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RESEARCH ARTICLE

A Multi-Valued Rough Neutrosophic Approach for Multi-Criteria Decision-Making: Application in Hospital Selection

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ABSTRACT

Decision-making problems are often characterized by complexity and uncertainty, presenting significant challenges for individuals or organizations. Traditional aggregation methods often struggle with the imprecise, inconsistent, and unclear data found in these situations. To address these issues, this study proposes a new multi-criteria decision-making method based on Multi-Valued Rough Neutrosophic Numbers. We first introduce the fundamental operational laws for multi-valued rough neutrosophic numbers, based on t-norm and t-conorm. Subsequently, we establish four new multi-valued rough neutrosophic number-based aggregation operators: the Multi-Valued Rough Neutrosophic Arithmetic Mean Operator, the Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator, the Multi-Valued Rough Neutrosophic Geometric Mean Operator, and the Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator. We also prove that these proposed aggregation operators satisfy desirable properties, including monotonicity, idempotency, and boundedness. To determine the optimal weights of criteria, we utilize the Shapley fuzzy measure, capturing the significance and contribution of each criterion. Following this, specific score and accuracy functions are developed to facilitate the ranking of alternatives. Finally, a numerical illustration derived from a real-world hospital selection context is presented to demonstrate the effectiveness and practical utility of the proposed approach.

Keywords: Aggregation operators, Hospital selection, Multi-criteria decision-making, Multi-valued rough neutrosophic numbers, Rough neutrosophic set, Shapley fuzzy measure

Introduction

Decision-making problems, particularly in healthcare, are highly complex and involve significant ambiguity and uncertainty.[1,2] Existing decision-making techniques struggle to handle imprecise information, which necessitates the development of more advanced methods. Zadeh[3] introduced Fuzzy Sets to represent uncertainty through membership values. Later, Atanassov[4] extended this concept by proposing Intuitionistic Fuzzy Sets with both membership and non-membership degrees. Smarandache[5] introduced Neutrosophic Sets, characterized by truth, indeterminacy, and falsity components, which further led to the development of Single-Valued and Interval-Valued Neutrosophic Sets applicable in real-world situations. Peng and Wang[6] introduced Multi-Valued Neutrosophic Sets (MVNS) with multiple values for each component.

Simultaneously, Pawlak[7] introduced Rough Sets to handle incomplete information by using lower and upper approximations to define sets. Later, Dubois and Prade[8] combined fuzzy and rough set methods. Broumi et al.[9] then expanded rough sets to include neutrosophy, which led to the creation of neutrosophic rough

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sets.[10] These developments show the ongoing effort to improve how uncertainty is modeled. To enhance decision-making under these complex environments, various aggregation operators have been developed. Yao[11] introduced relational interpretations for rough sets. Key operators like t-norms and t-conorms were developed for fuzzy sets and extended by Atanassov to intuitionistic fuzzy sets, and by Smarandache[12] for neutrosophic sets. Rough neutrosophic aggregation operators were proposed by Mondal.[13] Additionally, the Single-Valued Neutrosophic Dombi Weighted Aggregation Operators (SVNDWAO) have been introduced for multiple attribute decision-making, offering flexible aggregation structures.[14] Other researchers also contributed by proposing new models, such as those combining ideas for Multi-Valued Rough Neutrosophic Sets (MVRNS).[15,16] Awang et al.[17] proposed a normalized Bonferroni mean operator using Shapley fuzzy measures. The Shapley value, introduced by Shapley,[18] helps distribute gains fairly and was adapted to fuzzy environments for evaluating criteria importance.

Further advancements include geometric aggregation for single-valued neutrosophic hesitant fuzzy elements,[19] hesitant complex neutrosophic sets (HCNS) and their distance measures,[20] and neutrosophic hesitant fuzzy multi-objective programming models.[21] Matrix games with pay-offs have been applied in various neutrosophic settings, including single-valued,[22] trapezoidal,[23] and interval neutrosophic matrices.[24] New models like correlation coefficients[25] and hypersoft sets[26] have improved handling of complex decision problems. Aggregation operators for material selection were developed by Farid and Riaz.[27] Fuzzy normed spaces help model uncertain data.[28] Multi-valued neutrosophic sets with power operators improve decision-making.[29]

Neutrosophic rough sets continue to evolve for better decisions. Topological rough sets manage vague data, and cost-based models increase accuracy.[30] Neutrosophic matrices assist in site selection,[31] and fuzzy rough sets support smart city plans.[32] Group decisions are improved using confidence and two-universe models,[33,34] and rough sets help in manufacturing choices.[35,36] Jun Ye used algebraic and Einstein operators for complex decisions.[37] Kamran introduced Z-rough sine operators for industry evaluation.[38] Elsayed studied MCDM for economic assessment.[39] Xu and Zhao proposed a distance measure and used TOPSIS–TODIM for better results.[40] Ali and Bibi worked on waste reduction using neutrosophic rough sets.[41] Alias et al. applied entropy-based multisets,[42] and Halim et al. developed Shapley-Einstein operators for uncertain problems.[43] Yumashev et al.[44] proposed a rough neutrosophic and deep learning-based approach to improve kidney disease diagnosis. In a similar way, Pratheesha et al.[45] employed a TOPSIS technique based on quadripartitioned neutrosophic aggregation to make decisions with unclear data.

Despite these significant advancements, processing multi-valued, rough, and neutrosophic data within a coherent framework for complex decision-making processes remains a significant issue for current techniques. For instance, even when researching advanced neutrosophic clustering using distance measures[46] and sustainable management models,[47] or prioritizing options using type-2 neutrosophic numbers,[48] some studies might not accurately capture the multi-valued and rough components of expert judgments. While approaches using Muirhead Mean operators under complex single-valued neutrosophic values[49] or interdependency of complex fuzzy neighborhood operators[50] offer insights, a full framework combining all three dimensions is still being investigated. This gap is crucial because real-world expert assessments often require a more robust mathematical model due to their inherent roughness and incorporation of many uncertain opinions.

Furthermore, recent research on fuzzy linguistic Muirhead mean operators, which are used in neural networks,[51] and soft covering-based sets,[52] highlights the ongoing need for new aggregation methods in various areas. Research opportunities are also presented by the possibility of combining ideas such as Fuzzy Superior Mandelbrot Sets with Neutrosophic Sets, however this integration presents its own set of theoretical and practical difficulties. The range of issues needing complex MCDM tools is further illustrated by advanced aggregation operators such as those for renewable energy analysis,[53] disturbance observer-based control,[54] trust-driven consensus models,[55] and food waste treatment assessments.[56] Senapati[57] offered a strong framework for managing uncertainty by proposing an Aczel-Alsina aggregation-based outranking technique for multiple attribute decision-making utilizing single-valued neutrosophic numbers. Wang et al.[58] developed the Dual Generalized Single-Valued Neutrosophic Bonferroni Mean and the Dual Hesitant Single-Valued Neutrosophic Weighted Geometric Bonferroni Mean Operators. Liu and Zhang[59] introduced Neutrosophic Hesitant Fuzzy Heronian Mean Aggregation Operators, which are especially beneficial for capturing interrelationships among attributes in group decision-making.

To address this gap, this study proposes an MCDM method using MVRNNs. It introduces new aggregation operators, proves their properties, and uses the Shapley measure for optimal weights. New score and accuracy functions are developed, and a hospital selection example demonstrates their effectiveness.

