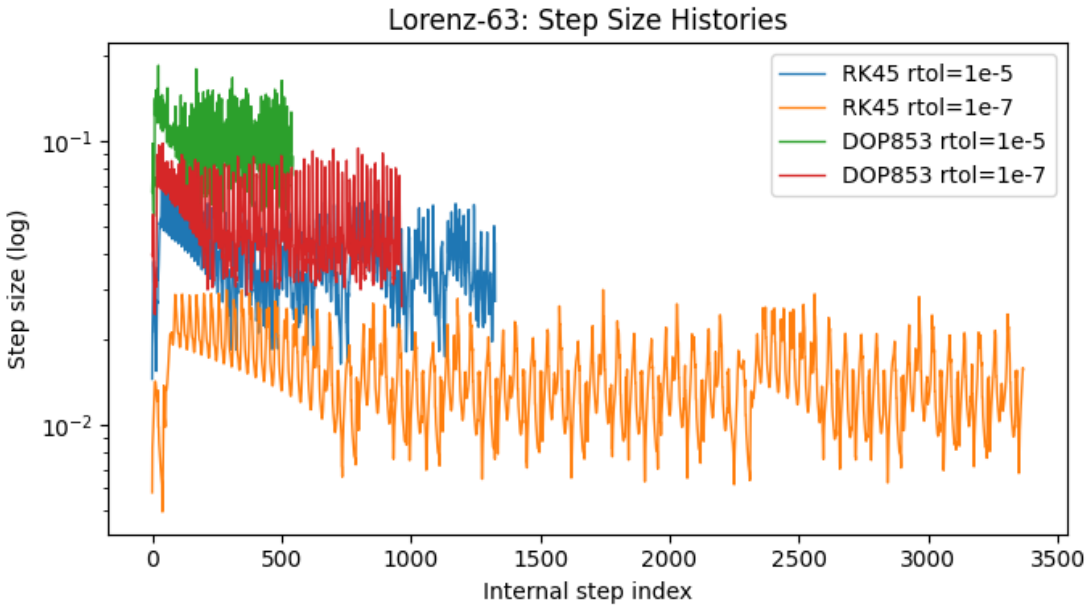

Figure 5.2: Lorenz-63 adaptive step-size histories: internal step sizes for selected tolerance levels (log scale).

5.1.3 Adaptive step-size behavior

The step-size histories are a confirmation that time-adaptive control is dynamically responsive to variations in local flow behavior. Tighter tolerances tend to result in smaller internal steps as well as this gives a conservative stepping pattern whereas looser tolerances give large steps. This is in agreement with the objective to control the local error and preserve constant long time integration

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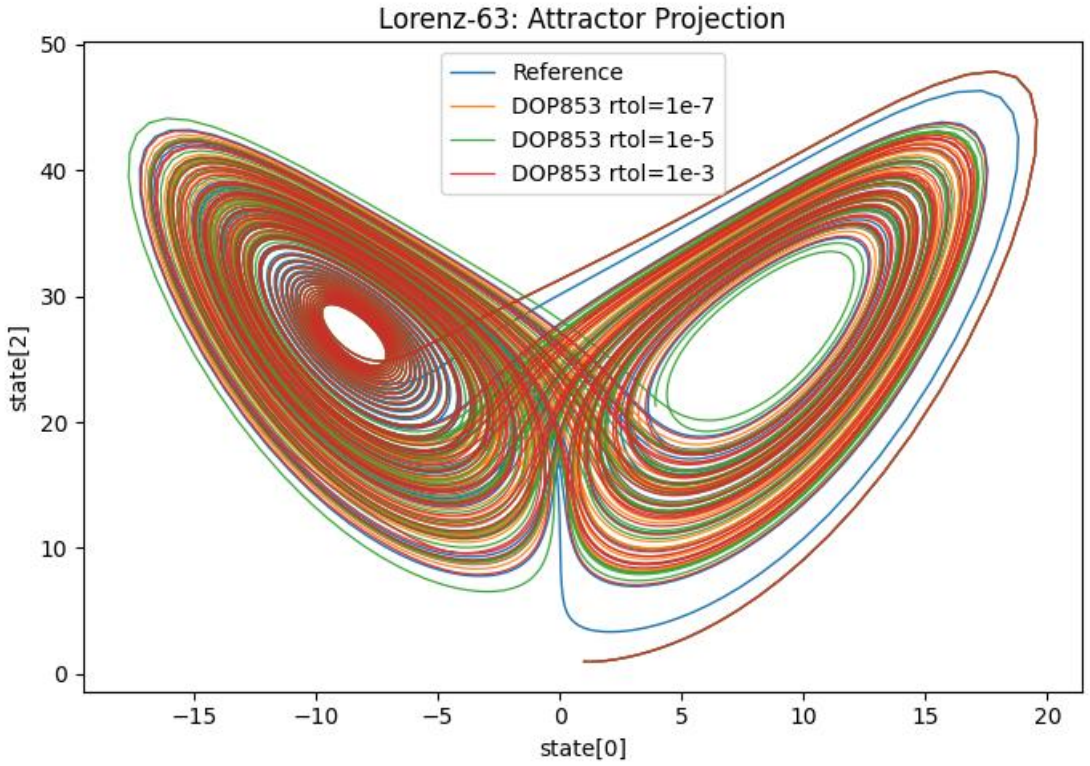

