Invariant S-best Coapproximation in 2-normed Spaces

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ABSTRACT

The purpose of this paper is to discuss invariant S-best coapproximation concept for non-empty compact starshaped subset of linear 2-normed spaces, with respect to two linear contractive commuting operators.

Keywords: 2-normed space, S- best coapproximation, common fixed point, invariant approximation.

ثبات افضل اقتراب مشترك S في الفضاءات المعيارية الثنائية

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المستخلص

الهدف من هذا البحث مناقشة مفهوم افضل اقتراب ثابت مشترك S في المجموعات الجزئية المرصوصة النجمية للفضاءات المعيارية الثنائية، بالنسبة الى المؤثرات الخطية الانكماشية المتبادلة.

1- INTRODUCTION

The concepts of linear 2-normed spaces were initially introduced by White[1]. Then the subject has got great attention of researched many researchers (see [2-3]). This space has subsequently been studied by showing the existence of fixed point of contractive, nonexpensive, asymptotically nonexpensive mappings. As in other spaces, the fixed point theory has been developed in such space also. Fixed point theorems have been used at many places in approximation theory, in the literature, most studies on fixed point theorem to approximation theory consider with normed linear spaces like [4-5-6-7]. On the other hand, some mathematicians applied the idea of best approximation in 2-normed spaces such as [8-9-10-11].

The concepts of best coapproximation are another kind of approximation theory was first introduced by Franchettei and Furi [12], to study some characteristic properties of real Hilbert spaces. Subsequently, Vijayaragaavan[13], developed this problem and dealt with some fundamental properties of the set of strongly unique best coapproximation in linear 2-normed spaces. Delphi [14] recently, introduced and study new concept namely, S-best coapproximation in linear 2-normed spaces, where introduced the notions S-best coapproximation and S-orthogonality in 2-normed spaces and then, some characterization and important theorem about existence of S-best coapproximation in convex subset of 2-normed linear spaces were proved. For this paper, we give two results about invariant S-best coapproximation in linear 2-normed spaces, for this purpose we recall some definition and facts as follows:

Definition (1.1) [9]: Let X be a linear space over real number with dimension d, where $2 \le d \le \infty$ and let $\|.,.\|$: be a non-negative real valued function on $X \times X$ satisfying the following properties for all a, b, c in X :

1- $\|x, y\| = 0 \leftrightarrow x, y$ are linearly dependent

2- $\|x, y\| = \|y, x\|$

3- $\|\alpha x, y\| = |\alpha|\|x, y\|$ where $\alpha \in R$

4- $\|x, y + z\| \le \|x, y\| + \|x, z\|$

Then $\|.,.\|$ is called 2-norm and the pair $(X, \|.,.\|)$ linear space X is called a linear 2-normed space.

A standard example of a 2-normed space is R^2 equipped with the following 2-norm, $\|x, y\| :=$ the area of the triangle having vertices $0, x$ and y.

Observe that in any 2-normed space $(X, \|.,.\|)$ we have $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in R$. Also, if x, y, z are linearly dependent(this happens for instance, when $d = 2$) then $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ [6]. Every subspace A of 2-normed spaces X is convex. In particular every 2-normed spaces X is convex. Since, if A is a subspace of X and $a_1, a_2 \in A$ then $\alpha a_1 + \beta a_2 \in A$, for all scalars α, β, thus in particular if we put $\alpha = 1 - \lambda$ and $\beta = \lambda$, for all $\lambda \in [0,1]$, then we have $(1-\lambda)a_1 + \lambda b_2 \in A$, and so A is convex.

Definition(1.2)[13]: Let A be a subset of real linear 2-normed space X and $x \in X$,then $a_\circ \in A$ is said to be a best coapproximation to $x \in X$ from the element of A, if for every $a \in A$ $\|a - a_\circ, z\| \leq \|x - a, z\|$, $\forall z \in X / V(x, A)$, where $V(x, A)$ is the subspace generated by x and A.

The set of all elements of best coapproximation to $x \in X$ from A with respect to z is denoted by $R_A(x, z)$ where $R_A(x, z) = \{a_\circ \in G \mid \|a - a_\circ, z\| \leq \|x - a, z\|\}$ and it is called coproximinal set.