Definition 1 (Neutrosophic Sets (NS)[12]**):** Let $\mathbb{U}$ be the universal set, and each element $x \in \mathbb{U}$ has a degree of membership in the neutrosophic set $\mathbb{S}$ that is Truth, Indeterminacy, and False. $\mathbb{S}$ can be expressed as follows:

$$\mathbb{S} = \{(x, \mathbb{T}_{\mathbb{S}}(x), \mathbb{I}_{\mathbb{S}}(x), \mathbb{F}_{\mathbb{S}}(x)) : x \in \mathbb{U}\},$$

Where $0 \leqslant \mathbb{T}_{\mathbb{S}}(x) + \mathbb{I}_{\mathbb{S}}(x) + \mathbb{F}_{\mathbb{S}}(x) \leqslant 3$, and the membership functions are defined as follows: Truth Membership function $\mathbb{T}_{\mathbb{S}} : \mathbb{U} \to [0, 1]$, Indeterminacy Membership function $\mathbb{I}_{\mathbb{S}} : \mathbb{U} \to [0, 1]$, False Membership function $\mathbb{F}_{\mathbb{S}} : \mathbb{U} \to [0, 1]$.

Definition 2 (Rough Set[7]**):** Let $\mathbb{R}$ be an equivalence relation defined on the universal set $\mathbb{U}$. An approximation space is defined as $\mathbb{S} = \mathbb{U}/\mathbb{R}$, which is a collection of all equivalence classes of $\mathbb{U}$ over $\mathbb{R}$.

Let $\mathbb{Y} \subseteq \mathbb{U}$ be a subset of $\mathbb{U}$. The terms $\underline{\mathbb{S}}(\mathbb{Y})$ and $\overline{\mathbb{S}}(\mathbb{Y})$, which stand for the lower and higher approximations of $\mathbb{U}$ in $\mathbb{S}$ respectively, are defined below.

$$\underline{\mathbb{S}}(\mathbb{Y}) = \{x \in \mathbb{U} : [x]_{\mathbb{R}} \subseteq \mathbb{Y}\}, \overline{\mathbb{S}}(\mathbb{Y}) = \{x \in \mathbb{U} : [x]_{\mathbb{R}} \cap \mathbb{Y} \neq \varnothing\}$$

where $[x]_{\mathbb{R}}$ represents the equivalence class of x under $\mathbb{R}$ that contains an element x. The pair $\mathbb{S}(\mathbb{Y}) = (\underline{\mathbb{S}}(\mathbb{Y}), \overline{\mathbb{S}}(\mathbb{Y}))$ is known as a rough set of $\mathbb{Y}$ in $\mathbb{S}$.

Definition 3 (Rough Neutrosophic Set[9]**):** Let $\mathbb{U}$ be a universal set, and let each element $x \in \mathbb{U}$ be a member of a neutrosophic set $\mathbb{S}$, which includes membership functions for truth $\mathbb{T}_{\mathbb{S}}$, indeterminacy $\mathbb{I}_{\mathbb{S}}$, and falsity $\mathbb{F}_{\mathbb{S}}$ in $\mathbb{U}$. Let $\mathbb{R}$ be an equivalence relation on $\mathbb{U}$. Within the approximation space $\mathbb{U}/\mathbb{R}$, the lower and upper approximations of $\mathbb{S}$ are denoted by $\underline{\mathbb{N}}(\mathbb{S})$ and $\overline{\mathbb{N}}(\mathbb{S})$, respectively, and have the following definitions:

$$\underline{\mathbb{N}}(\mathbb{S}) = \left\{\left\langle x, \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}(x)\right\rangle : y \in [x]_{\mathbb{R}}, x \in \mathbb{U}\right\}$$
$$\overline{\mathbb{N}}(\mathbb{S}) = \left\{\left\langle x, \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x)\right\rangle : y \in [x]_{\mathbb{R}}, x \in \mathbb{U}\right\}$$

where,

$$\begin{aligned} \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}(x) &= \bigwedge_{y\in[x]_{\mathbb{R}}} \mathbb{T}_{\mathbb{S}}(y) & \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x) &= \bigvee_{y\in[x]_{\mathbb{R}}} \mathbb{T}_{\mathbb{S}}(y) \\ \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}(x) &= \bigvee_{y\in[x]_{\mathbb{R}}} \mathbb{I}_{\mathbb{S}}(y) & \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x) &= \bigwedge_{y\in[x]_{\mathbb{R}}} \mathbb{I}_{\mathbb{S}}(y) \\ \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}(x) &= \bigvee_{y\in[x]_{\mathbb{R}}} \mathbb{F}_{\mathbb{S}}(y) & \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x) &= \bigwedge_{y\in[x]_{\mathbb{R}}} \mathbb{F}_{\mathbb{S}}(y) \end{aligned}$$

Additionally, the following conditions hold:

$$0 \leqslant \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}(x) \leqslant 3 \text{ and } 0 \leqslant \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x) \leqslant 3$$

$\mathbb{T}_{\mathbb{S}}(x)$, $\mathbb{I}_{\mathbb{S}}(x)$, and $\mathbb{F}_{\mathbb{S}}(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively, for the element x in the neutrosophic set $\mathbb{S}$. The symbols for the minimum and maximum operations are $\wedge$ and $\vee$, respectively. Therefore $\underline{\mathbb{N}}(\mathbb{S})$ and $\overline{\mathbb{N}}(\mathbb{S})$ are two neutrosophic sets in $\mathbb{U}$. The rough neutrosophic set in the approximation space $\mathbb{U}/\mathbb{R}$ is the pair $(\underline{\mathbb{N}}(\mathbb{S}), \overline{\mathbb{N}}(\mathbb{S}))$. If, for every $x \in \mathbb{U}$, $\underline{\mathbb{N}}(\mathbb{S}) = \overline{\mathbb{N}}(\mathbb{S})$, then $\mathbb{S}$ is a defined neutrosophic set.

Definition 4 (Shapley Fuzzy Measure[17]**):** Let $\mathbb{Q} = \{1, 2, \ldots, n\}$ be a set of criteria, and let $\mathbb{P}(\mathbb{Q})$ be its power set. A fuzzy measure μ is a function $\mu : \mathbb{P}(\mathbb{Q}) \to [0, 1]$ satisfying:

1. **Boundary:** $\mu(\varnothing) = 0$, $\mu(\mathbb{Q}) = 1$.
2. **Monotonicity:** If $\mathbb{A} \subseteq \mathbb{B}$, then $\mu(\mathbb{A}) \leqslant \mu(\mathbb{B})$.

A specific fuzzy measure, called the λ-fuzzy measure, is defined for disjoint sets $\mathbb{A}$ and $\mathbb{B}$ as:

$$\mu(\mathbb{A} \cup \mathbb{B}) = \mu(\mathbb{A}) + \mu(\mathbb{B}) - \lambda\mu(\mathbb{A})\mu(\mathbb{B}),$$

where $\lambda > -1$.

$$\mu_\lambda(\mathbb{A}) = \begin{cases} \frac{1}{\lambda}\left(\prod_{\mathbb{S}_i \subseteq \mathbb{A}} \left(1 + \lambda\mu(\{\mathbb{S}_i\})\right) - 1\right), & \text{if } \lambda \neq 0, \\ \sum_{\mathbb{S}_i \subseteq \mathbb{A}} \mu(\{\mathbb{S}_i\}), & \text{if } \lambda = 0. \end{cases}$$

The parameter λ can be computed as:

$$\prod_{\mathbb{S}_i \subseteq \mathbb{A}} \left(1 + \lambda\mu\left(\{\mathbb{S}_i\}\right)\right) = \lambda + 1. \tag{1}$$

The Shapley fuzzy measure, representing the importance of each criterion, is computed as:

$$w_i = \sum_{\mathbb{W} \subseteq \mathbb{Q} \setminus \{i\}} \frac{(|\mathbb{Q}| - |\mathbb{W}| - 1)!\,|\mathbb{W}|!}{|\mathbb{Q}|!} \left(\mu\left(\mathbb{W} \cup \{i\}\right) - \mu\left(\mathbb{W}\right)\right). \tag{2}$$

It satisfies the property: $\sum_{i \in \mathbb{Q}} w_i = 1$.