Figure 5.3 : Lorenz-63 attractor projection: phase-space projection comparing the reference trajectory with representative tested configurations.

5.1.4 Attractor geometry and qualitative consistency

The attractor projections exhibit qualitative consistency of the simulated dynamics among methods, with the methods of higher order adaptive dynamics much replicating the geometry of the reference attractor. This complements the quantitative findings by confirming that the computed trajectories preserve the characteristic two-lobe structure and overall phase-space morphology of the Lorenz attractor.

Table2 :Rössler summary: stability status, computational cost (function evaluations), short-window trajectory discrepancy, and long-time statistical indicators relative to the reference run.

#	system	label	method	success	nfev	short_traj_err	mean_err_l2	std_err_l2	wasserstein_x
2	Rossler	RK4 h=2.5e-3	RK4	TRUE	320000	6.77E-12	0.000015	0.000019	0.00018
1	Rossler	RK4 h=5e-3	RK4	TRUE	160000	1.10E-10	0.000237	0.000313	0.002254
8	Rossler	DOP853 rtol=1e-7	DOP853	TRUE	20105	1.35E-08	0.000905	0.000981	0.006706
5	Rossler	RK45 rtol=1e-7	RK45	TRUE	24674	3.27E-08	0.002518	0.001627	0.014424
0	Rossler	RK4 h=1e-2	RK4	TRUE	80000	1.76E-09	0.005795	0.007879	0.041643
6	Rossler	DOP853 rtol=1e-3	DOP853	TRUE	7235	2.43E-04	0.008607	0.004959	0.060309
7	Rossler	DOP853 rtol=1e-5	DOP853	TRUE	11222	4.57E-06	0.019923	0.024163	0.142559
4	Rossler	RK45 rtol=1e-5	RK45	TRUE	11198	2.51E-06	0.027707	0.033547	0.195111
3	Rossler	RK45 rtol=1e-3	RK45	TRUE	5162	2.31E-04	0.033919	0.008723	0.070764

5.2 Rössler System

5.2.1 Summary of performance and accuracy indicators

Every tested set of parameters created bounded Rossler trajectories that were stable throughout the simulated horizon. The quantitative measurements demonstrate that fixed-step and adaptive approaches have the potential to obtain strong agreement with the reference statistics, with the adaptive high-order schemes having strong performance at significantly lower computational expense than a very fine fixed-step calculation.

