Posterior Moments and Bayes Estimators of the Scale Parameter in Half-Normal Distribution Under Different Priors Based on SELF and DLF with Simulation Study

العزوم اللاحقة ومقدرات بيز لمعلمة القياس للتوزيع الطبيعي النصفي باستعمال معلومات أولية مختلفة بالاعتماد على SELF وDLF مع دراسة محاكاة

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Abstract

The Half-Normal distribution is used for modeling non-negative data, particularly in reliability and lifetime analysis to study product failure times and measurement error. In this study, we investigated Bayesian analysis of the Half-Normal distribution to obtain different estimators of unknown scale parameter (φ) using three different informative priors which are represented by the inverse Maxwell–Boltzmann (IMB), the inverse Nakagami-b (INK) and the inverse half-normal (IHN), based on two different loss functions which are represented by the squared error loss function (SELF), which is known as quadratic loss function and De Groot loss function (DLF). In addition to the maximum likelihood estimator (MLE). We introduced close-form by deriving the posterior distribution and the J^{th} posterior moments where j=1,2,3,4.

In order to evaluate the performance of different estimators of unknown scale parameter (φ), we conducted Simulation Study by generating data for different sample sizes of the HND with several values for the true value of the scale parameter (φ). The accuracy of the different estimates for φ was compared based on three different criteria: the mean relative estimate (MRE) and the mean biased estimate (Bias) and the mean square error (MSE). We obtain the best estimators for the scale parameter when the MRE and Bias and MSE tend to one (zero). In general, the simulation results showed that Bayes estimators better performance than the maximum likelihood estimator (MLE) for all sample sizes. Also, we concluded that Bayes estimators for scale parameter where the true value for scale parameter is $\varphi = 1.25, 1.5$ under the Inverse Nakagami-b (INK) with (2,2) as prior distribution, according to the smallest value for MSE based on DLF for all n.

Keywords: The Half-Normal Distribution (HND), the Maximum Likelihood Estimation (MLE), Bayesian estimation, Posterior distribution, Posterior moments, squared error loss function (SELF), De Groot loss function (DLF).

المستخلص

استعمل توزيع Half-Normal للبيانات ذات القيم الموجبة, بالتحديد في تحليل بيانات العمر والثقة لدراسة أوقات الفشل وخطأ القياس. في هذه الدراسة, تحرينا عن تحليل بيز لتوزيع Half-Normal لاستحصال مقدرات لمعلمة القياس المجهولة (φ) باستعمال ثلاثة مقدرات أولية مختلفة مثلت باستعمال توزيع معكوس ماكسويل- بولتزمان (inverse Maxwell–Boltzmann (IMB)) وتوزيع معكوس Nakagami-b (inverse Nakagami-b (INK)) وتوزيع معكوس inverse Half-Normal (half-normal (IHN)), اعتمادا على دالتين مختلفتين التي مثلت بدالة خسارة الخطأ التربيعي (SELF) و التي تعرف بدالة الخسارة التربيعية ودالة خسارة De Groot. بالإضافة الى مقدر الإمكان الأعظم (MLE). قدمنا صيغة تقريبية من خلال اشتقاق التوزيع اللاحق واشتقاق الصيغة العامة للعزوم اللاحقة من الرتبة J^{th} عندما تكون j=1,2,3,4. لتقييم أداء المقدرات المختلفة لمعلمة القياس (φ), أجرينا دراسة محاكاة بتوليد بيانات لاحجام مختلفة من العينات لتوزيع Half-Normal بعدة قيم للقيمة

الحقيقية لمعلمة القياس (φ). اعتمادنا على ثلاثة معايير لمقارنة دقة التقديرات لمعلمة القياس: مثلت بمتوسط التقدير النسبي (MRE) ومتوسط تقدير التحيز (Bias) ومتوسط مربع الخطأ (MSE).

نحصل على افضل المقدرات لمعلمة القياس عندما تقترب قيم متوسط التقدير النسبي (MRE) من الواحد وعندما تقترب قيم متوسط تقدير التحيز (Bias) وقيم متوسط مربع الخطأ (MSE) من الصفر.

بشكل عام بينت نتائج المحاكاة بان مقدرات بيز افضل من مقدر الإمكان الأعظم (MLE) لكل حجوم العينات. أيضا استنتجنا بان مقدرات بيز لمعلمة القياس عندما تكون القيمة الحقيقية لمعلمة القياس تكون مساوية لـ $\varphi=1.25,1.5$ تكون افضل باستعمال توزيع معكوس Nakagami-b بقيم لمعلمتي التوزيع مساوية لـ $(2,2)$ كتوزيع اولي, وفقا لاقل قيمة لمتوسط تقدير التحيز (Bias) اعتمادا على دالة الخسارة التربيعة (SELF) لكل حجوم العينات (n). و ان افضل مقدرات لمعلمة القياس عندما تكون القيمة الحقيقية لمعلمة القياس تكون مساوية لـ $\varphi=1.25,1.5$ باستعمال توزيع معكوس Nakagami-b بقيم لمعلمتي التوزيع مساوية لـ $(2,2)$ كتوزيع اولي, وفقا لاقل قيمة لمتوسط مربع الخطأ (MSE) اعتمادا على دالة الخسارة التربيعة (DLF) لكل حجوم العينات (n).

الكلمات المفتاحية: توزيع Half-Normal, تقدير الإمكان الأعظم (MLE), تقدير بيز, التوزيع اللاحق, العزوم اللاحقة, دالة الخسارة التربيعة (SELF), دالة خسارة De Groot (DLF).

1. Introduction

The Half-Normal distribution is a continuous probability distribution that is derived from the Normal distribution but is restricted to only positive values. It is characterized by a single scale parameter (φ), which determines the width of the distribution. So , the half- normal (HN(φ)) distribution is generated by taking the absolute value of the data generating from the normal distribution with mean equal to zero and sigma (φ). The half- normal (HN(φ)) distribution is linked with skewed positive data, and it to describe lifetime process under fatigue. The half-normal distribution (HND) applied to the probability–severity risk analysis traditionally performed through risk matrices, also used as a model for (left) truncated data from application areas as diverse as fibre buckling, blowfly dispersion, sports science [1] physiology and stochastic frontier modeling [2]. Many studies are done on the characteristics of the half- normal (HN(φ)) distribution using Bayesian estimation under different informative priors based on different Loss Functions. For example, Pewsey in 2002 [1] used likelihood function for the original parametrization of the general half-normal distribution. He used the maximum likelihood (ML) estimate to obtain the estimators for the parameters of the general half-normal distribution, and the distributions of the ML estimators. Also, he introduced large-sample methods for the construction of confidence sets for the parameters, and he discussed the use of such confidence sets as the basis for hypothesis testing. He proved an application of some of the new inferential procedures to a data set. And in 2008 Wiper and Giron and Pewsey [2] fitted a half-normal distribution to the data using both the maximum likelihood and Bayesian estimation for both the location and the scale parameter with and interval estimators. They derived Bayes estimators for the location and the scale parameter under the natural uninformative prior distribution, and they concluded that sampling based procedures such as Gibbs sampling are not necessary for the implementation of Bayesian inference for the half-normal distribution. Also, Cooray and Ananda in 2008 [3] investigated the importance of the generalized half-normal (GHN) Distribution through the Hazard Function of Generalized Gamma distribution. And they introduced the properties of the GHN Distribution such as mode, median, inflection points, kth moment, variance, k^{th} limited expected value. Also, they examined how the values of the shape parameter and the scale parameter of the GHN Distribution affect the shape of the probability density and the hazard rate function. They used maximum likelihood estimation to estimate the two parameters of the GHN Distribution. And they obtained the confidence intervals for parameters via large sample theory using the expected Fisher information matrix.

In 2010 Kang and Kim and Lee [4] used Bayes approach to obtain the estimators for the common location parameters in half-normal distributions for different groups of orderings of $(\xi;\eta_1;\eta_2)$ based on an appropriate penalty term of Ghosh and Mukerjee [5] (1992) priors according to the following formulas:

1- $\pi_1(\xi;\eta_1;\eta_2)\,\alpha\,\eta_1^{-1}\eta_2^{-1}$

2- $\pi_2(\xi;\eta_1;\eta_2)\ \alpha\ \eta_1^{-1}\eta_2^{-1}(n_1\eta_1^{-1}+n_2\eta_2^{-1})$ with $n_1, n_2>0$

3- $\pi 3(\xi;\eta_1;\eta_2)\ \alpha\,(n_1\eta_1^{-1}+n_2\eta_2^{-1})^c$ with c >0.

Also, they derived posterior distributions for the location parameters in the half-normal distributions. They investigated the frequentist coverage probabilities for the three reference priors through simulation study. They concluded that the reference priors π_1 and π_3 perform better than the reference prior π_2 in matching the target coverage probabilities.

Also, Alzaatreh and Knight in 2013 [6] proposed the gamma-half normal distribution as new distribution, was derived using the method of the T-X families. They studied the properties of the distribution which are including the moments, mean deviations from the mean and median, hazard function, modality and Shannon entropy. They proposed the maximum likelihood method to estimate the parameters of the gamma-half normal distribution.