Definition 5 (Multi-Valued Rough Neutrosophic Set[2]**):** Let $\mathbb{U}$ be a universal set, and let each element $x \in \mathbb{U}$ be associated with a neutrosophic set $\mathbb{S}$, characterized by the truth membership degree $\mathbb{T}_{\mathbb{S}}(x)$, indeterminacy membership degree $\mathbb{I}_{\mathbb{S}}(x)$, and falsity membership degree $\mathbb{F}_{\mathbb{S}}(x)$. Let $\mathbb{R}$ denote an equivalence relation on $\mathbb{U}$. Within the approximation space $\mathbb{U}/\mathbb{R}$, the lower and approximations of the neutrosophic set $\mathbb{S}$ are represented as $\underline{\mathbb{N}}(\mathbb{S})$ and $\overline{\mathbb{N}}(\mathbb{S})$, respectively. The following is the definition of these approximations:

$$\underline{\mathbb{N}}(\mathbb{S}) = \left\{\left\langle x, \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}(x)\right\rangle : \;\; y \in [x]_{\mathbb{R}} \subseteq \mathbb{S}, \;\; x \in \mathbb{U}\right\},$$

$$\overline{\mathbb{N}}(\mathbb{S}) = \left\{\left\langle x, \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x), \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x)\right\rangle : \;\; y \in [x]_{\mathbb{R}} \cap \mathbb{S} \neq \varnothing, \;\; x \in \mathbb{U}\right\}$$

The components $\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x)$, $\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x)$, and $\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x)$ for the lower approximation, as well as $\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x)$, $\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x)$, and $\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x)$ for the upper approximation, the values are computed as follows:

$$\begin{array}{llll} \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}(x) = \bigwedge_{y \in [x]_{\mathbb{R}}} \mathbb{T}_{\mathbb{S}}(y) & \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x) = \bigvee_{y \in [x]_{\mathbb{R}}} \mathbb{T}_{\mathbb{S}}(y) \\ \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}(x) = \bigvee_{y \in [x]_{\mathbb{R}}} \mathbb{I}_{\mathbb{S}}(y) & \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x) = \bigwedge_{y \in [x]_{\mathbb{R}}} \mathbb{I}_{\mathbb{S}}(y) \\ \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}(x) = \bigvee_{y \in [x]_{\mathbb{R}}} \mathbb{F}_{\mathbb{S}}(y) & \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x) = \bigwedge_{y \in [x]_{\mathbb{R}}} \mathbb{F}_{\mathbb{S}}(y) \end{array}$$

Here, $\wedge$ and $\vee$ denote the minimum and maximum operations, respectively. The following conditions hold:

$$0 \leqslant \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}(x) \leqslant 3, \;\; 0 \leqslant \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}(x) + \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}(x) \leqslant 3.$$

Furthermore, for any $x \in \mathbb{U}$, let $\gamma \in \mathbb{T}_{\mathbb{S}}(x)$, $\eta \in \mathbb{I}_{\mathbb{S}}(x)$, $\xi \in \mathbb{F}_{\mathbb{S}}(x)$, and let $\gamma^+ = \sup\mathbb{T}_{\mathbb{S}}(x)$, $\eta^+ = \sup\mathbb{I}_{\mathbb{S}}(x)$, $\xi^+ = \sup\mathbb{F}_{\mathbb{S}}(x)$. These values satisfy:

$$0 \leqslant \gamma, \eta, \xi \leqslant 1, \;\; 0 \leqslant \gamma^+ + \eta^+ + \xi^+ \leqslant 3.$$

The pair $(\underline{\mathbb{N}}(\mathbb{S}), \overline{\mathbb{N}}(\mathbb{S}))$ represents an MVRNS in $\mathbb{U}/\mathbb{R}$. If $\underline{\mathbb{N}}(\mathbb{S}) = \overline{\mathbb{N}}(\mathbb{S})$ for any $x \in \mathbb{U}$, then $\mathbb{S}$ is termed a definable MVNS.

Methodology

Multi-valued rough neutrosophic numbers and their operations

A Multi-Valued Rough Neutrosophic Number (MVRNN) builds on the concepts of rough and neutrosophic sets. This helps to better handle uncertainty, indeterminacy, and inconsistency in complex decision-making. Each element is assigned a set of truth, indeterminacy, and falsity values. To represent the inherent uncertainty, lower and upper approximations are then used. This feature makes MVRNNs especially useful when dealing with incomplete or conflicting information. This section will explain the basic operational rules and important properties of MVRNNs, specifically using Einstein operators.

We also establish several aggregation operators for MVRNNs, including the Multi-Valued Rough Neutrosophic Arithmetic Mean Operator, the Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator, the Multi-Valued Rough Neutrosophic Geometric Mean Operator, and the Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator.

Definition 6: Algebraic Operations on MVRNNs:

Let $\mathbb{N}(\mathbb{S}_1) = \left\langle (\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_1)}, \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_1)}, \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_1)}),\ (\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_1)}, \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_1)}, \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_1)}) \right\rangle$ and

$\mathbb{N}(\mathbb{S}_2) = \left\langle (\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_2)}, \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_2)}, \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_2)}),\ (\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_2)}, \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_2)}, \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_2)}) \right\rangle$,

be two MVRNNs. Then, the algebraic operations are defined as follows:

$$1.\ \lambda(\mathbb{N}(\mathbb{S})) = \left\langle \begin{array}{l} \left(\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S})} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} - (1-\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (1-\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S})} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S})} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\} \right), \\ \left(\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S})} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} - (1-\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (1-\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S})} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S})} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\} \right) \end{array} \right\rangle.$$

$$2.\ [\mathbb{N}(\mathbb{S})]^{\lambda} = \left\langle \begin{array}{l} \left(\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S})} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{2(\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(2-\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (\gamma_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S})} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{(1+\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} - (1-\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(1+\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (1-\eta_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S})} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{(1+\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} - (1-\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(1+\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (1-\xi_{\underline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\} \right), \\ \left(\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S})} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{2(\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(2-\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (\gamma_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S})} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{(1+\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} - (1-\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(1+\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (1-\eta_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\}, \bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S})} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})}} \left\{ \frac{(1+\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} - (1-\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}}{(1+\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda} + (1-\xi_{\overline{\mathbb{N}}(\mathbb{S})})^{\lambda}} \right\} \right) \end{array} \right\rangle.$$

3. $\mathbb{N}(\mathbb{S}_1) \oplus \mathbb{N}(\mathbb{S}_2)$

$$= \left\langle \begin{array}{l} \left(\bigcup\limits_{\substack{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)} \in T_{\underline{\mathbb{N}}(\mathbb{S}_1)} \\ \gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)} \in T_{\underline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)} + \gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}} \right\}, \bigcup\limits_{\substack{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)} \in I_{\underline{\mathbb{N}}(\mathbb{S}_1)} \\ \eta_{\underline{\mathbb{N}}(\mathbb{S}_2)} \in I_{\underline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\left(1-\eta_{\mathbb{N}(\mathbb{S}_1)}\right)\left(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)} \right\}, \bigcup\limits_{\substack{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)} \in F_{\underline{\mathbb{N}}(\mathbb{S}_1)} \\ \xi_{\underline{\mathbb{N}}(\mathbb{S}_2)} \in F_{\underline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\left(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)\left(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)} \right\} \right), \\ \left(\bigcup\limits_{\substack{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)} \in T_{\overline{\mathbb{N}}(\mathbb{S}_1)} \\ \gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)} \in T_{\overline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)} + \gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}} \right\}, \bigcup\limits_{\substack{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)} \in I_{\overline{\mathbb{N}}(\mathbb{S}_1)} \\ \eta_{\overline{\mathbb{N}}(\mathbb{S}_2)} \in I_{\overline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\left(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)\left(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)} \right\}, \bigcup\limits_{\substack{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)} \in F_{\overline{\mathbb{N}}(\mathbb{S}_1)} \\ \xi_{\overline{\mathbb{N}}(\mathbb{S}_2)} \in F_{\overline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\left(1-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)\left(1-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)} \right\} \right) \end{array} \right\rangle.$$