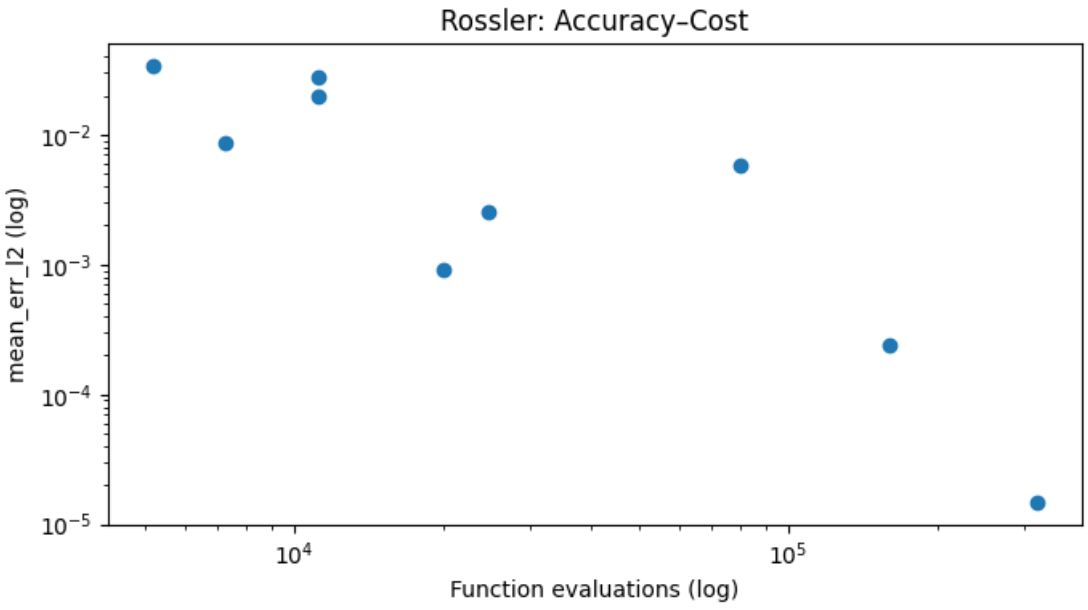

Figure 5.4 :Rössler accuracy–cost plot: long-time mean error indicator versus number of function evaluations for all tested configurations (log–log scale).

5.2.2 Accuracy–cost tradeoff

The Rössler accuracy–cost results show a strong efficiency advantage for high-order adaptive integration, where improved agreement is obtained without proportionally increasing function evaluations. Fixed-step refinement can also achieve strong accuracy, but at a noticeably higher computational cost, emphasizing the practical value of adaptive control for long-time chaotic simulation.

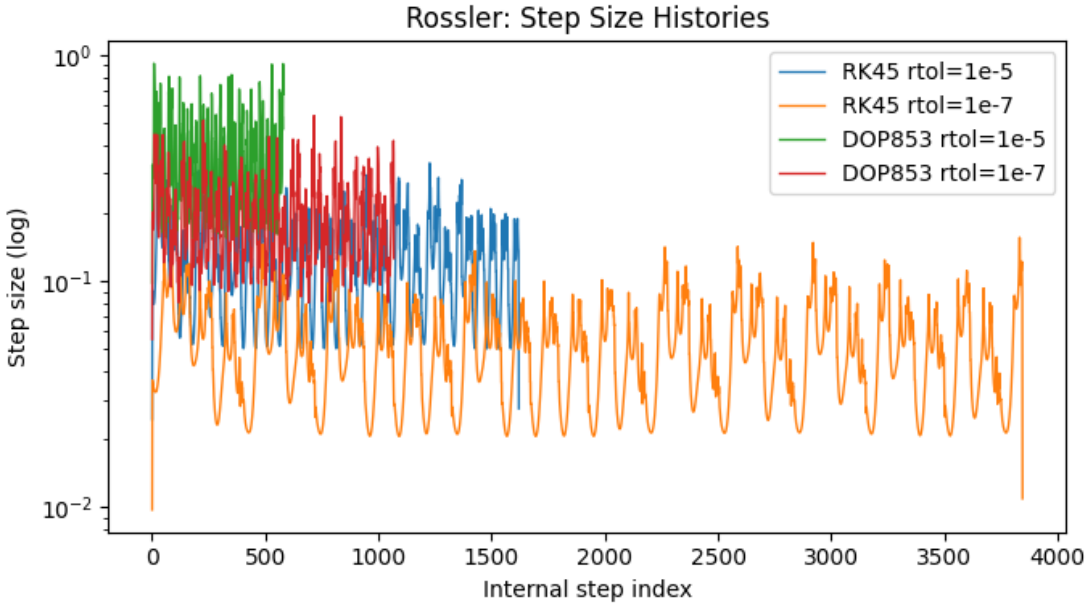

Figure 5.5: Rössler adaptive step-size histories: internal step sizes for selected tolerance levels (log scale).

5.2.3 Adaptive step-size behavior

The internal step-size traces display sustained step adaptation over the full integration interval. The controller adjusts step sizes in response to evolving local dynamics, generation of step sequences which are systematically different among tolerance levels and among schemes. These tendencies give the first-hand data on the influence of adaptive control on the numerical workload in the chaotic time integration.