And in 2020 Wallner [7] study the sufficient conditions for a half-normal distribution and he used the normally distributed random variable *X* with mean zero to generate the half-normal by computing the absolute value |X|. Also Bosch-Badia and Serrats and Rodon in 2020 [8] applied the half-normal distribution to estimate the probability–severity risk analysis traditionally accomplished by using risk matrices and continuous probability–consequence diagrams (CPCDs). They developed a model that applied to the financial risk measures Value-at-Risk (VaR) and Conditional Value at Risk (CVaR) to risky scenarios. And in 2022 Alruwaili and Alharbi and Hafez and Riad [9] used the mixture of normal and half-normal distributions and they derived the mean, median, and mode of the mixture of normal and half-normal distributions. They used Bayesian estimation for the mixture of normal and half normal distributions using type-I censoring by using different methods (Maximum Likelihood Function and Bayesian Estimation by Using the Prior Function). Also, Kiani and Aslam and Bhatti in 2023 [10] derived The posterior distribution for the unknown scale parameter of half-normal distribution when location parameter equal to zero using informative priors (squared root inverted gamma prior and inverted chi-square prior and inverse Raleigh prior). Also, they discussed the properties of posterior distribution (mean, median, mode, variance and coefficient of variation) using simulation and real data set, they gave the derivations of the prior predictive distribution using informative priors and they derived Bayes estimators and posterior risks for the unknown scale parameter of half-normal distribution under four different loss functions which are represented by the squared error loss function (SELF), the quadratic loss function (QLF), the modified loss function (MLF) and the Degroot loss function (DLF).

Our aim in this study is to derive closed-form expressions for the posterior distributions of for the unknown scale parameter of half-normal distribution under three different informative priors, which are represented by the inverse Maxwell–Boltzmann (IMB) and the inverse Nakagami-b (INK) and the inverse half-normal (IHN), also we derived closed-form for the J^{th} posterior moments where j=1,2,3,4. And we derived closed-form the Bayes estimators of unknown scale parameter (φ) of the HND based on two different loss functions which are represented by the squared error loss function (SELF), which is known as quadratic loss function and De Groot loss function (DLF), in addition to the maximum likelihood estimator (MLE). We obtained the results by using Simulation study by assuming different values for the scale parameter of the HND. We compared the accuracy of the different estimates for φ, based on three different criteria: the mean relative estimate (MRE) and The mean biased estimate (Bias) and mean square error (MSE).

2. The Half- Normal Distribution

Let us assuming we have a random sample of size n, $y = (y_1, y_2, \dots, y_n)$, from the half-Normal (HN(φ)) distribution where location parameter equal to zero and scale parameter is $(\varphi > 0)$. And the probability density function (pdf) with the cumulative distribution function (cdf) of the half normal distribution are given by [9,10]:

$$g(y, \varphi) = \frac{1}{\varphi}\sqrt{\frac{2}{\pi}}\exp(-\frac{y^2}{2\varphi^2}) \quad , \ y \geq 0 \ \text{and} \quad \varphi > 0 \qquad \dots (1)$$

$$G(y, \varphi) = \text{erf}(\frac{y}{\varphi\sqrt{2}}) = \frac{2}{\pi}\int_0^{y/\varphi\sqrt{2}} \exp(-t^2)\, dt \quad , \ y \geq 0 \ \text{and} \quad \varphi > 0 \qquad \dots (2)$$

Where erf(y) is the error function, and it can be defined as follows [11]:

$$\text{erf}(y) = \frac{2}{\pi}\int_0^{y} \exp(-t^2)\, dt \qquad \dots (3)$$

With mean is $\mu_y = \varphi\sqrt{\frac{2}{\pi}} \approx 0.797885\varphi$, and variance is $\text{var}(y) = \varphi^2(1 - \frac{2}{\pi})$ [6].

3. Maximum Likelihood Estimation (MLE)

For a random sample $y = (y_1, y_2, \dots, y_n)$ which is taken from the HND with unknown scale parameter (φ), pdf and according to the pdf in equation (1) we can define its likelihood function is [12]:

$$L(\varphi \setminus \underline{y}) = \prod_{i=1}^{n} g(y_i, \varphi) = \prod_{i=1}^{n}\frac{1}{\varphi}\sqrt{\frac{2}{\pi}}\exp(-\frac{y_i^2}{2\varphi^2}) = \frac{1}{\varphi^n}(\sqrt{\frac{2}{\pi}})^n \exp(-\frac{\sum_{i=1}^{n} y_i^2}{2\varphi^2}) \qquad \dots (4)$$

Then the log-likelihood function is maximized by taking the derivative of log of the likelihood with respect to the scale parameter (φ) and setting to zero, i.e. $(\frac{\partial}{\partial\varphi}(\log L(\varphi \setminus \underline{y})) = 0)$, it yields the maximum likelihood (ML) estimate of (φ) as

$$\hat{\varphi}_{MLE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} y_i^2} \qquad \dots (5)$$

4. Bayesian estimation

We derive the posterior distributions under three different informative priors which are the inverse Maxwell–Boltzmann (IMB) and the inverse Nakagami-b (INK) and the inverse half-normal (IHN), then we derive posterior moments and Bayes estimates of unknown scale parameter (φ) of the HND under the squared error loss function (SELF), which is known as quadratic loss function and De Groot loss function (DLF).

4.1 Posterior Distribution of Unknown Scale Parameter of The HND

We can obtain the posterior distribution of the scale parameter (φ) of the HND, under three different informative priors will be as follow[13, 14]:

$$w_i(\varphi \mid \underline{y}) = \frac{L(\varphi \setminus \underline{y})k_i(\varphi)}{\int_{\varphi=0}^{\infty} L(\varphi \setminus \underline{y})k_i(\varphi)d\varphi} \qquad \text{for} \quad i = 1,2,3 \qquad \dots (6)$$

By using the equation (4) and combined it with the equation of the prior distribution as shown below in the equation (6). So we can derive the posterior distribution and the expression for the Jth posterior moments as follows:

4.1.1 Using The Inverse Maxwell–Boltzmann (IMB) as Prior Distribution

Let us assumed the informative prior for the scale parameter (φ) of the HND is the inverse Maxwell–Boltzmann (IMB) with the scale parameter (a > 0) with the pdf written as [15,16]:

$$k_1(\varphi; a) = \frac{4a^3}{\sqrt{\pi}} \varphi^{-4} \exp(-\frac{a}{\varphi})^2 \quad , \varphi > 0 \qquad \dots (7)$$

By using the equation (4) and combined it with the equation (7) in the equation (6), it yields

$$w_1(\varphi \mid \underline{y}) = \frac{(\frac{1}{\varphi^n} (\sqrt{\frac{2}{\pi}})^n \exp(-\frac{\sum_{i=1}^n y_i^2}{2\varphi^2}))(\frac{4a^3}{\Gamma(\pi)} \varphi^{-4} \exp(-\frac{a}{\varphi})^2)}{\int_{\varphi=0}^{\infty} (\frac{1}{\varphi^n} (\sqrt{\frac{2}{\pi}})^n \exp(-\frac{\sum_{i=1}^n y_i^2}{2\varphi^2}))(\frac{4a^3}{\Gamma(\pi)} \varphi^{-4} \exp(-\frac{a}{\varphi})^2)\, d\varphi} \qquad \dots (8)$$

$$w_1(\varphi \mid \underline{y}) = \frac{\varphi^{-(n+4)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2} + a^2))}{\int_{\varphi=0}^{\infty} \varphi^{-(n+4)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2} + a^2))\, d\varphi} \qquad \dots (9)$$

Rewriting the $(\varphi^{-(n+4)})$ as $(\varphi^{-(2(\frac{n+3}{2})+1)})$. And multiplying the integral in equation (9) by $(\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)^{(\frac{n+3}{2})}}{\Gamma((n+3)/2)} \frac{\Gamma((n+3)/2)}{2(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)^{(\frac{n+3}{2})}})$. Where $\Gamma(.)$ is the gamma function.

$$w_1(\varphi \mid \underline{y}) = \frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)^{(\frac{n+3}{2})}}{\Gamma((n+3)/2)\, B_1(\varphi \mid \underline{y})} \varphi^{-(2(\frac{n+3}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)) \;, n>0,\; a>0 \text{ and } \varphi>0 \dots (10)$$

Where $B_1(\varphi \mid \underline{y})$ is the integral in equation (9) will be the pdf of the square root Gamma distribution(SRIG) [17] which equals to one, as follows:

$B_1(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)^{(\frac{n+3}{2})}}{\Gamma((n+3)/2)} \varphi^{-(2(\frac{n+3}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+a^2))\, d\varphi = 1$. So the posterior distribution of the scale parameter (φ) of the HND, under the inverse Maxwell–Boltzmann (IMB) is the pdf of the SRIG distribution with the parameters $((\frac{n+3}{2}),\ (\frac{\sum_{i=1}^n y_i^2}{2}+a^2))$ written as[17]:

$$w_1(\varphi \mid \underline{y}) = \frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)^{(\frac{n+3}{2})}}{\Gamma((n+3)/2)} \varphi^{-(2(\frac{n+3}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)) \,, n>0,\; a>0 \text{ and } \varphi>0 \quad \dots (10)$$

