The Tilling Using Ω-Class of Nested Abacus

Authors Names	**ABSTRACT**
[1]Eman F. mohommed , [2]Jalal Abd jassim Publication date: 25 /12 /2025 *Keywords: Tiling, Class, Partition theory, k-connected ominoes, Abacus*	This study applies a class of Abacus diagrams, referred to as the Ω-class of Nested Abacus (N.C.A), to the problem of tiling within finite regions. An algorithm was developed using mapping notation to represent the movement of subsets containing multiple β-positions and empty β-positions. The new class was embedded to facilitate tiling in a finite rectangular region. Furthermore, a theoretical model was formulated to describe the tiling process in finite regions based on the proposed Nested Abacus approach.

1.Introduction

This study examines the representation of several types of polyominoes (k-connected ominoes) through the lens of partition theory. A polyomino consists of k-ominoes joined edge to edge in a plane. The k-connected ominoes have been widely used to construct various puzzles. Since at least 1907, several findings involving segments from k = 1 to 6 ominoes were first reported in Fairy Chess papers between 1937 and 1954 under the title "Dissection Issues" [2]. Golomb (1954) proposed linking adjacent squares into polyominoes, defining a single square as a monomino with an interior that is connected [3]. Klarner (1966) defined a connection as a collection of a finite number of square units without a cut point, as this field further developed [4]. Many combinatorial problems involve k-connected ominoes, which can generally be classified into two primary categories: enumeration and plane tiling [5–8]. These problems relate to the broader challenge of articulating the concept of shape beyond conventional examples such as circles or squares, encompassing complex forms that may contain one or more voids, including polyominoes (k-connected ominoes). Let σ denote a region in the plane. A tiling of σ refers to the configuration of polyominoes situated within σ, such that each cell of σ is covered by exactly one of these polyominoes, each referred to as a tile. This study further investigates k-connected ominoes designed for finite configurations using a graphical representation known as the nested Abacus [9–11], to define k-connected ominoes with one or more holes. Finally, we provide remarks and preliminary definitions related to the contracting tiling algorithm presented in this section..

1.1 FUNDAMENTAL DEFINITIONS

Some fundamental definitions needed for this research have been established.

Definition 1. A point lattice is an array of points that are uniformly spaced; the array is often called a grid.

Remark 2. The (N.C.A) is engraved in the grid with $\bar{g}$ columns and $\bar{h}$ rows so that every place in the (N.C.A) is a point lattice, every point in the plan lattice is the vertices of a unit square.

Remark 3. The (N.C.A) is engraved in the grid with $\bar{g}$-columns and $\bar{h}$-rows, then every place in the (N.C.A) is a point lattice, every point in the plan lattice is a vertices of unit squares.

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According to Definition 1 and Remark 2, the (N.C.A) is represented as a rectangular diagram containing bead and empty positions. The bead positions form the image, while the empty positions create the background. The next figure shows the (N.C.A) embedded in a grid with $\bar{g}$-columns and $\bar{h}$-rows, Fig.1.(a) shows an original (N.C.A) with vacant and bead position, colored in blue and gray, respectively.

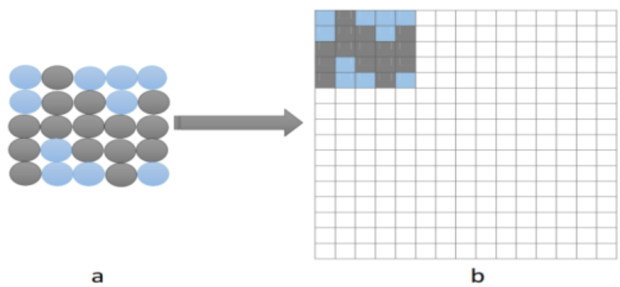

Figure 1 (a) N.C.A for 15-connected beads and (b) The N.C.A for 15- beads embedded in a finite grid.

Next, we construct a new class of k-connected beta numbers. Files must be in MS Word only and formatted for direct printing using the CRC MS Word provided. Figures and tables should be embedded and not supplied.

2. Ω- Class

Definition 4: adjusted beta numbers (β) are an aggregation of beta numbers such that

β_(j-) β_i=1 or β_(j-) β_i=b.

Definition 5: A sequence of beta is an aggregation of adjusted beta numbers located in a row or column.

Definition 6: A set-rows sequences (SRS) is a sequence of beta numbers located in rows.

Definition 7. Ω- Class (N.C.A.) consists of a full chain (chain number n) and adjusted set-rows sequences of adjusted beta in each row, as shown in the next figure.