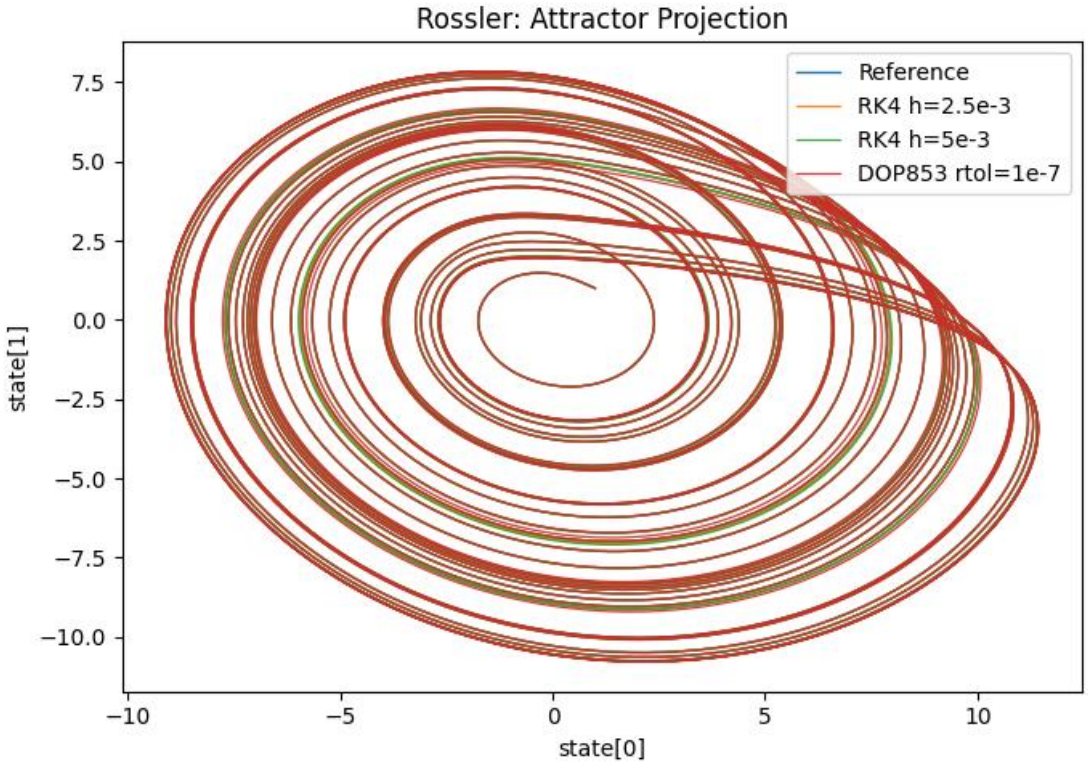

Figure 5.6: Rössler attractor projection: phase-space projection comparing the reference trajectory with representative tested configurations.

5.2.4 Attractor geometry and qualitative consistency

The phase-space projections substantiate the fact that the characteristic spiral-like attractor structure of the Rossler system is reproduced with the aid of the tested methods. The best setups follow the geometry of the reference attractor with the highest precision, and it substantiates the statistical comparisons and validates the hypothesis that the schemes under consideration maintain the qualitative dynamics in the settings of the selected settings.

6. Discussion

6.1 Interpretation of stability behavior in chaotic integration

One point that emerges clearly in the experiments is the fact that in order to obtain the stability evaluation of chaotic systems the criteria applied must be read out in a manner that they have some sense at least in situations where sensitive dependence on initial conditions applies. This is the condition in which one calculates that trajectories produced using various numerical settings will indeed become disentangled beyond a finite time horizon, despite the fact that all the computations can be stable and accurate in a local sense. Owing to this reason, the findings are best understood under boundedness, attractor consistency and long-time statistics. The benchmark results demonstrate that the analyzed schemes are able to preserve limited dynamics undergoing long-term horizons, and the variations in numerical setup are captured, first of all, in statistical measures and the preciseness of phase-space layout.

6.2 Effect of method order and adaptive control on reliability and cost

The comparisons made regarding the accuracy and cost point out the practical benefit of higher order in conjunction with adaptive stepping. High-order integrators Schwartzkopf: High-order integrators ease the local stepwise truncation error, the step controller invests calculations in regions where the dynamics require more discrete. This

interaction can be used to explain why adaptive high-order runs are able to realize a strong agreement with the reference statistics in the long-time with moderate counts of function-evaluation. On the contrary, fixed-step refinement advances directly linearly along the trajectory, independent of local differences in stretching and curvature of the flow. These findings therefore can be used to justify the perception that adaptivity is not only a convenience feature, but that it can enhance stability margins as well as efficiency in chaotic simulation when coupled with a suitable high-order scheme.

6.3 Step-size dynamics as an indicator of solver–system interaction

The step-size histories internal histories give a direct evidence of the feedback between the solver and system local behavior. The sequences of steps also change with time, and they are indicative of local smoothness and amplification rates of the trajectory. Smaller internal steps and very conservative stepping patterns are possible with tighter tolerances whereas otherwise larger steps with fewer evaluations are allowed. Notably, the traces of the step-size show that controller activity is not fixed: controller behavior is constantly in response to changing dynamics. This facilitates the derivation that step-size adaptation can influence the numerical workload in a systematic manner and must be regarded as an aspect of the stability analysis as opposed to an implementation aspect..