Suppose that $(X, \|.,.\|)$ is a 2-normed space, with dimension d, where $2 \leq d < \infty$, and $\{z_1, \ldots, z_d\}$ be its basis. we start with :

Definition(1.3)[14]: let A be a non-empty subset of linear 2-normed spaces X . An element $a_\circ \in A$ is said to be an S-best coapproximation of $x \in X$ from A if $a_\circ \in \bigcap_{i=1}^{d} R_G(x, z_i)$. The set of all elements of S-best coapproximation of X from A is denoted by $SR_A(x, z)$, this means $SR_A(x, z) = \bigcap_{i=1}^{d} R_A(x, z_i)$. Also if each $x \in X$ has at least (respectively exactly) one S-best coapproximation in A, then $SR_A(x, z)$ is called S-best coaproximinal(respectively S-coChebyshev) set.

Remark(1.4))[14]: If each $x \in X$ has at least (respectively exactly) one S-best coapproximation in A, then $SR_A(x, z)$ is called S-best coaproximinal(respectively S-coChebyshev) set.

Example(1.5)[14]: Suppose $X = R^2$ with usual basis $e = \{e_1, e_2\}$ and the norm $\|x,z\| = \begin{vmatrix} x_{11} & x_{12} \\ z_{21} & z_{22} \end{vmatrix} = |x_{11}z_{22} - x_{12}z_{21}|$,let $A = \{a = (a_1, a_2) \mid a \geq 0\}$ be a subset of X , to prove that

$a_\circ = (0,1) \in \bigcap_{i=1}^{2} R_{(1,1)}((-1,1), e_i) = SR_{(1,1)}((-1,1), e)$, for any $a = (a_1, a_2) \in A$ we have,

$$\|a_\circ - a, e_1\| \leq \|a - x, e_1\| = \begin{vmatrix} a_{\circ 1} - a_1 & a_{\circ 2} - a_2 \\ 0 & 1 \end{vmatrix} \leq \begin{vmatrix} a_1 - x_1 & a_2 - x_2 \\ 0 & 1 \end{vmatrix} \Rightarrow$$

$$\|(0,1) - (1,1), e_1\| \leq \|(1,1) - (-1,1), e_1\| \Rightarrow \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \leq \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \Rightarrow |1| \leq |2|,$$

and so $a_\circ = (0,1) \in R_{(1,1)}((-1,1), e_1)$, we have the same result if replace e_1 by e_2

$$\|a_\circ - a, e_2\| \leq \|g - x, e_2\| \Rightarrow \begin{vmatrix} a_{\circ 1} - a_1 & a_{\circ 2} - a_2 \\ 1 & 0 \end{vmatrix} \leq \begin{vmatrix} a_1 - x_1 & a_2 - x_2 \\ 1 & 0 \end{vmatrix} \Rightarrow$$

$$\|(0,1) - (1,1), e_2\| \leq \|(1,1) - (-1,1), e_2\| \Rightarrow \begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} \leq \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \Rightarrow |0| = |0|,$$

and so $a_\circ = (0,1) \in R_{(1,1)}((-1,1), e_2)$,therefore

$a_\circ = (0,1) \in \bigcap_{i=1}^{2} R_{(1,1)}((-1,1), e_i)$, and so $a_\circ = (0,1) \in SR_{(1,1)}((-1,1), e)$, and so $SR_A(x,z)$ is S-coChebyshev set.

Definition(1.6)[8]: let X be a non-empty set and $T : X \to X$ a self map. We say that $x \in X$ is a fixed point of T, if $T(x) = x$ and denote by F(T) the set of all fixed point of T.

Definition(1.7)[15]: let X be a non-empty set and T, S are self mapping on X. We say that $x \in X$ is a common fixed point of T and S, if $T(x) = S(x) = x.$

Definition(1.8)[8]: let X be a linear 2-normed spaces then the mapping $T : X \to X$ is said to be a contractive if T satisfies $\|T(x) - T(y), z\| < \|x - y, z\|$ for all $x, y, z \in X$.

Definition(1.9) [15]: let X be a non-empty set and T,S are self mapping on X. We say that T,S commuting mapping on X, if $TS(x)=ST(x)$ for each $x\in X$.

2- MAIN RESULTS

In this section, we introduce some fixed point theorems and its application to invariant S-best coapproximation in 2-normed spaces.