4. $\mathbb{N}(\mathbb{S}_1) \otimes \mathbb{N}(\mathbb{S}_2)$

$$= \left\langle \begin{array}{l} \left(\bigcup\limits_{\substack{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)} \in T_{\underline{\mathbb{N}}(\mathbb{S}_1)} \\ \gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)} \in T_{\underline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)} \cdot \gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})} \right\}, \bigcup\limits_{\substack{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)} \in I_{\underline{\mathbb{N}}(\mathbb{S}_1)} \\ \eta_{\underline{\mathbb{N}}(\mathbb{S}_2)} \in I_{\underline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)} + \eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}} \right\}, \bigcup\limits_{\substack{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)} \in F_{\underline{\mathbb{N}}(\mathbb{S}_1)} \\ \xi_{\underline{\mathbb{N}}(\mathbb{S}_2)} \in F_{\underline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)} + \xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}} \right\} \right), \\ \left(\bigcup\limits_{\substack{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)} \in T_{\overline{\mathbb{N}}(\mathbb{S}_1)} \\ \gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)} \in T_{\overline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)} \cdot \gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})} \right\}, \bigcup\limits_{\substack{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)} \in I_{\overline{\mathbb{N}}(\mathbb{S}_1)} \\ \eta_{\overline{\mathbb{N}}(\mathbb{S}_2)} \in I_{\overline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)} + \eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}} \right\}, \bigcup\limits_{\substack{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)} \in F_{\overline{\mathbb{N}}(\mathbb{S}_1)} \\ \xi_{\overline{\mathbb{N}}(\mathbb{S}_2)} \in F_{\overline{\mathbb{N}}(\mathbb{S}_2)}}} \left\{ \frac{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)} + \xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}} \right\} \right) \end{array} \right\rangle.$$

Definition 7. Properties of MVRNNs:

Let $\mathbb{N}(\mathbb{S}_1) = \langle (T_{\underline{\mathbb{N}}(\mathbb{S}_1)}, I_{\underline{\mathbb{N}}(\mathbb{S}_1)}, F_{\underline{\mathbb{N}}(\mathbb{S}_1)}), (T_{\overline{\mathbb{N}}(\mathbb{S}_1)}, I_{\overline{\mathbb{N}}(\mathbb{S}_1)}, F_{\overline{\mathbb{N}}(\mathbb{S}_1)}) \rangle$, $\mathbb{N}(\mathbb{S}_2) = \langle (T_{\underline{\mathbb{N}}(\mathbb{S}_2)}, I_{\underline{\mathbb{N}}(\mathbb{S}_2)}, F_{\underline{\mathbb{N}}(\mathbb{S}_2)}), (T_{\overline{\mathbb{N}}(\mathbb{S}_2)}, I_{\overline{\mathbb{N}}(\mathbb{S}_2)}, F_{\overline{\mathbb{N}}(\mathbb{S}_2)}) \rangle$, and $\mathbb{N}(\mathbb{S}_3) = \langle (T_{\underline{\mathbb{N}}(\mathbb{S}_3)}, I_{\underline{\mathbb{N}}(\mathbb{S}_3)}, F_{\underline{\mathbb{N}}(\mathbb{S}_3)}), (T_{\overline{\mathbb{N}}(\mathbb{S}_3)}, I_{\overline{\mathbb{N}}(\mathbb{S}_3)}, F_{\overline{\mathbb{N}}(\mathbb{S}_3)}) \rangle$ be three MVRNNs. Then, the following properties hold:

1. $\mathbb{N}(\mathbb{S}_1) \oplus \mathbb{N}(\mathbb{S}_2) = \mathbb{N}(\mathbb{S}_2) \oplus \mathbb{N}(\mathbb{S}_1)$;

2. $\mathbb{N}(\mathbb{S}_1)\otimes\mathbb{N}(\mathbb{S}_2)=\mathbb{N}(\mathbb{S}_2)\otimes\mathbb{N}(\mathbb{S}_1)$;
3. $\lambda(\mathbb{N}(\mathbb{S}_1)\oplus\mathbb{N}(\mathbb{S}_2))=\lambda\mathbb{N}(\mathbb{S}_1)\oplus\lambda\mathbb{N}(\mathbb{S}_2),\quad \lambda>0$;
4. $(\mathbb{N}(\mathbb{S}_1)\otimes\mathbb{N}(\mathbb{S}_2))^{\lambda}=\mathbb{N}(\mathbb{S}_1)^{\lambda}\otimes\mathbb{N}(\mathbb{S}_2)^{\lambda},\quad \lambda>0$;
5. $\lambda_1\mathbb{N}(\mathbb{S}_1)\oplus\lambda_2\mathbb{N}(\mathbb{S}_1)=(\lambda_1+\lambda_2)\mathbb{N}(\mathbb{S}_1),\quad \lambda_1>0,\quad \lambda_2>0$;
6. $\mathbb{N}(\mathbb{S}_1)^{\lambda_1}\otimes\mathbb{N}(\mathbb{S}_1)^{\lambda_2}=\mathbb{N}(\mathbb{S}_1)^{\lambda_1+\lambda_2},\quad \lambda_1>0,\quad \lambda_2>0$;
7. $(\mathbb{N}(\mathbb{S}_1)\oplus\mathbb{N}(\mathbb{S}_2))\oplus\mathbb{N}(\mathbb{S}_3)=\mathbb{N}(\mathbb{S}_1)\oplus(\mathbb{N}(\mathbb{S}_2)\oplus\mathbb{N}(\mathbb{S}_3))$;
8. $(\mathbb{N}(\mathbb{S}_1)\otimes\mathbb{N}(\mathbb{S}_2))\otimes\mathbb{N}(\mathbb{S}_3)=\mathbb{N}(\mathbb{S}_1)\otimes(\mathbb{N}(\mathbb{S}_2)\otimes\mathbb{N}(\mathbb{S}_3))$.

Proof:

1. $\mathbb{N}(\mathbb{S}_1)\oplus\mathbb{N}(\mathbb{S}_2)$

$$=\left\langle\begin{array}{c}\left(\bigcup\limits_{\substack{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_1)}\\ \gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\right\},\bigcup\limits_{\substack{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_1)}\\ \eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})}\right\},\bigcup\limits_{\substack{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_1)}\\ \xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})}\right\}\right),\\ \left(\bigcup\limits_{\substack{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_1)}\\ \gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\right\},\bigcup\limits_{\substack{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_1)}\\ \eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})}\right\},\bigcup\limits_{\substack{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_1)}\\ \xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})(1-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})}\right\}\right)\end{array}\right\rangle$$