And we can derive the J^th^ posterior moments of the scale parameter (φ) of the HND, under the inverse Maxwell–Boltzmann (IMB) distribution using equation (10), these moments can be obtain by the following equation [18,19]:

$$\varphi_{1J} = E(\varphi_1^J \mid \underline{y}) = \int_{\varphi=0}^{\infty} \varphi^J w_1(\varphi \mid \underline{y})\, d\varphi \qquad \dots (11)$$

$$\varphi_{1J} = E(\varphi_1^J \mid \underline{y}) = \int_{\varphi=0}^{\infty} \varphi^J \frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)^{(\frac{n+3}{2})}}{\Gamma((n+3)/2)} \varphi^{-(2(\frac{n+3}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+a^2)) d\varphi \qquad \dots (12)$$

$$\varphi_{1J} = E(\varphi_1^J \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{n+3}{2}\right)}}{\Gamma((n+3)/2)} \varphi^{-(2(\frac{n+3-J}{2})+1)} \exp\left(-\frac{1}{\varphi^2}\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)\right) d\varphi \quad \text{... (13)}$$

Rewriting the quantity $\left(\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{n+3}{2}\right)}\right)$ as $\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{n+3-J+J}{2}\right)} = \left(\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{n+3-J}{2}\right)} \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{J}{2}\right)}\right)$. And multiplying the integral in equation (13) by $\left(\frac{\Gamma((n+3-J)/2)}{\Gamma((n+3-J)/2)}\right)$, and the integral in equation (13) will be the pdf of the square root Gamma distribution (SRIG) [17] which equals to one.

$$\varphi_{1J} = E(\varphi_1^J \mid \underline{y}) = \left(\frac{\Gamma((n+3-J)/2)}{\Gamma((n+3)/2)}\right) \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{J}{2}\right)} A_1(\varphi \mid \underline{y}), n > 0,\ a > 0, J = 1,2,3,... \quad \text{... (14)}$$

Where $A_1(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{n+3-J}{2}\right)}}{\Gamma((n+3-J)/2)} \varphi^{-(2(\frac{n+3-J}{2})+1)} \exp\left(-\frac{1}{\varphi^2}\left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)\right) d\varphi = 1$. Then the expression for the Jth posterior moments for the scale parameter (φ) of the HND, under the inverse Maxwell–Boltzmann (IMB) distribution is given by:

$$\varphi_{1J} = E(\varphi_1^J \mid \underline{y}) = \left(\frac{\Gamma((n+3-J)/2)}{\Gamma((n+3)/2)}\right) \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{J}{2}\right)}, \quad n > 0,\ a > 0, J = 1,2,3,... \quad \text{... (15)}$$

So, we can drive the first four moments form equation (15) are given by the following:
For j=1, we obtain

$$\varphi_{11} = E(\varphi_1^1 \mid \underline{y}) = \left(\frac{\Gamma((n+2)/2)}{\Gamma((n+3)/2)}\right) \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{1}{2}\right)}, n > 0,\ a > 0 \quad \text{... (16)}$$

For j=2, we obtain

$$\varphi_{12} == E(\varphi_1^2 \mid \underline{y}) = \left(\frac{\Gamma((n+1)/2)}{\Gamma((n+3)/2)}\right) \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right), n > 0,\ a > 0 \quad \text{... (17)}$$

For j=3, we obtain

$$\varphi_{13} = E(\varphi_1^3 \mid \underline{y}) == \left(\frac{\Gamma((n)/2)}{\Gamma((n+3)/2)}\right) \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^{\left(\frac{3}{2}\right)}, n > 0,\ a > 0 \quad \text{... (18)}$$

For j=4, we obtain

$$\varphi_{14} == E(\varphi_1^4 \mid \underline{y}) == \left(\frac{\Gamma((n-1)/2)}{\Gamma((n+3)/2)}\right) \left(\frac{\sum_{i=1}^n y_i^2}{2} + a^2\right)^2, n > 0,\ a > 0 \quad \text{... (19)}$$

4.1.2 Using The Inverse Nakagami-b (INK) as Prior Distribution

And, let the informative prior for the scale parameter (φ) of the HND is the inverse Nakagami-b (INK) with the shape parameter (b > 0) and the scale parameter (c > 0) with the pdf is given by [21, 22]:

$$k_2(\varphi; b, c) = \frac{2}{\Gamma(b)} \left(\frac{b}{c}\right)^b \varphi^{-2b-1} \exp\left(-\frac{b}{c\varphi^2}\right) \quad , \varphi > 0 \quad \text{... (20)}$$

By using the equation (4) and combined it with the equation (20) in the equation (6), it yields:

$$w_2(\varphi|\underline{y})=\frac{\left(\frac{1}{\varphi^n}(\sqrt{\frac{2}{\pi}})^n\exp(-\frac{\sum_{i=1}^n y_i^2}{2\varphi^2})\right)\left(\frac{2}{\Gamma(b)}(\frac{b}{c})^b\varphi^{-2b-1}\exp(-\frac{b}{c\varphi^2})\right)}{\int\limits_{\varphi=0}^{\infty}\left(\frac{1}{\varphi^n}(\sqrt{\frac{2}{\pi}})^n\exp(-\frac{\sum_{i=1}^n y_i^2}{2\varphi^2})\right)\left(\frac{2}{\Gamma(b)}(\frac{b}{c})^b\varphi^{-2b-1}\exp(-\frac{b}{c\varphi^2})\right)d\varphi} \quad \dots (21)$$

$$w_2(\varphi|\underline{y})=\frac{\varphi^{-2b-n-1}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c}))}{\int\limits_{\varphi=0}^{\infty}\varphi^{-2b-n-1}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c}))\,d\varphi} \quad \dots (22)$$

Rewriting the $(\varphi^{-2b-n-1})$ as $(\varphi^{-(2(\frac{n+2b}{2})+1)})$. And multiplying the integral in equation (22) by $\left(\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})^{(\frac{n+2b}{2})}}{\Gamma(\frac{n+2b}{2})}\frac{\Gamma(\frac{n+2b}{2})}{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})^{(\frac{n+2b}{2})}}\right)$. Where $\Gamma(.)$ is the gamma function.

$$w_2(\varphi|\underline{y})=\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})^{(\frac{n+2b}{2})}}{\Gamma((n+2b)/2)\,B_2(\varphi|\underline{y})}\varphi^{-(2(\frac{n+2b}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})) \quad \dots(23)$$

Where $B_2(\varphi|\underline{y})$ is the integral in equation (22) will be the pdf of the square root Gamma distribution(SRIG) [17] which equals to one, as follows:

$B_2(\varphi|\underline{y})=\int\limits_{\varphi=0}^{\infty}\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})^{(\frac{n+2b}{2})}}{\Gamma((n+2b)/2)}\varphi^{-(2(\frac{n+2b}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c}))\,d\varphi=1$. So the posterior distribution of the scale parameter (φ) of the HND, under the inverse Nakagami-b (INK) of the SRIG distribution with the parameters $((\frac{n+2b}{2}),\ (\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c}))$ written as[17]:

$$w_2(\varphi|\underline{y})=\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})^{(\frac{n+2b}{2})}}{\Gamma((n+2b)/2)}\varphi^{-(2(\frac{n+2b}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})),$$
$$n>0, b>0, c>0, \quad \varphi>0 \quad \dots (23)$$

And we can derive the J^{th} posterior moments of the scale parameter (φ) of the HND, under the inverse Nakagami-b (INK) distribution using equation (23), these moments can be obtain by the following equation [18, 19]:

$$\varphi_{2J}=E(\varphi_2^J|\underline{y})=\int\limits_{\varphi=0}^{\infty}\varphi^J w_2(\varphi|\underline{y})\,d\varphi \quad \dots (24)$$

$$\varphi_{2J}=E(\varphi_2^J|\underline{y})=\int\limits_{\varphi=0}^{\infty}\varphi^J\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c})^{(\frac{n+2b}{2})}}{\Gamma((n+2b)/2)}\varphi^{-(2(\frac{n+2b}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{b}{c}))d\varphi \quad \dots (25)$$

$$\varphi_{2J} = E(\varphi_2^J \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{n+2b}{2})}}{\Gamma((n+2b)/2)} \varphi^{-(2(\frac{n+2b-J}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c}))d\varphi \quad \text{... (26)}$$

Rewriting the quantity $((\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{n+2b}{2})})$ as $(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{n+2b-J+J}{2})} = ((\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{n+2b-J}{2})} (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{J}{2})})$. And multiplying the integral in equation (26) by $(\frac{\Gamma((n+2b-J)/2)}{\Gamma((n+2b-J)/2)})$, and the integral in equation (26) will be the pdf of the square root Gamma distribution (SRIG) [17] which equals to one.

$$\varphi_{2J} = E(\varphi_2^J \mid \underline{y}) = (\frac{\Gamma((n+2b-J)/2)}{\Gamma((n+2b)/2)}) (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{J}{2})} A_2(\varphi \mid \underline{y}) \quad , n>0,\ b>0,\ c>0,$$
$$J = 1,2,3,... \quad \text{... (27)}$$

Where $(A_2(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{n+2b-J}{2})}}{\Gamma((n+2b-J)/2)} \varphi^{-(2(\frac{n+2b-J}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c}))d\varphi = 1$. Then, the expression for the Jth posterior moments for the scale parameter (φ) of the HND, under the inverse Nakagami-b (INK) distribution is given by:

$$\varphi_{2J} = E(\varphi_2^J \mid \underline{y}) = (\frac{\Gamma((n+2b-J)/2)}{\Gamma((n+2b)/2)}) (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{J}{2})} \quad , n>0,\ b>0,$$
$$c>0, J = 1,2,3,... \quad \text{... (28)}$$