Theorem (2.1):let T be a linear contractive operator on 2-normed linear space X, let A be a T-invariant subset of X and a a T-invariant point. If the set of S-best coapproximation to $x\in X$ is non-empty, compact, and starshaped, then $SR_A(x,z)\cap F(T)\neq\phi$.

Proof:

let Q be the set of S-best coapproximation to $x\in X$, then $T:Q\to Q$, since if $a_\circ\in Q$ then then for all $(i=1,\ldots,d)$,

$\|T(a_\circ)-a,z_i\|=\|T(a_\circ)-T(a),z_i\|\leq\|a_\circ-a,z_i\|\leq\|a-x,z_i\|$, then $T(a_\circ)\in Q$,

Take $q\in Q$ such that $\lambda q+(1-\lambda)h\in Q$ for all $h\in Q$ and $0\leq\lambda\leq 1$, let $k_n, 0\leq k_n<1$, be a real number such that $k_n\to 1$ as $n\to\infty$. Then define $T_n:Q\to Q$ by $T_n(h)=k_nT(h)+(1-k_n)q$ for all $h\in Q$, since T maps Q into Q, also T_n maps Q into Q for each n., also for all $(i=1,\ldots,d)$, we have $\|T_n(h)-T_n(y),z_i\|=k_n\|T(h)-T(y),z_i\|\leq k_n\|h-y,z_i\|<\|h-y,z_i\|$ for all $h,y\in Q, h\neq y$. Then, since Q is compact and T_n contractive, then T_n has a unique fixed point, say h_n for each n. thus, $T_n(h_n)=h_n$. since Q is compact, h_n has a convergent subsequence h_{n_i} converging to h. We claim that $T(h)=h$. Now since $h_{n_i}=T_{ni}(h_{n_i})=(1-k_{n_i})q+k_{n_i}T(h_{n_i})$, by the following inequality $\|T(h)-h,z_i\|\leq\|T(h)-T(h_{n_i}),z_i\|+\|h_{n_i}-h,z_i\|$ and by taking limit as $i\to\infty, k_n\to 1$, we have $T(h)=h$ ($h_{n_i}\to h$ then $T(h_{n_i})\to T(h)$ as T is contractive) thus h is a T-invariant.■

To illustrate theorem(2.1), we give the following example:

Example(2.2): let T be a linear continues contractive contractive operator on $X=R^2$ with usual basis $e=\{e_1,e_2\}$ and the norm

$\|x,z\|=\begin{vmatrix}x_{11} & x_{12}\\ z_{21} & z_{22}\end{vmatrix}=|x_{11}z_{22}-x_{12}z_{21}|$ and $A=\{a=(a_1,a_2)\mid -1\leq a\leq 1\}$ be

a T-invariant subset of X and a a T-invariant point, then with a simple calculus can be shown that $(-1,1),(0,1)\in\bigcap_{i=1}^{2}R_{(-1,1)}((-2,2),e_i)=SR_{(-1,1)}((-2,2),e)$, and so $SR_A(x,z)$ is non-empty and compact, let $Q=SR_{(-1,1)}((-2,2),e)$, the $T:Q\to Q$, since $(0,1)\in SR_{(-1,1)}((-2,2),e)$, then for $(i=1,2)$ we have,
$\|T((0,1))-(-1,1),e_i\|=\|T((0,1))-T((-1,1)),e_i\|\le\|(0,1)-(-1,1),e_i\|\le\|(-1,1)-(-2,2),e_i\|$,
then $T((0,1))\in Q$, since $\lambda(0,1)+(1-\lambda)(-1,1)\in Q$ for all $\lambda\in[0,1]$, then Q is starshaped, let $k_n, 0\le k_n<1$, be a real number such that $k_n\to 1$ as $n\to\infty$. Then define $T_n:Q\to Q$ by $T_n((0,1))=k_nT((0,1))+(1-k_n)(-1,1)$, since T maps Q into Q, also T_n maps Q into Q for each n., also for $(i=1,2)$, we have
$\|T_n((0,1))-T_n((-1,1)),e_i\|=k_n\|T((0,1))-T((-1,1),e_i\|\le k_n\|(0,1)-(-1,1),e_i\|<\|(0,1)-(-1,1),e_i\|$,
and so T_n a contractive operator on Q and Q is compact, then T_n has a unique fixed point, thus, $T_n((0,1))=(0,1)$ for each n. Now, by the inequality $\|T((0,1))-(0,1),z_i\|\le\|T((0,1))-T_n((0,1)),e_i\|+\|T_n((0,1))-(0,1),e_i\|$, and by taking limit as $n\to\infty, k_n\to 1$, we have $T((0,1)=(0,1)$, thus $(0,1)$ is a T-invariant.
Now, we extend Theorem(2.1) for pair of contractive operator:

Theorem (2.3): let T,I two linear contractive commuting self operators on 2-normed linear space X , let A be subset of X such that $T:\partial A\to A$, and $a\in F(T)\cap F(I)$. Further, T and I satisfy $\|T(x)-T(y),z\|\le\|I(x)-I(y),z\|$ for all $x,y\in SR_A(x,z)$, and let I be linear, continuous, on $SR_A(x,z)$, and $I(T(x))=T(I(y))$ for all $x,y\in SR_A(x,z)$, if the set of S-best coapproximation to $x\in X$ is non-empty, compact, and starshaped, with respect to a point $q\in F(I)$ and if $I(SR_A(x,z))=SR_A(x,z)$, then $SR_A(x,z)\cap F(T)\cap F(I)\ne\phi$.

Proof: let Q be the set of S-best coapproximation to $x\in X$, then $T:Q\to Q$ (since if $a_\circ\in Q$ and hence $I(Q)=Q$. Further, $y\in\partial(A)$ since $T(\partial A)\subseteq A$.
From $\|T(x)-T(y),z\|\le\|I(x)-I(y),z\|$ it follows that
$\|T(a_\circ)-a,z\|=\|T(a_\circ)-T(a),z\|\le\|I(a_\circ)-I(a),z\|\le\|a-x,z\|$, then
$T(a_\circ)\in Q$,thus T maps Q into Q. let $\{k_n\}$ be a sequence of real numbers such that $0\le k_n<1$, be a real number such that $k_n\to 1$ as $n\to\infty$. Define $T_n:Q\to Q$ by $T_n(h)=k_nT(h)+(1-k_n)q$ for all $h\in Q$, since T maps Q into Q, also T_n maps Q into Q for each n. , also for all $(i=1,\ldots,d)$, we have
$\|T_n(h)-T_n(y),z_i\|=k_n\|T(h)-T(y),z_i\|\le k_n\|h-y,z_i\|<\|h-y,z_i\|$ for all

$h, y \in Q, h \neq y$. Then, since Q is compact and T_n contractive, then T_n has a unique fixed point, say h_n for each n. thus, $T_n(h_n) = h_n$. since Q is compact, h_n has a convergent subsequence h_{n_i} converging to h. We claim that $T(h) = h$. Now since $h_{n_i} = T_{ni}(h_{n_i}) = (1-k_{n_i})q + k_{n_i}T(h_{n_i})$, by the following inequality $\|T(h)-h, z_i\| \leq \|T(h)-T_{n_i}(h_{n_i}), z_i\| + \|h_{n_i} - h, z_i\|$ and by taking limit as $i \to \infty, k_n \to 1$, we have $T(h) = h$ ($h_{n_i} \to h$ then $T_{n_i}(h_{n_i}) \to T(h)$ as T is contractive) thus $h \in Q \cap F(T)$. Now , since I is linear and commutes with T on Q , we have that

$T_n(I(h)) = k_n T(I(h)) + (1-k_n)I(q) = k_n I(T(h)) + (1-k_n)I(q)$

$= I(k_n T(h) + (1-k_n)q) = I(T_n(h))$ for all $h \in Q$. Thus I commutes with T_n on Q for each n, $T_n(Q) \subseteq Q = I(Q)$. Further , then for all $(i = 1,\ldots,d)$, we have $\|T_n(h) - T_n(y), z_i\| = k_n \|T(h) - T(y), z_i\| \leq k_n \|I(h) - I(y), z_i\| < \|I(h) - I(y), z_i\|$ whenever $I(h) \neq I(y)$, since Q is compact and I is continuous, we deduce that $F(T) \cap F(I) = \{h_n\}$ for each n. Further, the continuity of I implies that $I(h) = I(\lim_{i\to\infty} h_{n_i}) = \lim_{i\to\infty} I(h_{n_i}) = \lim_{i\to\infty}(h_{n_i}) = h,$ i.e., $h \in Q \cap F(I)$, and so $Q \cap F(T) \cap F(I) \neq \phi$. ■