$$=\left\langle\begin{array}{c}\left(\bigcup\limits_{\substack{\gamma_{\mathbb{N}(\mathbb{S}_2)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_2)}\\ \gamma_{\mathbb{N}(\mathbb{S}_1)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_1)}}}\left\{\frac{\gamma_{\mathbb{N}(\mathbb{S}_2)}+\gamma_{\mathbb{N}(\mathbb{S}_1)}}{1+\gamma_{\mathbb{N}(\mathbb{S}_2)}\gamma_{\mathbb{N}(\mathbb{S}_1)}}\right\},\bigcup\limits_{\substack{\eta_{\mathbb{N}(\mathbb{S}_2)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_2)}\\ \eta_{\mathbb{N}(\mathbb{S}_1)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_1)}}}\left\{\frac{\eta_{\mathbb{N}(\mathbb{S}_2)}\eta_{\mathbb{N}(\mathbb{S}_1)}}{1+(1-\eta_{\mathbb{N}(\mathbb{S}_2)})(1-\eta_{\mathbb{N}(\mathbb{S}_1)})}\right\},\bigcup\limits_{\substack{\xi_{\mathbb{N}(\mathbb{S}_2)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_2)}\\ \xi_{\mathbb{N}(\mathbb{S}_1)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_1)}}}\left\{\frac{\xi_{\mathbb{N}(\mathbb{S}_2)}\xi_{\mathbb{N}(\mathbb{S}_1)}}{1+(1-\xi_{\mathbb{N}(\mathbb{S}_2)})(1-\xi_{\mathbb{N}(\mathbb{S}_1)})}\right\}\right),\\ \left(\bigcup\limits_{\substack{\gamma_{\mathbb{N}(\mathbb{S}_2)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_2)}\\ \gamma_{\mathbb{N}(\mathbb{S}_1)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_1)}}}\left\{\frac{\gamma_{\mathbb{N}(\mathbb{S}_2)}+\gamma_{\mathbb{N}(\mathbb{S}_1)}}{1+\gamma_{\mathbb{N}(\mathbb{S}_2)}\gamma_{\mathbb{N}(\mathbb{S}_1)}}\right\},\bigcup\limits_{\substack{\eta_{\mathbb{N}(\mathbb{S}_2)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_2)}\\ \eta_{\mathbb{N}(\mathbb{S}_1)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_1)}}}\left\{\frac{\eta_{\mathbb{N}(\mathbb{S}_2)}\eta_{\mathbb{N}(\mathbb{S}_1)}}{1+(1-\eta_{\mathbb{N}(\mathbb{S}_2)})(1-\eta_{\mathbb{N}(\mathbb{S}_1)})}\right\},\bigcup\limits_{\substack{\xi_{\mathbb{N}(\mathbb{S}_2)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_2)}\\ \xi_{\mathbb{N}(\mathbb{S}_1)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_1)}}}\left\{\frac{\xi_{\mathbb{N}(\mathbb{S}_2)}\xi_{\mathbb{N}(\mathbb{S}_1)}}{1+(1-\xi_{\mathbb{N}(\mathbb{S}_2)})(1-\xi_{\mathbb{N}(\mathbb{S}_1)})}\right\}\right)\end{array}\right\rangle$$

$=\mathbb{N}(\mathbb{S}_2)\oplus\mathbb{N}(\mathbb{S}_1)$.

Similarly, it can be proved that the associative property holds:

$(\mathbb{N}(\mathbb{S}_1)\oplus\mathbb{N}(\mathbb{S}_2))\oplus\mathbb{N}(\mathbb{S}_3)=\mathbb{N}(\mathbb{S}_1)\oplus(\mathbb{N}(\mathbb{S}_2)\oplus\mathbb{N}(\mathbb{S}_3))$

2. $\mathbb{N}(\mathbb{S}_1)\otimes\mathbb{N}(\mathbb{S}_2)$

$$=\left\langle\begin{array}{c}\left(\bigcup\limits_{\substack{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_1)}\\ \gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\cdot\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})}\right\},\bigcup\limits_{\substack{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_1)}\\ \eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}+\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\right\},\bigcup\limits_{\substack{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_1)}\\ \xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}+\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}}{1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\right\}\right),\\ \left(\bigcup\limits_{\substack{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_1)}\\ \gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\cdot\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})}\right\},\bigcup\limits_{\substack{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_1)}\\ \eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}+\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\right\},\bigcup\limits_{\substack{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_1)}\\ \xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_2)}}}\left\{\frac{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}+\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}}{1+\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\right\}\right)\end{array}\right\rangle$$

$$=\left\langle \begin{array}{l} \left(\bigcup_{\substack{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_2)}\\ \gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in T_{\underline{\mathbb{N}}(\mathbb{S}_1)}}} \left\{ \frac{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\cdot\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}}{1+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})}\right\}, \bigcup_{\substack{\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_2)}\\ \eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in I_{\underline{\mathbb{N}}(\mathbb{S}_1)}}} \left\{ \frac{\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}+\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}}{1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}}\right\}, \bigcup_{\substack{\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_2)}\\ \xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in F_{\underline{\mathbb{N}}(\mathbb{S}_1)}}} \left\{ \frac{\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}+\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}}{1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}}\right\} \right), \\ \left(\bigcup_{\substack{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_2)}\\ \gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in T_{\overline{\mathbb{N}}(\mathbb{S}_1)}}} \left\{ \frac{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\cdot\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}}{1+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})}\right\}, \bigcup_{\substack{\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_2)}\\ \eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in I_{\overline{\mathbb{N}}(\mathbb{S}_1)}}} \left\{ \frac{\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}+\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}}{1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}}\right\}, \bigcup_{\substack{\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_2)}\\ \xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in F_{\overline{\mathbb{N}}(\mathbb{S}_1)}}} \left\{ \frac{\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}+\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}}{1+\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}}\right\} \right) \end{array} \right\rangle$$

$$= \mathbb{N}\left(\mathbb{S}_2\right)\otimes\mathbb{N}(\mathbb{S}_1)$$

Similarly, it can be proved that the associative property holds:

$(\mathbb{N}(\mathbb{S}_1)\otimes\mathbb{N}(\mathbb{S}_2))\otimes\mathbb{N}(\mathbb{S}_3) = \mathbb{N}(\mathbb{S}_1)\otimes(\mathbb{N}(\mathbb{S}_2)\otimes\mathbb{N}(\mathbb{S}_3))$.

(3) Since $\lambda > 0$,

$$\lambda\mathbb{N}\left(\mathbb{S}_1\right)\oplus\lambda\mathbb{N}\left(\mathbb{S}_2\right) = \left(\begin{array}{l} \left(\bigcup_{\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)}\in\mathbf{T}_{\underline{\mathbf{N}}(\mathbf{s}_1)}\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)}\in\mathbf{T}_{\underline{\mathbf{N}}(\mathbf{s}_2)}} \left\{ \frac{\frac{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}-(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}-\frac{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}-(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}}{\mathbf{1}-\frac{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}-(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}\cdot\frac{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}-(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(1+\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(1-\gamma_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}} \right\}, \right. \\ \bigcup_{\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)}\in\mathbf{I}_{\underline{\mathbf{N}}(\mathbf{s}_1)}\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)}\in\mathbf{I}_{\underline{\mathbf{N}}(\mathbf{s}_2)}} \left\{ \frac{\frac{2(\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}-\frac{2(\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}}{\mathbf{1}-\left(\mathbf{1}-\frac{2(\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\eta_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}\right)\cdot\left(\mathbf{1}-\frac{2(\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\eta_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}\right)} \right\}, \\ \left. \bigcup_{\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)}\in\mathbf{F}_{\underline{\mathbf{N}}(\mathbf{s}_1)}\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)}\in\mathbf{F}_{\underline{\mathbf{N}}(\mathbf{s}_2)}} \left\{ \frac{\frac{2(\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}-\frac{2(\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}}{\mathbf{1}-\left(\mathbf{1}-\frac{2(\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\xi_{\underline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}\right)\cdot\left(\mathbf{1}-\frac{2(\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\xi_{\underline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}\right)} \right\} \right), \\ \left(\bigcup_{\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)}\in\mathbf{T}_{\overline{\mathbf{N}}(\mathbf{s}_1)}\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)}\in\mathbf{T}_{\overline{\mathbf{N}}(\mathbf{s}_2)}} \left\{ \frac{\frac{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}-(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}-\frac{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}-(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}}{\mathbf{1}-\frac{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}-(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}\cdot\frac{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}-(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(1+\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(1-\gamma_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}} \right\}, \right. \\ \bigcup_{\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)}\in\mathbf{I}_{\overline{\mathbf{N}}(\mathbf{s}_1)}\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)}\in\mathbf{I}_{\overline{\mathbf{N}}(\mathbf{s}_2)}} \left\{ \frac{\frac{2(\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}-\frac{2(\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}}{\mathbf{1}-\left(\mathbf{1}-\frac{2(\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\eta_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}\right)\cdot\left(\mathbf{1}-\frac{2(\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\eta_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}\right)} \right\}, \\ \left. \bigcup_{\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)}\in\mathbf{F}_{\overline{\mathbf{N}}(\mathbf{s}_1)}\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)}\in\mathbf{F}_{\overline{\mathbf{N}}(\mathbf{s}_2)}} \left\{ \frac{\frac{2(\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}-\frac{2(\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}}{\mathbf{1}-\left(\mathbf{1}-\frac{2(\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}{(2-\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}+(\xi_{\overline{\mathbf{N}}(\mathbf{s}_1)})^{\lambda}}\right)\cdot\left(\mathbf{1}-\frac{2(\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}{(2-\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}+(\xi_{\overline{\mathbf{N}}(\mathbf{s}_2)})^{\lambda}}\right)} \right\} \right) \end{array} \right)$$

