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Novel Approach to Integral Equations and Population Growth using Yang Transform

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Abstract:

Integral transforms provide powerful tools for engineers, physicists, and mathematicians. Many issues that arise in different areas of science and engineering, can be solved with ease and effectiveness using the yang integral transform methods. for solution using these integral Therefore, that exact solution has been obtained using very less computational work and spending very little time as well. The main objective of this paper is to focus on studying and classifying integro-differential equations and integral equations, how to solve the integral equations using these transforms by taking some practical examples such as the equation, to illustrate how to use these integral transforms accurately and effectively.

Keywords: Convolution Theorem ; integro-differential equation; integral transforms ; integral equations.

منهجية جديدة لحل المعادلات التكاملية ونمو السكان باستخدام تحويل يانغ

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المديرية العامة لتربية النجف الاشرف

ملخص البحث :

تعد التحويلات التكاملية أدوات قوية تستخدم على نطاق واسع من قبل المهندسين والفيزيائيين وعلماء الرياضيات . اذا يمكن حل العديد من المشكلات التي تنشأ في مجالات مختلفة من العلوم والهندسة بسهولة وكفاءة باستخدام طرق تحويل يانغ التكاملية . لذلك يمكن الحصول على حلول دقيقة مع جهد حسابي قليل جدا وفي زمن قصير نسبيا . يتمثل الهدف الرئيس لهذه الدراسة في التركيز على تحليل وتصنيف المعادلات التكاملية - وبيان كيفية حل المعادلات التكاملية باستخدام هذه التحويلات , بتطبيقها على أمثلة عملية مثل بعض المعادلات النموذجية وذلك لتوضيح كيفية استخدام التحويلات التكاملية بدقة وفاعلية .

الكلمات المفتاحية : مبرهنة الالتفاف , المعادلات التكاملية – التفاضلية , التحويلات التكاملية , المعادلات التكاملية .

1.Introduction

The integral equation is encountered in many scientific domains and applications oscillation theory, electrical engineering, economics, and medicine, filtration theory, game theory, control, queuing theory, elasticity, including fluid dynamics, plasticity, heat and mass transfer, among others. In mathematics, equations that have unknown functions appearing under the integral sign, are known as that integral equations:

$$\eta(\dot{x}) = f(\dot{x}) + \grave{}\lambda \int_{s(\dot{x})}^{r(\dot{x})} N(\dot{x}, \dot{t})\, \eta(\dot{t})\, d\dot{t}$$

The kernel of an integral equation is denoted by $N(\dot{x}, \dot{t})$, which is a known function of two variables. $s(\dot{x})$ and $r(\dot{x})$ are the limits of integration, and λ is that a constant parameter. It will be found that then unknown function is inside the integral sign [9]. Inside and outside the integral sign, the unknown function $\eta(\dot{x})$can be found in numerous other situations. $N(\dot{x}, \dot{t})$ and $f(\dot{x})$ are predefined functions. Understanding the qualitative aspects of numerous processes and phenomena in a variety of natural science domains depends on the precise solution of these equations. Since they are frequently derived from differential equations, integral equations are used in many different contexts [8]. In applied mathematics, engineering science and mathematical physics, integral transforms have been effectively utilized for nearly 200 years to solve a wide range of problems. The best integrals transform technique, in particular, can be used to derives the closed-form.

$$L\{f(\dot{x})\} = \int_0^\infty e^{-v\dot{x}} f(\dot{x})\, d\dot{x}, \quad \text{for } \dot{x} \geq 0$$

wherea v is real numberand L is Laplace transform operator and in 2016 Xiao – Jung Yang introduced a new integral transform, It has been mainly applied to solve many problems such as a steady heat transfer problem but here we will use it to solution the integral equations [3].,The Yang transform of the function `$f(\dot{x})$ is symbolizes by $\gamma\{f(\dot{x})\}$ and is the defined as:

$$\gamma\{f(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}} f(\dot{x}) d\dot{x} \qquad , \quad \dot{x} \geq 0$$

where are γ is Yang 'transform operator with $v \in (-\dot{x}_1, \dot{x}_2)$, the equations as well and a novel of that relationship between of the differential equation`and integral equations is considered.