To illustrate theorem(2.3), we give the following example:

Example(2.4): let T be a linear contractive operator on $X = R^2$ with usual basis $e = \{e_1, e_2\}$ and the norm $\|x, z\| = \begin{vmatrix} x_{11} & x_{12} \\ z_{21} & z_{22} \end{vmatrix} = |x_{11}z_{22} - x_{12}z_{21}|$ and $A = \{a = (a_1, a_2) \mid -2 \leq a \leq 2\}$ be a T-invariant subset of X and a a T-invariant point, then with a simple calculus can be shown that $(-2,2),(0,1) \in \bigcap_{i=1}^{2} R_{(-2,2)}((-3,3),e_i) = SR_{(-2,2)}((-3,3),e)$, and so $SR_A(x,z)$ is non-empty and compact, let $Q = SR_{(-2,2)}((-3,3),e)$, the $T: Q \to Q$, $I: Q \to Q$ since $(0,1) \in SR_{(-2,2)}((-3,3),e)$, then for $(i = 1,2)$ we have,

$\|T((0,1)) - T(-2,2), e_i\| = \|T((0,1)) - (-2,2), e_i\| \leq \|I(0,1) - I(-2,2), e_i\| = \|(0,1) - (-2,2), e_i\|$ $\leq \|(-1,1) - (-3,3), e_i\|$, then $T((0,1)) \in Q$, since $\lambda(0,1) + (1-\lambda)(-2,2) \in Q$ for all $\lambda \in [0,1]$, then Q is starshaped, let $k_n, 0 \leq k_n < 1$, be a real number such that $k_n \to 1$ as $n \to \infty$. Then define $T_n : Q \to Q$ by $T_n((0,1)) = k_n T((0,1)) + (1-k_n)(-2,2)$, since T maps Q into Q, also T_n maps Q

into Q for each $n.$, also for $(i=1,2)$, we have
$\|T_n((0,1))-T_n((-2,2)),e_i\| = k_n\|T((0,1))-T((-2,2),e_i\| \le k_n\|(0,1)-(-2,2),e_i\| < \|(0,1)-(-2,2),e_i\|$,
and so T_n a contractive operator on Q and Q is compact, then T_n has a unique fixed point, thus, $T_n((0,1))=(0,1)$ for each n. Now ,
$T_n(0,1)=k_nT((0,1))+(1-k_n)(-2,2)$, taking limit as $n\to\infty, k_n\to 1$, we have
$T_n((0,1)=T((0,1))$, since T is contractive we have ,
$\|T((0,1))-(0,1),e_i\| \le \|T((0,1))-T_n((0,1)),e_i\| + \|T_n(0,1)-(0,1),e_i\|$ as $n\to\infty, k_n\to 1$ we have $T((0,1))=(0,1)$ then $Q\cap F(T)\neq\phi$. Now , since I is linear and commutes with T on Q , we have $T_n(I(0,1))=k_nT(I(0,1))+(1-k_n)(I(-2,2))=$
$k_nI(T(0,1))+(1-k_n)(I(-2,2))=I(T_n(0,1))$ Thus I commutes with T_n for each $n.$ on Q. And so for $(i=1,2)$, we have
$\|T_n((0,1))-T_n((-2,2)),e_i\| = k_n\|T((0,1))-T((-2,2),e_i\| \le k_n\|I(0,1)-I(-2,2),e_i\| < \|I(0,1)-I(-2,2),e_i\|$
, then, since Q is compact, and I is continuous, then
$T_n(I(0,1))=I(T_n(0,1))=(0,1)$ by taking limit as $n\to\infty, k_n\to 1$, we have
$F(T)\cap F(I)\neq\phi$. The continuity of I on Q implies that
$\|I((0,1))-(0,1),e_i\| \le \|I(0,1)-T_n(0,1),e_i\| + \|T_n(0,1)-(0,1),e_i\|$ by taking limit as
$n\to\infty, k_n\to 1$,we have $Q\cap F(I)\neq\phi$, and so $Q\cap F(T)\cap F(I)\neq\phi$.

3- Conclusion

In this paper two results have been proved about S-best coapproximation as a fixed point in a 2-normed spaces. This paper can be extended to other setting, such as S-best coapproximation as a common fixed point in a 2-normed spaces and S-best coapproximation as a fixed point in a n-normed spaces with different kinds of operators.

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