So, we can drive the first four moments form equation (28) are given by the following:

For j=1, we obtain

$$\varphi_{21} = E(\varphi_2^1 \mid \underline{y}) = (\frac{\Gamma((n+2b-1)/2)}{\Gamma((n+2b)/2)}) (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{1}{2})} \quad , n>0,\ b>0,\ c>0 \quad \text{... (29)}$$

For j=2, we obtain

$$\varphi_{22} = E(\varphi_2^2 \mid \underline{y}) = (\frac{\Gamma((n+2b-2)/2)}{\Gamma((n+2b)/2)}) (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c}) \quad , n>0,\ b>0,\ c>0 \quad \text{... (30)}$$

For j=3, we obtain

$$\varphi_{23} = E(\varphi_2^3 \mid \underline{y}) = (\frac{\Gamma((n+2b-3)/2)}{\Gamma((n+2b)/2)}) (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(\frac{3}{2})} , n>0,\ b>0,\ c>0 \quad \text{... (31)}$$

For j=4, we obtain

$$\varphi_{24} = E(\varphi_2^4 \mid \underline{y}) = (\frac{\Gamma((n+2b-4)/2)}{\Gamma((n+2b)/2)}) (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{b}{c})^{(2)} , n>0,\ b>0,\ c>0 \quad \text{... (32)}$$

4.1.3 Using The Inverse Half-Normal (IHN)

And, let the informative priors for the scale parameter (φ) of the HND is the inverse half-normal (IHN) with the scale parameter ($\upsilon > 0$) with the pdf is given by [16]:

$$k_3(\varphi; \upsilon) = \sqrt{\frac{2}{\pi\upsilon^2}}\ \varphi^{-2} \exp(-\frac{1}{2\upsilon^2 \varphi^2}) \qquad \upsilon > 0,\ \varphi > 0 \qquad \text{... (33)}$$

By using the equation (4) and combined it with the equation (32) in the equation (6), it yields:

$$w_3(\varphi\,|\,\underline{y})=\frac{(\frac{1}{\varphi^n}(\sqrt{\frac{2}{\pi}})^n\exp(-\frac{\sum_{i=1}^n y_i^2}{2\varphi^2}))(\sqrt{\frac{2}{\pi\upsilon^2}}\ \varphi^{-2}\exp(-\frac{1}{2\upsilon^2\varphi^2}))}{\int_{\varphi=0}^{\infty}(\frac{1}{\varphi^n}(\sqrt{\frac{2}{\pi}})^n\exp(-\frac{\sum_{i=1}^n y_i^2}{2\varphi^2}))(\sqrt{\frac{2}{\pi\upsilon^2}}\ \varphi^{-2}\exp(-\frac{1}{2\upsilon^2\varphi^2}))\,d\varphi}\quad ...\ (34)$$

$$w_3(\varphi\,|\,\underline{y})=\frac{\varphi^{-n-2}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))}{\int_{\varphi=0}^{\infty}\varphi^{-n-2}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))d\varphi}\quad ...\ (35)$$

Rewriting the (φ^{-n-2}) as $(\varphi^{-(2(\frac{n+1}{2})+1)})$. And multiplying the integral in equation (34) by $(\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}}{\Gamma((n+1)/2)}\frac{\Gamma((n+1)/2)}{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}})$. Where $\Gamma(.)$ is the gamma function.

$$w_3(\varphi\,|\,\underline{y})=\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}}{\Gamma((n+1)/2)\,B_3(\varphi\,|\,\underline{y})}\varphi^{-(2(\frac{n+1}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))\ ,n>0,\ \upsilon>0,\ \varphi>0\quad ...(35)$$

Where $B_3(\varphi\,|\,\underline{y})$ is the integral in equation (35) will be the pdf of the square root Gamma distribution(SRIG) [17] which equals to one, as follows:

$B_3(\varphi\,|\,\underline{y})=\int_{\varphi=0}^{\infty}\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}}{\Gamma((n+1)/2)}\varphi^{-(2(\frac{n+1}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))\ d\varphi=1$. So the posterior distribution of the scale parameter (φ) of the HND, under the inverse Nakagami-b (INK) of the SRIG distribution with the parameters $((\frac{n+1}{2}),\ (\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))$ written as[17]:

$$w_3(\varphi\,|\,\underline{y})=\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}}{\Gamma((n+1)/2)}\varphi^{-(2(\frac{n+1}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))\ ,n>0,\upsilon>0\ \text{and}\ \varphi>0\quad ...(36)$$

And we can derive the J^th posterior moments about the origin of the scale parameter (φ) of the HND, under the inverse half-normal (IHN) distribution using equation (36), these moments can be obtain by the following equation [18, 19]:

$$\varphi_{3J}=E(\varphi_3^J\,|\,\underline{y})=\int_{\varphi=0}^{\infty}\varphi^J w_3(\varphi\,|\,\underline{y})\ d\varphi\quad ...\ (37)$$

$$\varphi_{3J}=E(\varphi_3^J\,|\,\underline{y})=\int_{\varphi=0}^{\infty}\varphi^J\frac{2(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}}{\Gamma((n+1)/2)}\varphi^{-(2(\frac{n+1}{2})+1)}\exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2}+\frac{1}{2\upsilon^2}))\ d\varphi\quad ...\ (38)$$

$$\varphi_{3J} = E(\varphi_3^J \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{n+1}{2})}}{\Gamma((n+1)/2)} \varphi^{-(2(\frac{n+1-J}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2}))\ d\varphi \quad ... (39)$$

Rewriting the quantity $((\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{n+1}{2})})$ as $((\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{n+1-J+J}{2})} = (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{n+1-J}{2})} (\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{J}{2})})$. And multiplying the integral in equation (39) by $(\frac{\Gamma((n+1-J)/2)}{\Gamma((n+1-J)/2)})$, and the integral in equation (39) will be the pdf of the square root Gamma distribution (SRIG) [17] which equals to one.

$$\varphi_{3J} = E(\varphi_3^J \mid \underline{y}) = (\frac{\Gamma((n+1-J)/2)}{\Gamma((n+1)/2)})(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{J}{2})} A_3(\varphi \mid \underline{y}) \ , n>0, \upsilon>0, J=1,2,... \quad ... (40)$$

Where $(A_3(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \frac{2(\frac{\sum_{i=1}^n y_i^2}{2} - \frac{1}{2\upsilon^2})^{(\frac{n+1-J}{2})}}{\Gamma((n+1-J)/2)} \varphi^{-(2(\frac{n+1-J}{2})+1)} \exp(-\frac{1}{\varphi^2}(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2}))d\varphi = 1$.Then, the expression for the J^{th} posterior moments for the scale parameter (φ) of the HND, under the inverse half-normal (IHN) distribution is given by:

$$\varphi_{3J} = E(\varphi_3^J \mid \underline{y}) = (\frac{\Gamma((n+1-J)/2)}{\Gamma((n+1)/2)})(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{J}{2})} \ , n>0, \upsilon>0, \ J=1,2,3,... \quad ... (41)$$

So, we can drive the first four moments form equation (41) are given by the following:
For j=1, we obtain

$$\varphi_{31} = E(\varphi_3^1 \mid \underline{y}) = (\frac{\Gamma(n/2)}{\Gamma((n+1)/2)})(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{1}{2})} \ , n>0, \ \upsilon>0 \quad ... (42)$$

For j=2, we obtain

$$\varphi_{32} = E(\varphi_3^2 \mid \underline{y}) = (\frac{\Gamma((n-1)/2)}{\Gamma((n+1)/2)})(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2}) \ , n>0, \ \upsilon>0 \quad ... (43)$$

For j=3, we obtain

$$\varphi_{3J} = E(\varphi_3^3 \mid \underline{y}) = (\frac{\Gamma((n-2)}{\Gamma((n+1)/2)})(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(\frac{3}{2})} \ , n>0, \ \upsilon>0 \quad ... (44)$$

For j=4, we obtain

$$\varphi_{34} = E(\varphi_3^4 \mid \underline{y}) = (\frac{\Gamma((n-3)/2)}{\Gamma((n+1)/2)})(\frac{\sum_{i=1}^n y_i^2}{2} + \frac{1}{2\upsilon^2})^{(2)} \ , n>0, \ \upsilon>0 \quad ... (45)$$

4.2 Bayes Estimator Based on Loss Functions

In this section, we derive Bayes estimators of the scale Parameter in Half-Normal distribution under different priors based on SELF and DLF.

<u>First</u>: Bayes Estimator Based on the Squared Error Loss Function (SELF).