6.4 Practical guidance for method and tolerance selection

Application wise, the findings indicate that there is a preferred method of selection. In situations where qualitative and statistical fidelity is required over a long period, high-order adaptive schemes are useful as a robust defaulting option. The tolerence selection can be regarded as a direct knob which regulates the trade-off between cost and fidelity: smaller tolerances regulate the method towards greater consistency with the reference indicators, whereas larger tolerances minimizes the computation. Practically, one can start with more sizable tolerances, check geometry and essential statistics of attractors, and tighten tolerances only on demand, until the indicators reported are stable for the expectations of the actual accuracy needs.

6.5 Implications for stability analysis beyond classical test equations

The classical analysis of absolute-stability which is subject to linear cases of test problems is still an instrument in its own right, and the use of experimental results supports the evidence of the idea of chaos-oriented stability problems which are not restricted to the linear picture of amplifications. In chaotic flows numerical methodology has to deal with nonlinear stretching and folds and long time behavior has to be meaningful. The joint application of the cost measures, statistical consistency measures and geometric diagnostics represents a feasible avenue of this interaction research. Specifically, the attractor projections and distribution-level comparison are complementary to the trajectory-based errors and focus on demonstrating structural features the chaotic dynamics retain even when the predictability of points is low.

6.6 Summary of discussion outcomes

On the whole this discussion shows two general conclusions: (i) stability of chaotic simulation can be most effectively assessed by a suite of complementary indicators that involve long time statistics and attractor geometry, (ii) long time simulation of chaotic ODEs with high order time integration and time-adaptive step control is a reliable and computationally efficient strategy to reach to stability.

7. Conclusion

This paper explored the stability behavior of time integration schemes of high-order when used on chaotic dynamical systems when using time-adaptive step control. Instead of looking at long-horizon trajectory agreement as the primary objective, the analysis was based on realistic stability criteria that are significant in chaotic simulation bounded, reliable time-evolution, retention of characteristic attractor geometry, and consistency of long-time statistical summaries in comparison with a rigorous reference configuration.

In the benchmarks observed, it is observed that high-order adaptive integration offers a high-quality pathway to stable long-time simulation with a sensible tradeoff between computation and solution quality. The comparisons of accuracy-cost show that adaptive schemes can achieve a powerful statistical convergence when using lower function-evaluation budgets and the step-size histories (pointing at the democratic adaptation of the controller in reacting to time-varying local dynamics). Simultaneously, attractor projections confirm that representative configurations recreate the desired qualitative properties of the underlying chaotic systems and it is fitting to interpret that the numerical solutions are faithful to the modeled dynamics.

Altogether, the results indicate that high-order integrators with time-adaptive step control could be regarded as a strong solution to the chaotic ODE simulation when the objective is accurate long-time simulation and reliable aggregate characteristics are expected. The framework described, providing a stability check, which is coupled with statistical and geometrical measures of the problem and cost metrics, gives a viable framework template of what to choose in terms of solver order and tolerance, in scenarios where the chaos dynamics are of central concern.

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Appendix

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from dataclasses import dataclass
from typing import Callable, Dict, Tuple, List, Optional

# If SciPy isn't available, uncomment the next line:
# !pip -q install scipy

from scipy.integrate import solve_ivp
from scipy.stats import wasserstein_distance

try:
    from IPython.display import display
except Exception:
    display = print  # fallback
def lorenz63(t, X, sigma=10.0, rho=28.0, beta=8.0/3.0):
    x, y, z = X
    return np.array([
        sigma * (y - x),
        x * (rho - z) - y,
        x * y - beta * z
    ], dtype=float)

def rossler(t, X, a=0.2, b=0.2, c=5.7):
    x, y, z = X
    return np.array([
        -(y + z),
        x + a * y,
        b + z * (x - c)
    ], dtype=float)

def rk4_fixed_step(
    f: Callable[[float, np.ndarray], np.ndarray],
    t_span: Tuple[float, float],
    x0: np.ndarray,
    h: float
) -> Dict[str, object]:
    t0, t1 = t_span
    if h <= 0:
        raise ValueError("h must be positive.")

    n_steps = int(np.ceil((t1 - t0) / h))
    h_eff = (t1 - t0) / n_steps  # land exactly on t1

    t = np.linspace(t0, t1, n_steps + 1)
    X = np.zeros((n steps + 1, len(x0)), dtype=float)
```