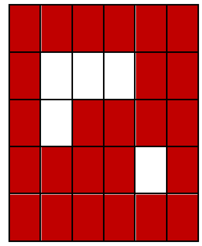

Figure 2 Ω- CLASS WITH 6 COLUMNS, 5 ROWS AND 25 BETA

In the present work, an algorithm using N.C.A. with k-beta to tiling a finite region was proposed using Ω- nested Abacus with b-columns and d-rows as follows:

2.1 Tilling With Ω- Class Of N.C.A.

For the present of this work, a tile is a set of nonempty compact $\sigma \subset R^2$ which is the closure of its interior. a tiling is a collection of tiles $\sigma = \{\sigma_i \cup | \sigma \in N\}$ that is a covering (i.e. $i \in N, \sigma_i = R^2$) as well as a packing (i.e. the intersection of the interiors of any two distinct tiles τ_i and τ_j is empty)Let σ be a set of Ω- Class (tiles)with b

Columns and d rows. Let σ be a region with $\grave{b}$ columns and $\grave{d}$ rows. Possibly, there are several ways to tile σ. In this work, we will tile the rectangles using the Ω-class mapping notation.

A symmetry of the tiling σ is a map isometry that maps every tile from σ to a tile of σ. Two tiles σ_1 & σ_2 of a tiling are equivalent if the has a symmetry map of the tiling from σ_1 onto σ_2. The set of all equivalents of the tiles σ_1 is the transitivity-class of σ_1. An isohedral tiling is one with only one transitivity class. A k-isohedral tiling is a tiling of transitivity classes with $k > 1$. If a prototile admits an isohedral tiling, it is said to be isohedral. If a prototile admits a k-isohedral tiling, it is k-isohedral.

3. FIRST STEP: CREATING Ω- CLASS

In this step, a Ω- Class of N.C.A. is constructed by fox the number of columns (b), rows (d) and k beta partitions into d

1. In Ω- Class, all ρ are beta numbers where $\rho = \{0,1,\ldots,b-1\} \cup \{(d-1)b,(d-1)b+1,\ldots,d(b-1)\} \cup \{b,2b,\ldots,(d-2)b\} \cup \{2(b-1),3(b-1),\ldots,(d-1)(b-1)\}$ (outer chain).
2. The set $S = [b, d, L_\sigma]$ be an initial parameter such that each is the number of columns, rows and beta positions located in row σ and inner chain where $2 \le \sigma \le d-1$.
3. Identify the head row beads to find the original Ω- Class of N.C.A.

Theorem 1: The number of Ω- nested Abacus Class of N.C.A. with $[b, d, k]$ such that each is the number of columns, rows and beta, is

$$\frac{x^{d-2}(d-2)!\,[(b-2)(d-1)-\alpha]}{(1-x)(1-x^2)\ldots..(1-x^{d-2})}$$

Where $\alpha = k - 2(b + d - 2)$.

Proof : Since the outer number has 2(b+d-2) positions and Ω- nested Abacus Class with full outer chain in, then the inner chains have $k - 2(b + d - 2)$ beta, where k is the number of beta in Ω- nested Abacus Class. The number of partitions α, beta into $d - 2$ is $(d - 2)!$. Every partition of $(d - 2)!$ can be product $[(b - 2)(d - 1) - \alpha]$ Ω- Class.

Since the number of partitions of any integer n with d parts is

$$\frac{x^{d-2}}{(1-x)(1-x^2)\ldots..(1-x^{d-2})}$$

Thus, the number of Ω- Class with $[b, d, k]$ is

$$\frac{x^{d-2}(d-2)!\,[(b-2)(d-1)-\alpha]}{(1-x)(1-x^2)\ldots..(1-x^{d-2})}.$$

SECOND STEP: CREATING MAPS IN ISOMETRY.

In this section, a new map isometry τ_1, τ_2 and τ_3 to tilling a finite region with $\grave{b}$ columns and $\grave{d}$ rows were construct.