Let us defined the squared error loss function as follows [23, 24]:

$$R_1(\varphi, \hat{\varphi}) = (\varphi - \hat{\varphi})^2 \quad ... (46)$$

We can obtain Bayes estimator under this loss function, by solving the following equation:

$$\frac{\partial R_1(\varphi, \hat{\varphi})}{\partial \hat{\varphi}} = \frac{\partial}{\partial \hat{\varphi}} \int_{\varphi=0}^{\infty} (\varphi - \hat{\varphi})^2 \ w(\varphi \mid \underline{y})\, d\varphi = -2E(\varphi \mid \underline{y}) + 2\hat{\varphi} = 0 \quad ... (47)$$

Then the Bayes estimator based on a squared loss function will be the mean of the posterior distribution according to the following expression:

$$\hat{\varphi}_{BE\_SELF} = E(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \varphi \, w(\varphi \mid \underline{y}) \, d\varphi \qquad \text{... (48)}$$

So, the Bayes estimator for the scale parameter (φ) of the HND, under the inverse Maxwell–Boltzmann (IMB) distribution can be derive as $\hat{\varphi}_{BE\_SELF(1)} = E(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \varphi \, w_1(\varphi \mid \underline{y}) \, d\varphi$, Which is equal to the first moment in equation (16):

$$\hat{\varphi}_{BE\_SELF(1)} = \left(\frac{\Gamma((n+2)/2)}{\Gamma((n+3)/2)}\right)\left(\frac{\sum_{i=1}^{n} y_i^2}{2} + a^2\right)^{(\frac{1}{2})}, \; n > 0, \; a > 0 \qquad \text{... (49)}$$

Also, The Bayes estimator for the scale parameter (φ) of the HND, under the inverse Nakagami-b (INK) distribution can be derive as $\hat{\varphi}_{BE\_SELF(2)} = E(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \varphi \, w_2(\varphi \mid \underline{y}) \, d\varphi$, Which is equal to the first moment in equation (29):

$$\hat{\varphi}_{BE\_SELF(2)} = \left(\frac{\Gamma((n+2b-1)/2)}{\Gamma((n+2b)/2)}\right)\left(\frac{\sum_{i=1}^{n} y_i^2}{2} + \frac{b}{c}\right)^{(\frac{1}{2})} \quad , n > 0, \; b > 0, \; c > 0 \qquad \text{... (50)}$$

And, The Bayes estimator for the scale parameter (φ) of the HND, under the inverse half-normal (IHN) can be derive as $\hat{\varphi}_{BE\_SELF(3)} = E(\varphi \mid \underline{y}) = \int_{\varphi=0}^{\infty} \varphi \, w_3(\varphi \mid \underline{y}) \, d\varphi$, Which is equal to the first moment in equation (42):

$$\hat{\varphi}_{BE\_SELF(3)} = \left(\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\right)\left(\frac{\sum_{i=1}^{n} y_i^2}{2} + \frac{1}{2\upsilon^2}\right)^{(\frac{1}{2})}, \; n > 0, \; \upsilon > 0 \qquad \text{... (51)}$$

<u>Second</u>: Bayes Estimator Based on the De Groot loss function (DLF).

The De Groot loss function is asymmetric loss function [25 ,26] which is defined as follows:

$$R_2(\hat{\varphi}, \varphi) = \left(\frac{\varphi - \hat{\varphi}}{\hat{\varphi}}\right)^2 \qquad \text{... (52)}$$

We can obtain Bayes estimator under this loss function, by solving the following equation:

$$\frac{\partial R_2(\varphi, \hat{\varphi})}{\partial \hat{\varphi}} = \frac{\partial}{\partial \hat{\varphi}} \int_{\varphi=0}^{\infty} \left(\frac{\varphi - \hat{\varphi}}{\hat{\varphi}}\right)^2 \; w(\varphi \mid \underline{y}) \, d\varphi = -\frac{2E(\varphi^2 \mid \underline{y})}{\hat{\varphi}^3} + \frac{2E(\varphi \mid \underline{y})}{\hat{\varphi}^2} = 0 \qquad \text{... (53)}$$

Then the Bayes estimator based on the De Groot loss function will be as follows [24 ,25]:

$$\hat{\varphi}_{BE\_DLF} = \frac{E(\varphi^2 \mid \underline{y})}{E(\varphi \mid \underline{y})} = \frac{\int_{\varphi=0}^{\infty} \varphi^2 \, w(\varphi \mid \underline{y}) \, d\varphi}{\int_{\varphi=0}^{\infty} \varphi \, w(\varphi \mid \underline{y}) \, d\varphi} \qquad \text{... (54)}$$

So, the Bayes estimator for the scale parameter (φ) of the HND, under the inverse Maxwell–Boltzmann (IMB) distribution can be derive as $\hat{\varphi}_{BE\_DLF(1)} = \frac{E(\varphi^2 \mid \underline{y})}{E(\varphi \mid \underline{y})} = \frac{\int_{\varphi=0}^{\infty} \varphi^2 \, w_1(\varphi \mid \underline{y}) \, d\varphi}{\int_{\varphi=0}^{\infty} \varphi \, w_1(\varphi \mid \underline{y}) \, d\varphi}$, and using the second and the first moment in equations (16), (17) ,it yields

$$\hat{\varphi}_{BE\_DLF(1)} = \left(\frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)}\right)\left(\frac{\sum_{i=1}^{n} y_i^2}{2} + a^2\right)^{(\frac{1}{2})}, \; n > 0, \; a > 0 \qquad \text{... (55)}$$

Also, The Bayes estimator for the scale parameter (φ) of the HND, under the inverse Nakagami-b (INK) distribution can be derive as

$$\hat{\varphi}_{BE\_DLF(2)} = \frac{E(\varphi^2 \mid \underline{y})}{E(\varphi \mid \underline{y})} = \frac{\int_{\varphi=0}^{\infty} \varphi^2\, w_2(\varphi \mid \underline{y})\, d\varphi}{\int_{\varphi=0}^{\infty} \varphi\, w_2(\varphi \mid \underline{y})\, d\varphi}$$, and using the second and the first moment in equations (29), (30), it yields

$$\hat{\varphi}_{BE\_DLF(2)} = \left(\frac{\Gamma((n+2b-2)/2)}{\Gamma((n+2b-1)/2)}\right)\left(\frac{\sum_{i=1}^{n} y_i^2}{2} + \frac{b}{c}\right)^{(\frac{1}{2})}, n > 0,\ b > 0,\ c > 0 \qquad \text{... (50)}$$

And, The Bayes estimator for the scale parameter (φ) of the HND, under the inverse half-normal (IHN) can be derive as can be derive as

$$\hat{\varphi}_{BE\_DLF(3)} = \frac{E(\varphi^2 \mid \underline{y})}{E(\varphi \mid \underline{y})} = \frac{\int_{\varphi=0}^{\infty} \varphi^2\, w_3(\varphi \mid \underline{y})\, d\varphi}{\int_{\varphi=0}^{\infty} \varphi\, w_3(\varphi \mid \underline{y})\, d\varphi}$$, and using the second and the first moment in equations (42), (43), it yields

$$\hat{\varphi}_{BE\_DLF(3)} = \left(\frac{\Gamma((n-1)/2)}{\Gamma((n)/2)}\right)\left(\frac{\sum_{i=1}^{n} y_i^2}{2} + \frac{1}{2\upsilon^2}\right)^{(\frac{1}{2})}, n > 0,\ \upsilon > 0 \qquad \text{... (51)}$$

5. Simulation and Discussion

We conducted simulation study to evaluate and compare the performance of the maximum likelihood estimator and Bayesian estimators for estimating the parameter of from the Half- Normal $(HN(\varphi))$ distribution where location parameter equal to zero and scale parameter is $(\varphi > 0)$. The simulation programs were written by using MatlabR2018b program, we generated y_i from the half-Normal $(HN(\varphi))$ distribution by taking the absolute value of the data generating from the normal distribution mean equal to zero and different values for sigma (i.e. we obtained $y = |X|$ where $x \sim \text{normal distribution}(\text{mean} = 0, \text{sigma} = \varphi)$) follows a half-normal distribution. We assumed different values for the true values of the scale parameter $(\varphi = 0.95, 1.25, 1.5, 2)$ with different sample sizes n= (25,50,100,150) replicated number of the experiments (r=10000) for each sample size(n).

Also, we assumed different values for the hyper parameters of the prior distributions can be chosen arbitrarily to compare the accuracy of the different estimates for (φ) as follows :

- The values for the scale parameter (a) of the Inverse Maxwell-Boltzmann (IMB) prior have been selected arbitrarily to be a =0.3, 0.7.
- The values for the shape parameter (b) and the scale parameter (c) of the inverse Nakagami-b (INK) priors have been selected arbitrarily to be (b, c) = (0.5, 2), (2,2).
- The values for the scale parameter (υ) of the inverse half-normal (IHN) prior have been selected arbitrarily to be $(\upsilon = 2.5, 4)$.

Also, we repeated the experiments for the true value of the scale parameter $(\varphi = 2)$ with all sample sizes n= (25,50,100,150) with r=10000. We choose large values for the parameters of prior distributions, as follows:

- The values for the scale parameter (a) of the (IMB) prior have been selected arbitrarily to be a =2, 3.5.
- The values for (b, c) of the inverse Nakagami-b (INK) priors have been selected arbitrarily to be (b, c) = (3, 4), (3.5, 2.5).

- The values for (υ) of the inverse half-normal (IHN) prior have been selected arbitrarily to be $(\upsilon = 5.5, 6)$.