Since the integral equations have many applications in real life, some important applications of these kinds of equations such as renewal equation, are solved using the previous integral transform.

2.Basic Concepts of Integral Equations and Integro – Differential Equations

2.1 Integral Equations:

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An integral equation is an equation containing an integral sign in which the unknown function $\boldsymbol{\eta}(\dot{t})$ [4]:

$$``\eta(\dot{x}) = `f(\dot{x}) + \lambda \int_{s(x)}^{r(\dot{x})} N(\dot{x},\dot{t}) \, '`\eta(\dot{t})\, d\dot{t}$$

… (1)

The kernel function of an integral equation is a known function of two variables, ẋ and ṫ, denoted by $N(\dot{x},\dot{t})$, since λ is a constant, $s(\dot{x})$ and $r(\dot{x})$ are the limits of integration. The driving term of the integral equation is often defined as that function $f(\dot{x})$,which is known beforehand. Furthermore, $\eta(\dot{x})$ does not only appear in the integral sign. Notably, (ẋ) and r(ẋ) ,the limits of integration, can be both variables, mixed variables, or constants.

2. 2 Integro – Differential'Equations:

Many scientific applications can be mathematically represented by an integral-differential equation, especially when initial value or boundary value problems are transformed into integral equations A general integro–differential equation involves both integrals and derivatives of the unknown function, and can be written in the following form[4]:

$$`\eta^{(k)}(\dot{x}) = i\mathrm{F}(\dot{x}) + `\lambda \int_a^b N(\dot{x},\dot{t})\eta(\dot{t})\,'d\dot{t} \qquad , \qquad \eta^{(k)} = \frac{d^k\eta}{d\dot{x}^k}$$

… (2)

2.3 Connection Between` Integral Equations and Differential Equations:

As previously discussed, many engineering and scientific applications, such as atmospheric radar, quantum mechanical scattering, and water waves, give rise to both integro–differential and integral equations. Importantly, integral equations and integro-differential equations can be obtained by converting initial value problems, even though they are derived from boundary value problems with specified boundary conditions. Although both conversions are reversible, the final conversion involved integral equations [4].

3. Some Useful Properties of Yang Transform :

3.1 YangTransform (γ – Transform

In 2016, Xiao–Jung Yang introduced a new integral transform. It has been mainly applied to solve many problems, such as steady heat transfer problems. In this section, we apply it to solve integral equations [2].

Definition: **[1]** The γ - transform of a function `$\digamma(\dot{x})$ is denoted by $\gamma\ \{\digamma(\dot{x})\}$ and is defined as :

$$\gamma\ \{\digamma(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}}\digamma(\dot{x})\ d\dot{x} \qquad , \quad \dot{x} \geq 0 \qquad \dots (1)$$

where γ is Yang transform operator.

This definition is valid provided that the integral exists for some values of $v \in (-\dot{x}_1, \dot{x}_2)$

If we substitute this condition $\frac{\dot{x}}{v} = \dot{t}$,the equation(1) becomes:

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$$\gamma\,\{F(x)\} = v\int_0^{\infty} e^{-t}\,F(vt)\,dt\,,\;\; t > 0\,.$$

3.2Convolution Theorem for γ - Transform

This is the main theorem to solve integral and differential equations, and it always plays an important role in a number of different physical applications [2].