1-A symmetry of the tiling T with isometry maps (τ) that every tile from $\mathfrak{I}$ to a tile of $\mathfrak{I}$ using beta number β as follows: Let β_v be a beta number located in Ω- Class with d rows and d columns such that $\beta_v = (m - 1)b + (j - 1)$ where $1 \leq m \leq d - 1$ and $1 \leq j \leq b - 1$

$$\tau_1(\beta_v) = \beta_v + b, \text{ where } 0 \leq m \leq r - 1,\ 0 \leq j \leq e - 1 \text{ and } 1 \leq n \leq \left\lceil \frac{\grave{b}}{b} \right\rceil - 1$$

$$\tau_2(\beta_{vn}) = \beta_v + nb\ ,$$

$$\tau_3(\beta_{vn}) = \tau_1(\beta_{vn}) + \grave{n}d\grave{b}\ , \quad \text{where } 1 \leq \grave{n} \leq \left\lceil \frac{\grave{d}}{d} \right\rceil - 1$$

$$\tau_4(\beta_{vn}) = \tau_2(\beta_{vn}) + \grave{n}d\grave{b}\ , \quad \text{where } 1 \leq \grave{n} \leq \left\lceil \frac{\grave{d}}{d} \right\rceil - 1.$$

Figure 3 illustrates how to use τ_1, τ_2, τ_3 and τ_4 to tilling a finite reign with 10 rows and 12 columns

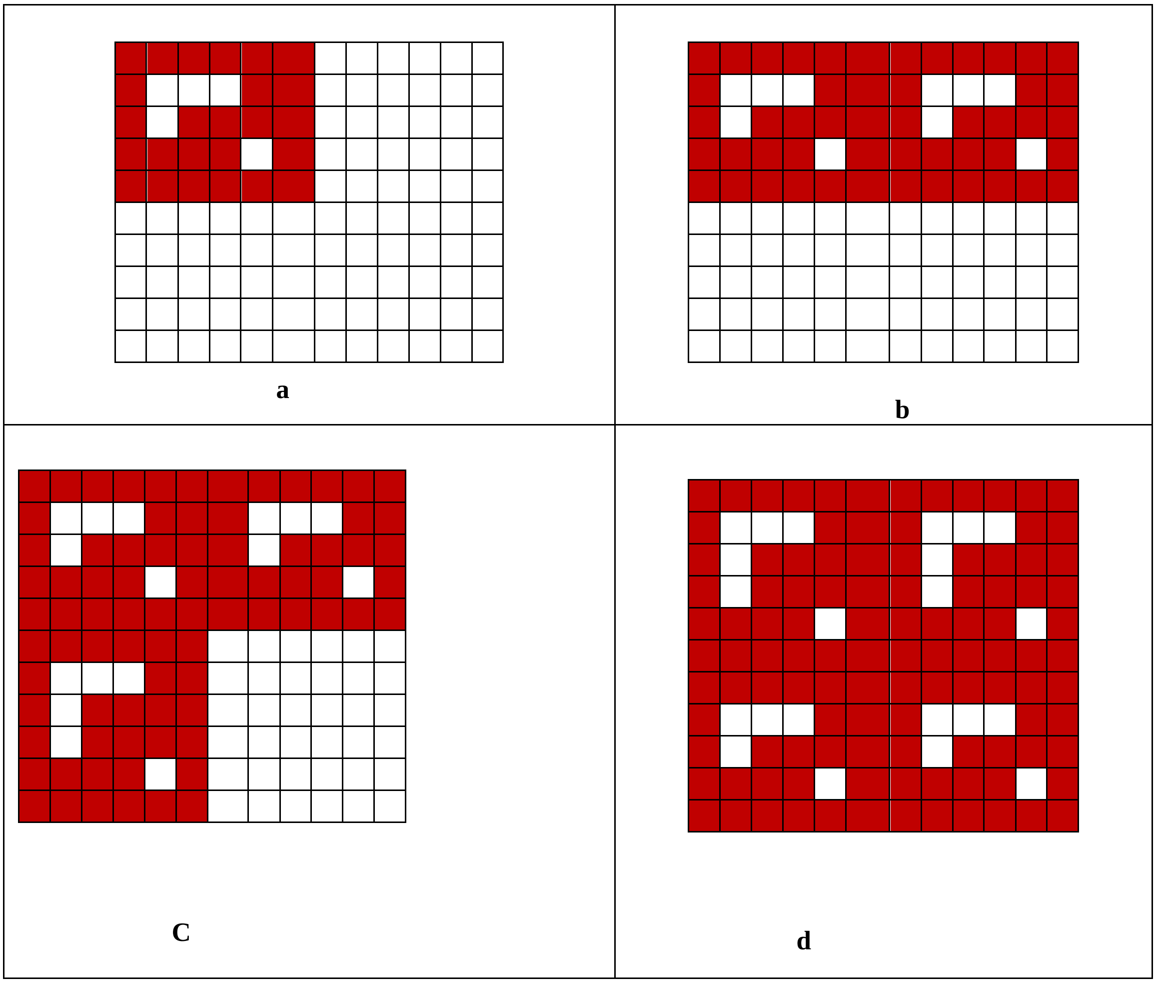

Figure 3 (A) Ω- CLASS OF N.C.A. WTH COLUMNS, 5 ROWS, 25 BETA (B) T_1 (C) T_2 (D) T_3 (E) T_4

1- A symmetry of the tiling T with isometry maps (Γ) that every tile from $\mathfrak{I}$ to a tile of $\mathfrak{I}$ using beta number β as follows: Let β_v be a beta number located in Ω- Class with d rows and d columns such that