Based on the squared error loss function (SELF) and the De Groot loss function (DLF). In order to compare the accuracy of the different estimates for φ, we depend on three different criteria: the mean relative estimate (MRE) and The mean biased estimate (Bias) and mean square error (MSE), which are defined, respectively, as follows:

- The mean relative estimate (MRE) given by

$$MRE(\varphi) = \frac{1}{10000}\sum_{r=1}^{10000}\frac{\hat{\varphi}(r)}{\varphi} \quad \text{... (52)}$$

- The mean biased estimate (Bias) given by

$$Bais(\varphi) = \frac{1}{10000}\sum_{r=1}^{10000}(\hat{\varphi}(r) - \varphi) \quad \text{... (53)}$$

- The mean square errors (MSE), given by

$$MSE(\varphi) = \frac{1}{10000}\sum_{r=1}^{10000}(\hat{\varphi}(r) - \varphi)^2 \quad \text{... (54)}$$

We obtain the asymptotically unbiased estimators for the parameters, when the MRE and Bias and MSE tend to one (zero). The results of the fits are provided in Table 1.1 to Table.4.2 for each estimator and for all sample sizes.

Table 1.1 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on SELF when the true value $(\varphi = 0.95)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a)) $\hat{\varphi}_{BE\_SELF(1)}$		Inverse Nakagami-b (INK (b,c)) $\hat{\varphi}_{BE\_SELF(2)}$		Inverse Half –Normal (IHN(υ)) $\hat{\varphi}_{BE\_SELF(3)}$	
	MLE	0.3	0.7	(0.5,2)	(2,2)	2.5	4
	true value $(\varphi = 0.95)$, ,n=25						
est. value	0.9407	0.9174	0.934	0.961	0.9372	0.9588	0.9556
MRE	0.9902	0.9656	0.9831	1.0116	0.9865	1.0093	1.0059
Bias	-0.0093	-0.0326	-0.016	0.011	-0.0128	0.0088	0.0056
MSE	0.0177	0.0175	0.0161	0.0177	0.0148	0.0177	0.0178
Table1.1	true value $(\varphi = 0.95)$, n=50						
est. value	0.9445	0.9325	0.9409	0.9546	0.9424	0.9535	0.9519
MRE	0.9942	0.9816	0.9904	1.0048	0.992	1.0037	1.002
Bias	-0.0055	-0.0175	-0.0091	0.0046	-0.0076	0.0035	0.0019
MSE	0.009	0.009	0.0086	0.009	0.0082	0.009	0.009
	true value $(\varphi = 0.95)$, n=100						
est. value	0.9485	0.9424	0.9466	0.9535	0.9472	0.953	0.9522
MRE	0.9984	0.992	0.9964	1.0037	0.9971	1.0031	1.0023
Bias	-0.0015	-0.0076	-0.0034	0.0035	-0.0028	0.003	0.0022
MSE	0.0046	0.0046	0.0045	0.0046	0.0044	0.0046	0.0046
	true value $(\varphi = 0.95)$, n=150						
est. value	0.948	0.9439	0.9467	0.9513	0.9471	0.951	0.9504
MRE	0.9979	0.9936	0.9965	1.0014	0.997	1.001	1.0005
Bias	-0.002	-0.0061	-0.0033	0.0013	-0.0029	0.001	0.0004
MSE	0.0029	0.0029	0.0029	0.0029	0.0029	0.0029	0.0029

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 1.2 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on DLF when the true value $(\varphi = 0.95)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a))		Inverse Nakagami-b (INK (b,c))		Inverse Half –Normal (IHN(υ))	
		$\hat{\varphi}_{BE\_DLF(1)}$		$\hat{\varphi}_{BE\_DLF(2)}$		$\hat{\varphi}_{BE\_DLF(3)}$	
	MLE	0.3	0.7	(0.5,2)	(2,2)	2.5	4
	true value $(\varphi = 0.95)$, n=25						
est. value	0.9407	0.9352	0.9521	0.9812	0.9547	0.979	0.9757
MRE	0.9902	0.9844	1.0022	1.0329	1.005	1.0305	1.027
Bias	-0.0093	-0.0148	0.0021	0.0312	0.0047	0.029	0.0257
MSE	0.0177	0.0173	0.0165	0.0193	0.0152	0.0192	0.0192
	true value $(\varphi = 0.95)$, n=50						
est. value	0.9445	0.9417	0.9502	0.9644	0.9515	0.9633	0.9617
MRE	0.9942	0.9913	1.0002	1.0151	1.0015	1.014	1.0123
Bias	-0.0055	-0.0083	0.0002	0.0144	0.0015	0.0133	0.0117
MSE	0.009	0.0089	0.0087	0.0094	0.0083	0.0094	0.0093
	true value $(\varphi = 0.95)$, n=100						
est. value	0.9485	0.947	0.9513	0.9583	0.9519	0.9578	0.957
MRE	0.9984	0.9969	1.0013	1.0087	1.002	1.0082	1.0073
Bias	-0.0015	-0.003	0.0013	0.0083	0.0019	0.0078	0.007
MSE	0.0046	0.0046	0.0045	0.0047	0.0044	0.0047	0.0047
	true value $(\varphi = 0.95)$, n=150						
est. value	0.948	0.947	0.9498	0.9545	0.9503	0.9542	0.9536
MRE	0.9979	0.9969	0.9998	1.0048	1.0003	1.0044	1.0038
Bias	-0.002	-0.003	-0.0002	0.0045	0.0003	0.0042	0.0036
MSE	0.0029	0.0029	0.0029	0.003	0.0029	0.003	0.003

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 2.1 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on SELF when the true value $(\varphi = 1.25)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a))		Inverse Nakagami-b (INK (b,c))		Inverse Half –Normal (IHN(υ))	
		$\hat{\varphi}_{BE\_SELF(1)}$		$\hat{\varphi}_{BE\_SELF(2)}$		$\hat{\varphi}_{BE\_SELF(3)}$	
	MLE	0.3	0.7	(0.5,2)	(2,2)	2.5	4
	true value $(\varphi = 1.25)$, n=25						
est. value	1.2391	1.2063	1.219	1.2598	1.2123	1.2582	1.2557
MRE	0.9913	0.965	0.9752	1.0079	0.9699	1.0066	1.0046
Bias	-0.011	-0.044	-0.031	0.0098	-0.038	0.0082	0.0057
MSE	0.0311	0.031	0.0294	0.0313	0.0281	0.0313	0.0314
	true value $(\varphi = 1.25)$, n=50						
est. value	1.2433	1.2264	1.2328	1.2536	1.229	1.2528	1.2515
MRE	0.9946	0.9812	0.9863	1.0029	0.9832	1.0022	1.0012
Bias	-0.007	-0.024	-0.0172	0.0036	-0.021	0.0028	0.0015
MSE	0.0153	0.0154	0.0149	0.0153	0.0146	0.0154	0.0154
	true value $(\varphi = 1.25)$, n=100						
est. value	1.2462	1.2377	1.2409	1.2513	1.2388	1.2509	1.2503
MRE	0.997	0.9901	0.9927	1.0011	0.9911	1.0007	1.0003
Bias	-0.004	-0.012	-0.0091	0.0013	-0.011	0.0009	0.0003
MSE	0.0081	0.0081	0.008	0.0081	0.0079	0.0081	0.0081
	true value $(\varphi = 1.25)$, n=150						
est. value	1.2487	1.243	1.2451	1.2522	1.2438	1.2519	1.2515
MRE	0.999	0.9944	0.9961	1.0017	0.995	1.0015	1.0012
Bias	-0.001	-0.007	-0.0049	0.0022	-0.006	0.0019	0.0015
MSE	0.0052	0.0052	0.0052	0.0052	0.0051	0.0052	0.0052

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 2.2 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on DLF when the true value $(\varphi = 1.25)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a)) $\hat{\varphi}_{BE\_DLF(1)}$		Inverse Nakagami-b (INK (b,c)) $\hat{\varphi}_{BE\_DLF(2)}$		Inverse Half –Normal (IHN(υ)) $\hat{\varphi}_{BE\_DLF(3)}$	
	MLE	0.3	0.7	(0.5,2)	(2,2)	2.5	4
	true value $(\varphi = 1.25)$, n=25						
est. value	1.2391	1.2297	1.2427	1.2864	1.235	1.2847	1.2821
MRE	0.9913	0.9838	0.9941	1.0291	0.988	1.0277	1.0257
Bias	-0.011	-0.02	-0.0073	0.0364	-0.015	0.0347	0.0321
MSE	0.0311	0.0306	0.0296	0.0338	0.028	0.0338	0.0338
	true value $(\varphi = 1.25)$, n=50						
est. value	1.2433	1.2385	1.245	1.2664	1.2409	1.2656	1.2644
MRE	0.9946	0.9908	0.996	1.0131	0.9927	1.0125	1.0115
Bias	-0.007	-0.012	-0.005	0.0164	-0.009	0.0156	0.0144
MSE	0.0153	0.0152	0.015	0.0159	0.0145	0.0159	0.0159
	true value $(\varphi = 1.25)$, n=100						
est. value	1.2462	1.2438	1.247	1.2577	1.2449	1.2573	1.2567
MRE	0.997	0.995	0.9976	1.0061	0.9959	1.0058	1.0053
Bias	-0.004	-0.006	-0.003	0.0077	-0.005	0.0073	0.0067
MSE	0.0081	0.0081	0.008	0.0082	0.0079	0.0082	0.0082
	true value $(\varphi = 1.25)$, n=150						
est. value	1.2487	1.2471	1.2493	1.2564	1.2479	1.2561	1.2557
MRE	0.999	0.9977	0.9994	1.0051	0.9983	1.0049	1.0046
Bias	-0.001	-0.003	-0.0007	0.0064	-0.002	0.0061	0.0057
MSE	0.0052	0.0052	0.0052	0.0053	0.0051	0.0053	0.0053

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 3.1 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on SELF when the true value $(\varphi = 1.5)$ with r=10000 .