Let f and g be functions with Yang transforms $T_1(v)$ and $T_2(v)$, respectively. Then, the Yang transform of their convolution is given by:

$$(`f * \bar{g})(\dot{x}) = '\int_0^{x} f(\dot{x})\,\bar{g}(\dot{x}-\dot{t})\,d\dot{t}$$

Then $$\gamma\{(`f * \bar{g})(\dot{x})\} = '\gamma\{f(\dot{x})\}.\gamma\{\bar{g}(\dot{x})\}$$

3.3Inverse of γ – Transform [2]

If $\gamma\,\{f(\dot{x})\}$ is the yang Transform of $f(\dot{x})$then the inverse of yang transform of $\gamma\,\{f(\dot{x})\}$ will be $f(\dot{x})$or mathematically:

$$\gamma^{-1}\{\gamma\ \{f(\dot{x})\}\} = f(\dot{x}).$$

where γ^{-1} is operator of the inverse γ - Transform and also then linearity property holds in the inverse γ – Transform . Thus:

$$\gamma^{-1}\left\{a\,\gamma\left\{f_1(\dot{x})\right\} + b\,\gamma\{f_2\,(\dot{x})\}\right\} = a\,\gamma^{-1}\{\gamma\{f_1(\dot{x})\}\} + b\,\gamma^{-1}\{\gamma\{f_2(\dot{x})\}\}$$

$$= a\,f_1(\dot{x}) + b f_2\,(\dot{x})\ .$$

Since a few basic functions, the following tables display the previously well-known transforms.

Table 1: (Y) –Transforms of some elementary functions

$S.N$	$f(\dot{x})$	$Y\{f(\dot{x})\}$
1	1	v
2	$\dot{x}$	v^2
3	$\dot{x}^2$	$2!\,v^3$
4	$\dot{x}^n, n \in N$	$n! \cdot v^{n+1}$
5	$e^{a\dot{x}}$	$\frac{v}{1-av}$
6	$\sin a\dot{x}$	$\frac{a\,v^2}{1+a^2v^2}$
7	$\cos a\dot{x}$	$\frac{v}{1+a^2v^2}$
8	$\sinh a\dot{x}$	$\frac{av^2}{1-a^2v^2}$

4. Dualities Between Transforms:

This section discusses the dualities between the Yang transform (γ-transform) and other well-known transforms, such as the Laplace transform, and highlights the relationships among them.

Examples are provided using φ to represent the results of other transforms, emphasizing the significance of these relationships in the context of integral transforms.

4.1 Laplace -Yang Duality

The L - Transform of the function $f(\dot{x})$, for $\dot{x} \geq 0$, denoted by $L\{f(\dot{x})\}$ and is defined as:

$$L\,[f(\dot{x})] = \int_0^\infty e^{-v\dot{x}}\, f(\dot{x})\, d\dot{x}\,,$$

and γ` - Transform of similarly function $f(\dot{x})$, for $\dot{x} \geq 0$ the defined as:

$$\gamma`\,\{f(\dot{x})\} = \int_0^\infty e^{-\frac{\dot{x}}{v}}\, f(\dot{x})\, d\dot{x}\,,$$

Consequently,

$$L`\,\{f(\dot{x})\} = \int_0^\infty e^{-v\dot{x}}\, f(\dot{x})\, d\dot{x} = \int_0^\infty e^{-\frac{\dot{x}}{\frac{1}{v}}}\, f(\dot{x})\, d\dot{x}.$$

From these definitions, the following relation holds:

$$L\{f(\dot{x})\} = \varphi(\tfrac{1}{\tilde{v}})$$

On the other hand, it can also be shown that:

$$\mathrm{i}\ \gamma\ \{f(\dot{x})\} = \int_0^{\infty} e^{-\frac{\dot{x}}{v}} f(\dot{x})\ d\dot{x}\ = \int_0^{\infty} e^{-\left(\frac{1}{v}\right)\dot{x}} f(\dot{x})\ d\dot{x} = \varphi\left(\frac{1}{\tilde{v}}\right).$$

Therefore, we obtain

$$\gamma\{f(\dot{x})\} = \varphi(\frac{1}{v})\ .$$

4.2Illustrative Examples

The following examples illustrate how Yang transforms can be applied to solve integral equations.