$$= \left\langle \left(\bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}} \right\}, \bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}} \right\}, \right. \right.$$

$$\left. \bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}} \right\} \right), \left(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}} \right\}, \right.$$

$$\left. \left. \bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}} \right\}, \bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{\lambda}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{\lambda}} \right\} \right) \right\rangle$$

$$= \lambda\mathbb{N}\left(\mathbb{S}_1\right)\oplus\lambda\mathbb{N}\left(\mathbb{S}_2\right).$$

Similarly, Eqs. (4) to (6) can be proved.

Aggregation operators

This section introduces the arithmetic and geometric mean aggregation operators specifically designed for MVRNNs. We also present a comprehensive methodology for applying these aggregation operators effectively in MCDM situations.

Definition 8 (Multi-Valued Rough Neutrosophic Arithmetic Mean Operator ($\mathbb{A}_{\mathbb{MVRN}}$)): Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $i = 1, 2, \ldots, n$ be a collection of MVRNNs. The Multi-Valued Rough Neutrosophic Arithmetic Mean Operator ($\mathbb{A}_{\mathbb{MVRN}}$) is then defined as:

$$\mathbb{A}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \underline{\mathbb{N}}(\mathbb{S}_i), \frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \overline{\mathbb{N}}(\mathbb{S}_i)\right)$$

Theorem 1: *Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $i = 1, 2, \ldots, n$ be a set of MVRNNs. The aggregated value obtained using the $\mathbb{A}_{\mathbb{MVRN}}$ operator, defined as:*

$$\mathbb{A}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \underline{\mathbb{N}}(\mathbb{S}_i), \frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \overline{\mathbb{N}}(\mathbb{S}_i)\right) \tag{3}$$

It is also an MVRNN.

$$\begin{aligned}
= \Bigg\langle \Bigg(& \bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} - \prod_{i=1}^{n}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} + \prod_{i=1}^{n}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} + \prod_{i=1}^{n}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \\
& \bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} + \prod_{i=1}^{n}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\} \Bigg), \Bigg(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} - \prod_{i=1}^{n}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} + \prod_{i=1}^{n}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \\
& \bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} + \prod_{i=1}^{n}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n} + \prod_{i=1}^{n}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\} \Bigg) \Bigg\rangle.
\end{aligned} \tag{4}$$

Definition 9: *Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator ($\mathbb{A}^{w}_{\mathbb{MVRN}}$)*

Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $i = 1, 2, \ldots, n$ be a collection of MVRNNs, and let $\mathrm{w} = (w_1, w_2, \ldots, w_n)$ be the weight vector for $\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator ($\mathbb{A}^{w}_{\mathbb{MVRN}}$) is then defined as:

$$\mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\overset{n}{\underset{i=1}{\oplus}} w_i \underline{\mathbb{N}}(\mathbb{S}_i), \overset{n}{\underset{i=1}{\oplus}} w_i \overline{\mathbb{N}}(\mathbb{S}_i)\right)$$

Theorem 2: *Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $(i = 1, 2, \ldots, n)$ be a collection of MVRNNs, and let $\mathrm{w} = (w_1, w_2, \ldots, w_n)$ be the weight vector for $\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The aggregated value obtained using the $\mathbb{A}^{w}_{\mathbb{MVRN}}$ operator, defined as:*

$$\mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\overset{n}{\underset{i=1}{\oplus}} w_i \underline{\mathbb{N}}(\mathbb{S}_i), \overset{n}{\underset{i=1}{\oplus}} w_i \overline{\mathbb{N}}(\mathbb{S}_i)\right) \tag{5}$$

is also an MVRNN.

$$\begin{aligned}
= \Bigg\langle \Bigg(& \bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} - \prod_{i=1}^{n}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} + \prod_{i=1}^{n}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} + \prod_{i=1}^{n}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \\
& \bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} + \prod_{i=1}^{n}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\} \Bigg), \Bigg(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} - \prod_{i=1}^{n}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} + \prod_{i=1}^{n}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \\
& \bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} + \prod_{i=1}^{n}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i} + \prod_{i=1}^{n}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\} \Bigg) \Bigg\rangle.
\end{aligned} \tag{6}$$

Proof: Using mathematical induction on n.

i) If $n = 2$, based on the operations Eqs. (1) and (3) in Definition 6,

$$w_1\mathbb{N}(\mathbb{S}_1)\oplus w_2\mathbb{N}(\mathbb{S}_2)$$

$$=\left\langle\begin{array}{l}\left(\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_1)}}\left\{\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right\},\bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_1)}}\left\{\frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right\},\bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_1)}}\left\{\frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right\}\right),\\ \left(\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_1)}}\left\{\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right\},\bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_1)}}\left\{\frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right\},\bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_1)}}\left\{\frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right\}\right)\end{array}\right\rangle$$

$$\oplus\left\langle\begin{array}{l}\left(\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\},\bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\},\bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\}\right),\\ \left(\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\},\bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\},\bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\}\right)\end{array}\right\rangle.$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_1)},\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}+\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}{1+\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\cdot\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}\right\},\\ \bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_1)},\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{\frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\cdot\frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}{1+\left(1-\frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right)\cdot\left(1-\frac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right)}\right\},\\ \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_1)},\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{\frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\cdot\frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}{1+\left(1-\frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right)\cdot\left(1-\frac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right)}\right\}\end{array}\right),\\ \left(\begin{array}{l}\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_1)},\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}+\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}{1+\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\cdot\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}\right\},\\ \bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_1)},\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{\frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\cdot\frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}{1+\left(1-\frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right)\cdot\left(1-\frac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right)}\right\},\\ \bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_1)},\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{\frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\cdot\frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}}{1+\left(1-\frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}}\right)\cdot\left(1-\frac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right)}\right\}\end{array}\right)\end{array}\right)$$

$$\mathbb{A}_{MVRN}(\mathbb{N}(\mathbb{S}_1),\mathbb{N}(\mathbb{S}_2))$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_1)}\,\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\},\\ \bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_1)}\,\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2\left(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}{\left(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}+\left(\eta_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\eta_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}\right\},\\ \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_1)}\,\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2\left(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}{\left(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}+\left(\xi_{\underline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\xi_{\underline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}\right\}\end{array}\right),\\ \left(\begin{array}{l}\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_1)}\,\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_1)})^{w_1}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_2)})^{w_2}}\right\},\\ \bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_1)}\,\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2\left(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}{\left(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}+\left(\eta_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\eta_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}\right\},\\ \bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_1)}\,\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_2)}}\left\{\frac{2\left(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}{\left(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}+\left(\xi_{\overline{\mathbb{N}}(\mathbb{S}_1)}\right)^{w_1}\left(\xi_{\overline{\mathbb{N}}(\mathbb{S}_2)}\right)^{w_2}}\right\}\end{array}\right)\end{array}\right).$$