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a)) $\hat{\varphi}_{BE\_SELF(1)}$		Inverse Nakagami-b (INK (b,c)) $\hat{\varphi}_{BE\_SELF(2)}$		Inverse Half –Normal (IHN(υ)) $\hat{\varphi}_{BE\_SELF(3)}$	
	MLE	0.3	0.7	(0.5,2)	(2,2)	2.5	4
	true value $(\varphi = 1.5)$, n=25						
est. value	1.4852	1.4449	1.4555	1.5071	1.442	1.5057	1.5036
MRE	0.9902	0.9632	0.9703	1.0047	0.9613	1.0038	1.0024
Bias	-0.0148	-0.0551	-0.0445	0.0071	-0.058	0.0057	0.0036
MSE	0.0441	0.0443	0.0426	0.0444	0.0418	0.0445	0.0446
	true value $(\varphi = 1.5)$, n=50						
est. value	1.4913	1.4706	1.4759	1.5022	1.4685	1.5015	1.5005
MRE	0.9942	0.9804	0.984	1.0015	0.979	1.001	1.0003
Bias	-0.0087	-0.0294	-0.0241	0.0022	-0.0315	0.0015	0.0005
MSE	0.0225	0.0226	0.0221	0.0225	0.0219	0.0225	0.0226
	true value $(\varphi = 1.5)$, n=100						
est. value	1.4958	1.4853	1.488	1.5013	1.4841	1.5009	1.5004
MRE	0.9972	0.9902	0.992	1.0008	0.9894	1.0006	1.0003
Bias	-0.0042	-0.0147	-0.012	0.0013	-0.0159	0.0009	0.0004
MSE	0.0111	0.0111	0.011	0.0111	0.0109	0.0111	0.0111
	true value $(\varphi = 1.5)$, n=150						
est. value	1.4978	1.4907	1.4925	1.5014	1.4898	1.5011	1.5008
MRE	0.9985	0.9938	0.995	1.0009	0.9932	1.0008	1.0005
Bias	-0.0022	-0.0093	-0.0075	0.0014	-0.0102	0.0011	0.0008
MSE	0.0076	0.0076	0.0075	0.0076	0.0075	0.0076	0.0076

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 3.2 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on DLF when the true value $(\varphi = 1.5)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a)) $\hat{\varphi}_{BE\_DLF(1)}$		Inverse Nakagami-b (INK (b,c)) $\hat{\varphi}_{BE\_DLF(2)}$		Inverse Half –Normal (IHN(υ)) $\hat{\varphi}_{BE\_DLF(3)}$	
	MLE	0.3	0.7	(0.5,2)	(2,2)	2.5	4
	true value $(\varphi = 1.5)$, n=25						
est. value	1.4852	1.4729	1.4837	1.5388	1.4689	1.5374	1.5353
MRE	0.9902	0.9819	0.9892	1.0259	0.9793	1.0249	1.0235
Bias	-0.0148	-0.0271	-0.0163	0.0388	-0.0311	0.0374	0.0353
MSE	0.0441	0.0436	0.0425	0.0478	0.0409	0.0478	0.0477
	true value $(\varphi = 1.5)$, n=50						
est. value	1.4913	1.4851	1.4905	1.5176	1.4827	1.5169	1.5159
MRE	0.9942	0.9901	0.9936	1.0117	0.9885	1.0113	1.0106
Bias	-0.0087	-0.0149	-0.0095	0.0176	-0.0173	0.0169	0.0159
MSE	0.0225	0.0224	0.0221	0.0233	0.0217	0.0233	0.0233
	true value $(\varphi = 1.5)$, n=100						
est. value	1.4958	1.4927	1.4954	1.5089	1.4914	1.5085	1.508
MRE	0.9972	0.9951	0.9969	1.0059	0.9943	1.0057	1.0053
Bias	-0.0042	-0.0073	-0.0046	0.0089	-0.0086	0.0085	0.008
MSE	0.0111	0.011	0.011	0.0113	0.0109	0.0113	0.0113
	true value $(\varphi = 1.5)$, n=150						
est. value	1.4978	1.4957	1.4974	1.5064	1.4948	1.5062	1.5059
MRE	0.9985	0.9971	0.9983	1.0043	0.9965	1.0041	1.0039
Bias	-0.0022	-0.0043	-0.0026	0.0064	-0.0052	0.0062	0.0059
MSE	0.0076	0.0076	0.0075	0.0077	0.0075	0.0077	0.0077

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 4.1 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on SELF when the true value $(\varphi = 2)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a)) $\hat{\varphi}_{BE\_SELF(1)}$		Inverse Nakagami-b (INK (b,c)) $\hat{\varphi}_{BE\_SELF(2)}$		Inverse Half –Normal (IHN(υ)) $\hat{\varphi}_{BE\_SELF(3)}$	
	MLE	2	3.5	(3,4)	(3.5,2.5)	5.5	6
	true value $(\varphi = 2)$, n=25						
est. value	1.9803	2.0017	2.1537	1.8371	1.8187	2.0021	2.002
MRE	0.9902	1.0008	1.0769	0.9185	0.9094	1.0011	1.001
Bias	-0.0197	0.0017	0.1537	-0.1629	-0.1813	0.0021	0.002
MSE	0.0785	0.0679	0.0822	0.0917	0.095	0.0795	0.0795
	true value $(\varphi = 2)$, n=50						
est. value	1.9884	1.9988	2.0781	1.9118	1.9007	1.9993	1.9992
MRE	0.9942	0.9994	1.0391	0.9559	0.9504	0.9997	0.9996
Bias	-0.0116	-0.0012	0.0781	-0.0882	-0.0993	-0.0007	-0.0008
MSE	0.0399	0.0371	0.0404	0.044	0.0452	0.0402	0.0402
	true value $(\varphi = 2)$, n=100						
est. value	1.9944	1.9995	2.04	1.9547	1.9486	1.9999	1.9999
MRE	0.9972	0.9998	1.02	0.9774	0.9743	0.9999	0.9999
Bias	-0.0056	-0.0005	0.04	-0.0453	-0.0514	-0.0001	-0.0001
MSE	0.0197	0.019	0.0198	0.0208	0.0211	0.0198	0.0198
	true value $(\varphi = 2)$, n=150						
est. value	1.997	2.0004	2.0275	1.9702	1.966	2.0006	2.0006
MRE	0.9985	1.0002	1.0137	0.9851	0.983	1.0003	1.0003
Bias	-0.003	0.0004	0.0275	-0.0298	-0.034	0.0006	0.0006
MSE	0.0135	0.0131	0.0136	0.0139	0.0141	0.0135	0.0135

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Table 4.2 The MRE and Bias and MSE of the Estimated Values of scale parameter (φ) of Half normal distribution by using MLE and Bayes estimation under different priors based on DLF when the true value $(\varphi = 2)$ with r=10000.

Criteria	Bayes est.	Inverse Maxwell-Boltzmann(IMB (a))		Inverse Nakagami-b (INK (b,c))		Inverse Half –Normal (IHN(υ))	
		$\hat{\varphi}_{BE\_DLF(1)}$		$\hat{\varphi}_{BE\_DLF(2)}$		$\hat{\varphi}_{BE\_DLF(3)}$	
	MLE	2	3.5	(3,4)	(3.5,2.5)	5.5	6
	true value $(\varphi = 2)$, n=25						
est. value	1.9803	2.0405	2.1955	1.869	1.8493	2.0443	2.0441
MRE	0.9902	1.0203	1.0978	0.9345	0.9246	1.0221	1.022
Bias	-0.0197	0.0405	0.1955	-0.131	-0.1507	0.0443	0.0441
MSE	0.0785	0.0722	0.0991	0.0845	0.087	0.0848	0.0848
	true value $(\varphi = 2)$, n=50						
est. value	1.9884	2.0185	2.0986	1.9296	1.9181	2.0198	2.0197
MRE	0.9942	1.0093	1.0493	0.9648	0.9591	1.0099	1.0099
Bias	-0.0116	0.0185	0.0986	-0.0704	-0.0819	0.0198	0.0197
MSE	0.0399	0.0382	0.0447	0.0419	0.0427	0.0414	0.0414
	true value $(\varphi = 2)$, n=100						
est. value	1.9944	2.0095	2.0501	1.9641	1.9579	2.01	2.01
MRE	0.9972	1.0047	1.025	0.9821	0.979	1.005	1.005
Bias	-0.0056	0.0095	0.0501	-0.0359	-0.0421	0.01	0.01
MSE	0.0197	0.0193	0.0209	0.0202	0.0205	0.0201	0.0201
	true value $(\varphi = 2)$, n=150						
est. value	1.997	2.007	2.0342	1.9766	1.9723	2.0074	2.0073
MRE	0.9985	1.0035	1.0171	0.9883	0.9862	1.0037	1.0037
Bias	-0.003	0.007	0.0342	-0.0234	-0.0277	0.0074	0.0073
MSE	0.0135	0.0133	0.0141	0.0137	0.0138	0.0136	0.0136

Note.1: The shadow cells represent the smallest value of Bias and MSE.