Example. Solve the following integral equation using integral transforms:

$$`\underset{\wedge}{u}(\dot{x}) = \frac{1}{6}\ \dot{x}^3 + \int_0^{\dot{x}} (\dot{x} - \dot{t})\ \underset{\wedge}{u}(\dot{t})\ d\dot{t}$$

Solution:

Applying the Yang transform to both sides of the equation gives:

$$\gamma\,\{\underset{\wedge}{u}(\dot{x})\} = \gamma\left\{\frac{1}{6}\,\dot{x}^3\right\} + \gamma\,\{\int_0^{\dot{x}}(\dot{x}-\dot{t})\,\underset{\wedge}{u}(\dot{t})\,d\dot{t}\}$$

$$\gamma\,\{\underset{\wedge}{u}(\dot{x})\} = \frac{1}{6}\,.3!\;\tilde{v}^4 + \tilde{v}^2\,.\gamma\,\{\underset{\wedge}{u}(\dot{x})\}$$

$$\gamma\,\{\underset{\wedge}{u}(\dot{x})\}(1-\tilde{v}^2) = \tilde{v}^4$$

$$\gamma`\,\{\underset{\wedge}{u}(\dot{x})\} = \frac{\tilde{v}^4}{(1-\tilde{v}^2)}$$

$$\underset{\wedge}{u}(\dot{x}) = \gamma^{-1}\left\{\frac{v^4}{1-\tilde{v}^2}\right\}$$

$$\underset{\wedge}{u}(\dot{x}) = \gamma^{-1}\left\{\frac{v^2}{1-\tilde{v}^2}\,.\tilde{v}^2\right\} = \dot{x}.\sinh\dot{x}.$$

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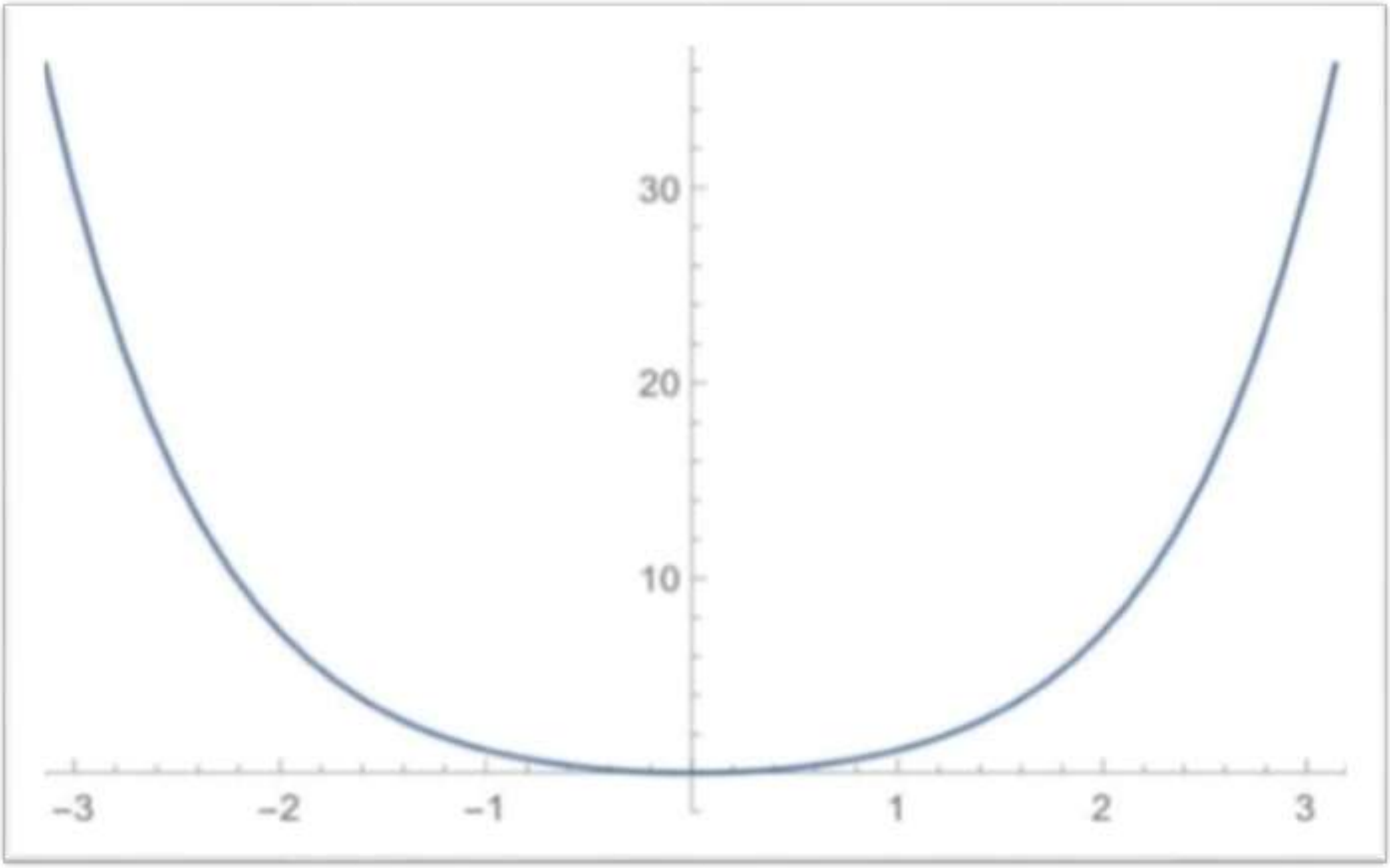