If Eq. (6) holds for $n = k$, then

$$\mathbb{A}^{w}_{MVRN}\left(\mathbb{N}\left(\mathbb{S}_1\right),\mathbb{N}\left(\mathbb{S}_2\right),\ldots,\mathbb{N}\left(\mathbb{S}_k\right)\right)$$

$$\begin{aligned}
&\Bigg(\bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{\prod_{i=1}^{k}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{k}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{k}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\\
=&\Bigg\langle\bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{k}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\}\Bigg),\Bigg(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{\prod_{i=1}^{k}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{k}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\Bigg\rangle.\\
&\bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{k}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{k}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\}\Bigg)
\end{aligned}$$

$$\mathbb{A}^{w}_{MVRN}(\mathbb{N}(\mathbb{S}_1),\mathbb{N}(\mathbb{S}_2),\ldots(,\mathbb{N}\mathbb{S}_k),\mathbb{N}(\mathbb{S}_{k+1}))$$

$$=\left(\begin{array}{l}
\left(\bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\dfrac{\dfrac{\prod_{i=1}^{k}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{k}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}+\dfrac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}{1+\dfrac{\prod_{i=1}^{k}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{k}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\cdot\dfrac{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}-(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}\right\},\right.\\
\bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\dfrac{\dfrac{2\prod_{i=1}^{k}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\cdot\dfrac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}{1+\left(1-\dfrac{2\prod_{i=1}^{k}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right)\cdot\left(1-\dfrac{2(\eta_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\eta_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}\right)}\right\},\\
\left.\bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\dfrac{\dfrac{2\prod_{i=1}^{k}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\cdot\dfrac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}{1+\left(1-\dfrac{2\prod_{i=1}^{k}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right)\cdot\left(1-\dfrac{2(\xi_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\xi_{\underline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}\right)}\right\}\right),\\
\left(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\dfrac{\dfrac{\prod_{i=1}^{k}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{k}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}+\dfrac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}{1+\dfrac{\prod_{i=1}^{k}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{k}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\cdot\dfrac{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}-(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}\right\},\right.\\
\bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\dfrac{\dfrac{2\prod_{i=1}^{k}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\cdot\dfrac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}{1+\left(1-\dfrac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right)\cdot\left(1-\dfrac{2(\eta_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\eta_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}\right)}\right\},\\
\left.\bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\dfrac{\dfrac{2\prod_{i=1}^{k}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\cdot\dfrac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}}{1+\left(1-\dfrac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{k}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{k}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right)\cdot\left(1-\dfrac{2(\xi_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}{(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}+(\xi_{\overline{\mathbb{N}}(\mathbb{S}_{k+1})})^{w_{k+1}}}\right)}\right\}\right)
\end{array}\right)$$

Thus, Eq. (6) holds for all n, then $\mathbb{A}^{w}_{MVRN}(\mathbb{N}(\mathbb{S}_1),\mathbb{N}(\mathbb{S}_2),\ldots,\mathbb{N}(\mathbb{S}_n))$

$$\begin{aligned}
&\Bigg(\bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{\prod_{i=1}^{n}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{n}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(1-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{n}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\\
=&\Bigg\langle\bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{n}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\Bigg),\Bigg(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{\prod_{i=1}^{n}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{n}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(1-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\Bigg\rangle.\\
&\bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{n}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\},\bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)}\in\mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}}\left\{\frac{2\prod_{i=1}^{n}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}\right\}\Bigg)
\end{aligned}$$

The proof ends.

Properties of aggregation operators

The $\mathbb{A}_{\mathbb{MVRN}}$ and $\mathbb{A}^{w}_{\mathbb{MVRN}}$ operators, along with their geometric counterparts, satisfy the following key properties:

Property 1 (Idempotency Law): If all MVRNNs are identical, i.e., $\mathbb{N}(\mathbb{S}_i) = \mathbb{N}(\mathbb{S})$ for $i = 1, 2, \ldots, n$, then:

$$\mathbb{A}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \mathbb{N}(\mathbb{S})$$

and

$$\mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n))= \mathbb{N}(\mathbb{S}).$$

Proof: For $\mathbb{N}(\mathbb{S}_i) = \mathbb{N}(\mathbb{S})$, by definition of $\mathbb{A}_{\mathbb{MVRN}}$ operator:

$$\mathbb{A}_{\mathbb{MVRN}}((\mathbb{N}\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\frac{1}{n}\bigoplus_{i=1}^{n} \underline{\mathbb{N}}(\mathbb{S}), \frac{1}{n}\bigoplus_{i=1}^{n} \overline{\mathbb{N}}(\mathbb{S})\right) = (\underline{\mathbb{N}}(\mathbb{S}),\overline{\mathbb{N}}(\mathbb{S})) = N(S).$$

For $\mathbb{A}^{w}_{\mathbb{MVRN}}$ operator, given $\sum_{i=1}^{n} w_i = 1$:

$$\mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\bigoplus_{i=1}^{n} w_i\underline{\mathbb{N}}(\mathbb{S}), \bigoplus_{i=1}^{n} w_i\overline{\mathbb{N}}(\mathbb{S})\right) = (\underline{\mathbb{N}}(\mathbb{S}),\overline{\mathbb{N}}(\mathbb{S})) = \mathbb{N}(\mathbb{S}).$$

This holds because $\oplus_{i=1}^{n} w_i\mathbb{X} = \mathbb{X}$ when $\sum w_i = 1$ for any MVRNN $\mathbb{X}$ (based on the definition of weighted aggregation and scalar multiplication).

Property 2 (Boundedness): Both operators are bounded.

Proof: For $i = 1, 2, \ldots, n$, let $\mathbb{N}(\mathbb{S}_i)$ be a set of MVRNNs. Construct the following lower and upper bounds: $\mathbb{N}(\mathbb{S})^-$ and $\mathbb{N}(\mathbb{S})^+$.

$$\mathbb{N}(\mathbb{S})^- = \left\langle \begin{array}{c} \left(\bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\min_i \gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\max_i \eta_{\underline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\max_i \xi_{\underline{\mathbb{N}}(\mathbb{S}_i)}\right\}\right), \\ \left(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\min_i \gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\max_i \eta_{\overline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\max_i \xi_{\overline{\mathbb{N}}(\mathbb{S}_i)}\right\}\right) \end{array} \right\rangle.$$

and

$$\mathbb{N}(\mathbb{S})^+ = \left\langle \begin{array}{c} \left(\bigcup_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\max_i \gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\min_i \eta_{\underline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\min_i \xi_{\underline{\mathbb{N}}(\mathbb{S}_i)}\right\}\right), \\ \left(\bigcup_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\max_i \gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\min_i \eta_{\overline{\mathbb{N}}(\mathbb{S}_i)}\right\}, \bigcup_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{\min_i \xi_{\overline{\mathbb{N}}(\mathbb{S}_i)}\right\}\right) \end{array} \right\rangle.$$

Then, the aggregated values are bounded as follows:

$$\mathbb{N}(\mathbb{S})^- \subseteq \mathbb{A}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) \subseteq \mathbb{N}(\mathbb{S})^+.$$

$$\mathbb{N}(\mathbb{S})^- \subseteq \mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) \subseteq \mathbb{N}(\mathbb{S})^+.$$