Form the results of the Bayes estimators which are listed in table1.1 to table 4.2, which are included the MRE and Bias and MSE of the estimated values of scale parameter (φ) of the Half normal distribution by using MLE and Bayes estimation under different priors based on SELF and DLF. In general, from simulation results, we obtained asymptotic unbiased estimators for the scale parameter (φ) according to the mean relative estimate (MRE) which are tend to one under all prior distributions and all sample sizes(n).

Also, from Table1.1 and Table1.2 when the true value $(\varphi = 0.95)$, we concluded that the best estimators based on the squared error loss function (SELF) of the scale parameter (φ), when the prior distribution are the Inverse Maxwell-Boltzmann (IMB(0.3)) according to the smallest values of the mean biased estimate (Bias) for all samples sizes (n) and the Inverse Nakagami-m (INK(2,2)) according to the smallest values of the mean square error (MSE) for all sample sizes (n). See Table .5. Also, From Table.5 we see that the best estimators based on (SELF) and (DLE) of the scale parameter (φ), when the prior distribution are the Inverse Maxwell-Boltzmann (IMB (0.3)) for all n and the IMB(0.7) for n=150. And the Inverse Nakagami-b (INK (2,2)) according to the smallest values of the mean square error (MSE) for n=150. See Table .5.

Table.5 The conclusion from Table1.1 and Table1.2 when the true value $(\varphi = 0.95)$.

Criteria	n	Prior dist^{n}.	Min. value for SELF	Min. value for DLF	Min. value	The best
Bias	25	IMB (0.3)	-0.0326	-0.0148	-0.0326	SELF
MSE		INK(2,2)	0.0148	0.0152	0.0148	SELF
Bias	50	IMB (0.3)	-0.0175	-0.0083	-0.0175	SELF
MSE		INK(2,2)	0.0082	0.0083	0.0082	SELF
Bias	100	IMB (0.3)	-0.0076	-0.003	-0.0076	SELF
MSE		INK(2,2)	-0.0076	0.0044	-0.0076	SELF and DLF
Bias	150	IMB (0.3)	-0.0061	-0.003	-0.0061	SELF
MSE		MLE	0.0029	0.0029	0.0029	SELF and DLF
		IMB (0.3) and IMB (0.7) and INK(2,2)	0.0029	0.0029	0.0029	SELF and DLF

Also, from Table2.1 and Table2.2 when the true value $(\varphi = 1.25)$, we obtained best estimators based on the SELF of the scale parameter (φ), when the prior distribution is the IMB(0.3) according to the smallest values of the mean biased estimate (Bias) for all sample sizes (n). And we obtained best estimators based on the DLF of the scale parameter (φ), when the prior distribution is the INK (2,2) according to the smallest values of the MSE for all sample sizes (n). Also we obtained same values for MSE when the estimators are based on the SELF and the DLF of the scale parameter (φ) for $n \geq 100$, See Table .6.

Table.6 The conclusion from Table 2.1 and Table 2.2 when the true value $(\varphi = 1.25)$.

Criteria	n	Prior dist^{n}.	Min. value for SELF	Min. value for DLF	Min.value	The best
Bias	25	IMB (0.3)	-0.0437	-0.02	-0.044	SELF
MSE		INK(2,2)	0.0281	0.028	0.028	DLF
Bias	50	IMB (0.3)	-0.0236	-0.012	-0.024	SELF
MSE		INK(2,2)	0.0146	0.0145	0.0145	DLF
Bias	100	IMB (0.3)	-0.0123	-0.006	-0.012	SELF
MSE		INK(2,2)	0.0079	0.0079	0.0079	SELF and DLF
Bias	150	IMB (0.3)	-0.007	-0.003	-0.007	SELF
MSE		INK(2,2)	0.0051	0.0051	0.0051	SELF and DLF

Also, from Table3.1 and Table3.2 when the true value $(\varphi = 1.5)$, we obtained best estimators based on the SELF of the scale parameter (φ), when the prior distribution is the INK (2,2) according to the smallest values of the mean biased estimate (Bias) for all sample sizes (n). And we obtained best estimators based on the DLF of the scale parameter (φ), when the prior distribution are the INK (2,2) according to the smallest values of the MSE for all n. Also we obtained same values for MSE when the estimators are based on the SELF and the DLF of the scale parameter (φ) for $n \geq 100$, See Table .7.

Table.7 The conclusion from Table 3.1 and Table 3.2 when the true value $(\varphi = 1.5)$.

Criteria	n	Prior dist^{n}.	Min. value for SELF	Min. value for DLF	Min. value	The best
Bias	25	INK(2,2)	-0.058	-0.0311	-0.058	SELF
MSE		INK(2,2)	0.0418	0.0409	0.0409	DLF
Bias	50	INK(2,2)	-0.0315	-0.0173	-0.0315	SELF
MSE		INK(2,2)	0.0219	0.0217	0.0217	DLF
Bias	100	INK(2,2)	-0.0159	-0.0086	-0.0159	SELF
MSE		INK(2,2)	0.0109	0.0109	0.0109	SELF and DLF
Bias	150	INK(2,2)	-0.0102	-0.0052	-0.0102	SELF
MSE		INK(2,2)	0.0075	0.0075	0.0075	SELF and DLF

Also, from Table4.1 and Table4.2 when the true value $(\varphi = 2)$, we obtained best estimators based on the SELF of the scale parameter (φ), when the prior distribution is the INK (3.5,2.5) according to the smallest values of the mean biased estimate (Bias) for all n. And we obtained best estimators based on the SELF of the scale parameter (φ), when the prior distribution is the IMB (2) according to the smallest values of the MSE for all n , See Table .7.

Table.8 The conclusion from Table 4.1 and Table 4.2 when the true value $(\varphi = 2)$.

Criteria	n	Prior dist^{n}.	Min. value for		Min. value	The best
			SELF	DLF		
Bias	25	INK (3.5,2.5)	-0.1813	-0.1507	-0.1813	SELF
MSE		IMB (2)	0.0679	0.0722	0.0679	SELF
Bias	50	INK (3.5,2.5)	-0.0993	-0.0819	-0.0993	SELF
MSE		IMB (2)	0.0371	0.0382	0.0371	SELF
Bias	100	INK (3.5,2.5)	-0.0514	-0.0421	-0.0514	SELF
MSE		IMB (2)	0.019	0.019	0.019	SELF
Bias	150	INK (3.5,2.5)	-0.034	-0.0277	-0.034	SELF
MSE		IMB (2)	0.0131	0.0133	0.0131	SELF

6. Conclusion

In this study, the half- Normal $(\mathrm{HN}(\varphi))$ distribution is defined and studied, we have proposed closed form estimators for the half- Normal $(\mathrm{HN}(\varphi))$ distribution where location parameter equal to zero and scale parameter is $(\varphi > 0)$.We used Bayes estimator in addition to the maximum likelihood estimator (MLE) of the scale parameter of the $\mathrm{HN}(\varphi)$ distribution. We have derived the posterior distributions under three different informative priors which are the inverse Maxwell–Boltzmann (IMB) and the inverse Nakagami-b (INK) and the inverse half-normal (IHN). Also, we have derived the expression for the Jthposterior moments when J=1,2,3,4. We have compared their performance according to the three different criteria: the mean relative estimate (MRE) and The mean biased estimate (Bias) and mean square error (MSE), using a simulation study.

In general, the simulation study showed that the Bayes estimators under three different informative priors based on SELF showed better performance than the maximum likelihood estimator (MLE) for all sample sizes (n) using the mean biased estimate (Bias) and mean square error criterion (MSE), as shown in Table 1.1 to Table 4.2, for all the true values of the scale parameter $(\varphi = 0.95, 1.25, 1.5, 2)$ of the half- Normal distribution and for all sample sizes.

Also, from the results in Table.5 when the true value of the scale parameter $(\varphi = 0.95)$ of the half- Normal distribution showed that the best estimators obtained based on SELF under the prior distribution are

- IMB with (0.3) and INK with (3.5,2.5) gave more accurate results in terms of smallest value for Bias for all n.
- the INK with (2,2) and IMB with (2) gave more accurate results in terms of smallest value for MSE for all n.

Also, from the results in Table.6 when the true value of the scale parameter $(\varphi = 1.25)$ and $(\varphi = 2)$ of the half- Normal distribution showed that the best estimators obtained under the prior distribution are

- IMB with (0.3) gave more accurate results in terms of smallest value for Bias based on SELF for all n.
- the INK with (2,2) gave more accurate results in terms of smallest value for MSE based on DLF for all n.

Also, from the results in Table.8 when the true value of the scale parameter $(\varphi = 1.5)$ of the half-Normal distribution showed that the best estimators obtained under the prior distribution are

- INK with (2,2) gave more accurate results in terms of smallest value for Bias based on SELF for all n.
- INK with (2,2) gave more accurate results in terms of smallest values for MSE based on DLF for all n.

7. Recommendation

We recommend to obtain the Bayes estimators for the scale parameter (φ) of the HND under the Inverse Nakagami-b (INK (2,2)) as prior distribution based on different loss functions such as Linex loss function and Entropy loss function and Minimum Expected Loss (MELO) Function to compare the accuracy of the different estimates.

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