Graphical representation of the solution to the integral equation using the Yang transform: Fig.1:` $\boldsymbol{u}(\dot{x}) = \dot{x} \,.\, \boldsymbol{sinh}\, \dot{x}$

4.Population Growth Equation

The population of a city grows at a rate proportional to the number of people presently living in this city or it includes the forecasting of any future surge in birthrate, which is a great importance for future planing throughout the world. Moreover, other examples are growth of plant, or bacteria, or a cell, or an organ, or a species.

Human population growth is one of the examples that we can express it as an integral equation. We consider the number of children born at time $x = 0$ is b (where b is arbitrary constant) and the survival function is x , so we have [3]:

$$f(x) = bx + \int_0^x N(x-y)f(y)dy$$

.... (2)

So, we will solve this equation with some transformations as shown below:

$$\boldsymbol{f(x) = bx + \int_0^x N(x-y)\,.\,f(y)dy.}$$

$$L\{f(x)\} = L\{bx\} + L\{\int_0^x N(x-y)\,.\,f(y)dy\}$$

$: \gamma - Transform$

$$\gamma\{f(x)\} = \gamma\{bx\} + \gamma\{\int_0^x N(x-y)\, f(y)\, dy\}$$

$$\gamma\{f(x)\} = b\,.\,v^2 + v^2\,.\gamma\{f(x)\}$$

$$\gamma\{f(x)\}\,(1-v^2) = b\,v^2$$

$$\gamma\{f(x)\} = \frac{b\,v^2}{1-v^2} = b\,.\frac{v^2}{1-v^2}$$

$$\therefore f(x) = \gamma^{-1}\{b\,.\frac{v^2}{1-v^2}\}$$

$$f(x) = b\,.\sinh x\,.$$

Fig.2: Graphical representation of the solution to the population growth equation using the Yang