$\beta_v = (m-1)b + (j-1)$ where $1 \leq m \leq d-1$ and $1 \leq j \leq b-1$

$$\Gamma_1(\beta_v) = \beta_v + (nb + o) \text{ , where } k = \{b, b-1, .., 1\}, o = \{1,2,5, \ldots\}, n = \{0,1,2, \ldots, \left[\frac{\grave{d}}{d}\right]\}$$

$$\Gamma_2(\beta_v) = \beta_v + (nd + o).$$

Figure 3 illustrates how to use Γ_1 and Γ_2 to tilling a finite reign with 10 rows and 12 columns

In table 1, we can calculate different nested Abacus Class of N.C.A. with b = 6, d = 5, and k = 25 using MATLAB programming.

Table 1. The different of Ω- nested Abacus Classof NCA with b = 6, d = 5 and k = 25

Ω- nested Abacus Class of N.C.A.	1 Beta in row 2	3 Beta in row 3	3 Beta in row 4
1	10	14	19
2	10	13	19
3	10	13	20
4	10	14	20
5	9	14	19
6	9	13	19
7	9	13	20
8	9	14	20
9	8	14	19
10	8	13	19
11	8	13	20

4. THEORETICAL RESULTS

In the next theorems, we found several theoretical results related with Ω- Class of N.C.A. using the tiling algorithm.

Theorem 1. Let b' be the number of empty bead positions. The generating function for the number of Ω- of N.C.A. has the following ordinary form:

$$xy\sum_{d\geq 1}(2n-1)^{d-1}y^{d-1}\sum_{b\geq 1}x^{b-1}.$$

Proof. based on the ordinary form, the generating function of the partition n beta on d columns is

$$\sum_{e,r\geq 1}(2n-1)^{d-1}x^{b}y^{r} = xy\sum_{e,r\geq 1}(2n-1)^{d-1}x^{b-1}y^{d-1}.$$

Since $\sum_{e,r\geq 1}x^{b-1} = 1 + x + x^2 * \dots\dots = \frac{1}{1-x}$

Then $xy\sum_{r\geq 1}(2n-1)^{d-1}y^{d-1}\sum_{e\geq 1}x^{b-1}$.

Theorem 2. Let Ω- Class be a NCH with b columns, d rows and q beta insecript in finite region, then there are

$$\left|\frac{pd}{\grave{b}d}\right|\prod_{i=1}^{\infty}\frac{1}{1-x^{i}}$$

Where Ω- Class can be tiling finite region with $\grave{b}$ columns and $\grave{d}$ rows where $\grave{b} > ub$ and $d \grave{>} ud$ such that $u \geq 2$.

Proof: first, the generating function $\pi(x) = \sum p(n)x^{n}$ to count all the partitions of n beta here $n = q - 2b - 2d + 4$, into d parts. We can choose an arbitrary λ-partition of n by independently determining how many times to include σin λ for each positive integer i. Every time σis used as a component, it adds σto the overall size n. The choice's generating function. Therefore $1 + x^{i} + x^{2i} + \cdots = \frac{1}{1-x^{i}}$, by multiplying for every i, we get $\prod_{i=1}^{n}\frac{1}{1-x^{i}}$. Since $\left|\frac{pd}{\grave{b}d}\right|$ is the number of Ω- Class where $\pi(x) = 1$. Thus there are

$$\left|\frac{pd}{\grave{b}d}\right|\prod_{i=1}^{\grave{b}d}\frac{1}{1-x^{i}}$$

Ω- Class can be tiling finite region with $\grave{b}$ columns and $\grave{d}$ rows where $\grave{b} > ub$ and $d \grave{>} ud$ such that $u \geq 2$.

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Ω- Class can be tiling finite region with $\grave{b}$ columns and $\grave{d}$ rows where $\grave{b} > ub$ and $d \grave{>} ud$ such that $u \geq 2$.

5. Conclusions

In this study, a new class of N.C.A., referred to as the Ω-Class of N.C.A., was generated using the full chains method and embedded in a finite grid. Furthermore, the proposed class was applied to develop an algorithm based on map isometry for tiling a rectangular region. Several theoretical results were also proposed, proved, and formulated.

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