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No.13A May 2024
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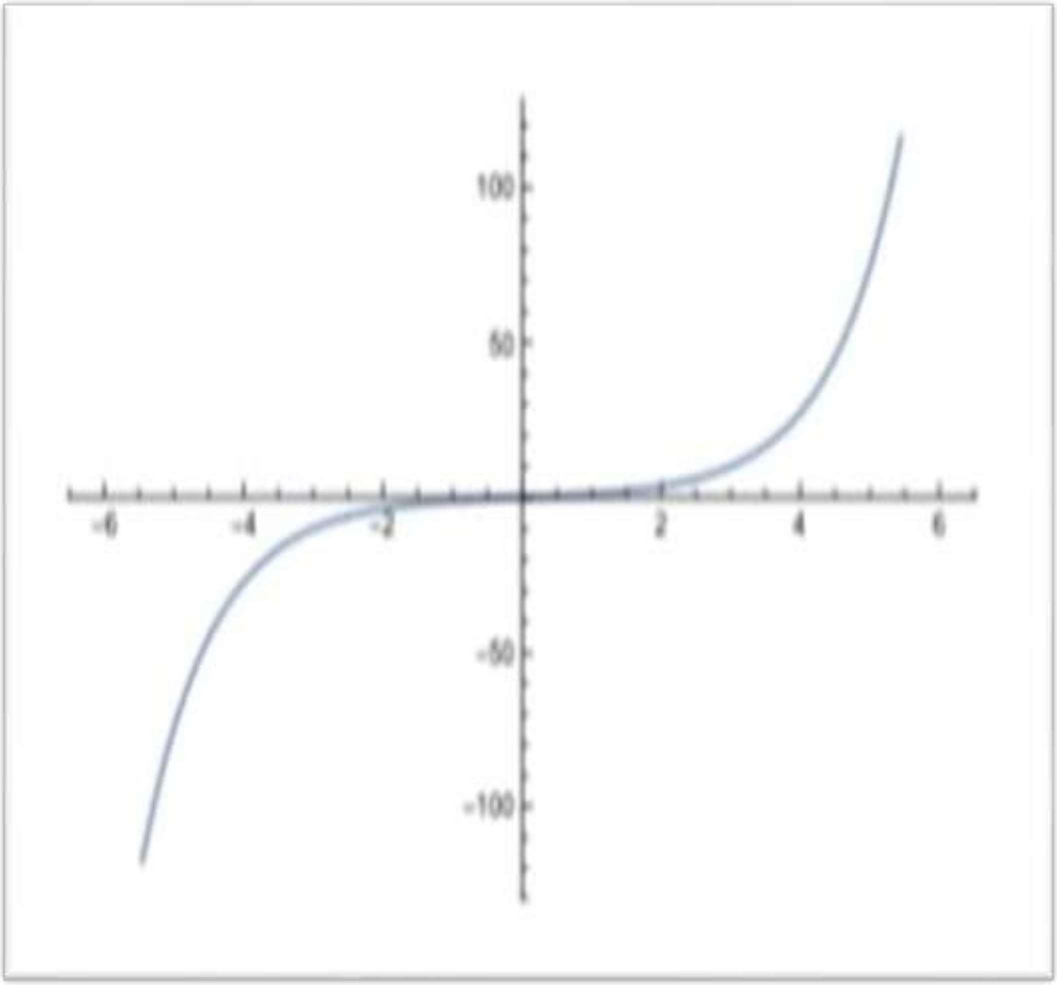
transform: $f(x) = b\,.\sinh x$



6.Conclusion and Future Work:

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In this paper, we focused on the study and classification of integral equations, particularly linear integral equations.

We demonstrated how well-known integral transforms, such as the Laplace and Yang transforms, can be used to solve such equations with improved accuracy and reduced error.

As an application, the population growth eq uation was discussed and solved using the Yang transform.

This study highlights the efficiency of the Yang transform as an alternative to traditional methods, and future research can further expand its applications to broader classes of integral and integro–differential equations. So, for future work, we recommend the following:

Apply these integral transforms to solve non-linear integral equations and explore their interrelationships.

1.Extend the method to integro–differential equations.

2.Investigate further applications of these transforms to non-linear problems and integro–differential equations.

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