Property 3 (Monotonicity): If $\mathbb{N}(\mathbb{S}_i) \leqslant \mathbb{N}(\mathbb{S}_i^*)$ for $i = 1, 2, \ldots, n$, then:

$$\mathbb{A}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{A}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1^*), \ldots, \mathbb{N}(\mathbb{S}_n^*)),$$

And:

$$\mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{A}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1^*), \ldots, \mathbb{N}(\mathbb{S}_n^*)).$$

Proof: Since $\mathbb{N}(\mathbb{S}_i) \leqslant \mathbb{N}(\mathbb{S}_i^*)$ for all $i = 1, 2, \ldots, n$, it follows that:

$$\mathbb{A}_{MVRN}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{A}_{MVRN}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n^*)),$$

And similarly:

$$\mathbb{A}^{w}_{MVRN}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{A}^{w}_{MVRN}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n^*)).$$

Definition 10: *Multi-Valued Rough Neutrosophic Geometric Mean Operator ($\mathbb{G}_{\mathbb{MVRN}}$)*

Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $(i = 1, 2, \ldots, n)$ be a collection of MVRNNs. The Multi-Valued Rough Neutrosophic Geometric Mean Operator ($\mathbb{G}_{\mathbb{MVRN}}$) is then defined as:

$$\mathbb{G}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\bigotimes_{i=1}^{n} (\underline{\mathbb{N}}(\mathbb{S}_i))^{\frac{1}{n}}, \bigotimes_{i=1}^{n} (\overline{\mathbb{N}}(\mathbb{S}_i))^{\frac{1}{n}} \right)$$

Theorem 3: *Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $(i = 1, 2, \ldots, n)$ be a collection of MVRNNs. The aggregated value obtained using the $\mathbb{G}_{\mathbb{MVRN}}$ operator, defined as:*

$$\mathbb{G}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\bigotimes_{i=1}^{n} (\underline{\mathbb{N}}(\mathbb{S}_i))^{\frac{1}{n}}, \bigotimes_{i=1}^{n} (\overline{\mathbb{N}}(\mathbb{S}_i))^{\frac{1}{n}} \right) \tag{7}$$

is also an MVRNN.

$$= \left\langle \begin{array}{c} \left(\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(2-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}+\prod_{i=1}^{n}(\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}-\prod_{i=1}^{n}(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}+\prod_{i=1}^{n}(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \right. \\ \left. \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}-\prod_{i=1}^{n}(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}+\prod_{i=1}^{n}(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\} \right), \left(\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(2-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}+\prod_{i=1}^{n}(\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \right. \\ \left. \bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}-\prod_{i=1}^{n}(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}+\prod_{i=1}^{n}(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\}, \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}-\prod_{i=1}^{n}(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}}{\prod_{i=1}^{n}(1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}+\prod_{i=1}^{n}(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{1/n}} \right\} \right) \end{array} \right\rangle . \tag{8}$$

Definition 11: *Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator ($\mathbb{G}^{w}_{\mathbb{MVRN}}$)*

Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $i = 1, 2, \ldots, n$ be a collection of MVRNNs, and let $\mathrm{w} = (w_1, w_2, \ldots, w_n)$ be the weight vector for $\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator ($\mathbb{G}^{w}_{\mathbb{MVRN}}$) is then defined as:

$$\mathbb{G}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\bigotimes_{i=1}^{n} (\underline{\mathbb{N}}(\mathbb{S}_i))^{w_i}, \bigotimes_{i=1}^{n} (\overline{\mathbb{N}}(\mathbb{S}_i))^{w_i} \right)$$

Theorem 4: *Let $\mathbb{N}(\mathbb{S}_i) = (\underline{\mathbb{N}}(\mathbb{S}_i), \bar{\mathbb{N}}(\mathbb{S}_i))$ for $(i = 1, 2, \ldots, n)$ be a collection of MVRNNs, and let $\mathrm{w} = (w_1, w_2, \ldots, w_n)$ be the weight vector for $\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The aggregated value obtained using the $\mathbb{G}^{w}_{\mathbb{MVRN}}$ operator, defined as:*

$$\mathbb{G}^{w}_{\mathbb{MVRN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\bigotimes_{i=1}^{n} (\underline{\mathbb{N}}(\mathbb{S}_i))^{w_i}, \bigotimes_{i=1}^{n} (\overline{\mathbb{N}}(\mathbb{S}_i))^{w_i} \right) \tag{9}$$

is also an MVRNN.

$$= \left\langle \begin{array}{l} \left(\bigcup\limits_{\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(\gamma_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \bigcup\limits_{\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{n}(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(1-\eta_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \right. \\ \left. \bigcup\limits_{\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{n}(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(1-\xi_{\underline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\} \right), \left(\bigcup\limits_{\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{2\prod_{i=1}^{n}(\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(2-\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(\gamma_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \right. \\ \left. \bigcup\limits_{\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{n}(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(1-\eta_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\}, \bigcup\limits_{\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)} \in \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S}_i)}} \left\{ \frac{\prod_{i=1}^{n}(1+\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}-\prod_{i=1}^{n}(1-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}}{\prod_{i=1}^{n}(1+\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}+\prod_{i=1}^{n}(1-\xi_{\overline{\mathbb{N}}(\mathbb{S}_i)})^{w_i}} \right\} \right) \end{array} \right\rangle. \tag{10}$$

Proof: Theorem 4 can be proved using mathematical induction.

Similarly, the $\mathbb{G}_{\mathrm{MV}\mathbb{RN}}$ operator satisfies the following properties.

Property 4 (Idempotency Law): If all MVRNNs are identical, i.e., $\mathbb{N}(\mathbb{S}_i) = \mathbb{N}(\mathbb{S})$ for $i = 1, 2, \ldots, n$, then:

$$\mathbb{G}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \mathbb{N}(\mathbb{S})$$

and

$$\mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) = \mathbb{N}(\mathbb{S}).$$

Proof: For $\mathbb{N}(\mathbb{S}_i) = \mathbb{N}(\mathbb{S})$, by definition of $\mathbb{G}_{\mathrm{MV}\mathbb{RN}}$ operator:

$$\mathbb{G}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\underset{i=1}{\overset{n}{\otimes}} (\underline{\mathbb{N}}(\mathbb{S}))^{\frac{1}{n}}, \underset{i=1}{\overset{n}{\otimes}} (\overline{\mathbb{N}}(\mathbb{S}))^{\frac{1}{n}} \right) = (\underline{\mathbb{N}}(\mathbb{S}), \overline{\mathbb{N}}(\mathbb{S})) = N(S).$$

For $\mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}$ operator, given $\sum_{i=1}^{n} w_i = 1$:

$$\mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) = \left(\underset{i=1}{\overset{n}{\otimes}} (\underline{\mathbb{N}}(\mathbb{S}))^{w_i}, \underset{i=1}{\overset{n}{\otimes}} (\overline{\mathbb{N}}(\mathbb{S}))^{w_i} \right) = (\underline{\mathbb{N}}(\mathbb{S}), \overline{\mathbb{N}}(\mathbb{S})) = N(S).$$

This holds because $\otimes_{i=1}^{n} \mathbb{X}^{w_i} = \mathbb{X}$ when $\sum w_i = 1$ for any MVRNN $\mathbb{X}$ (based on the definition of weighted aggregation and scalar multiplication).

Property 5 (Boundedness): Both the $\mathbb{G}_{\mathrm{MV}\mathbb{RN}}$ and $\mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}$ Operators are bounded. That is, for a given set of MVRNNs $\mathbb{N}(\mathbb{S}_i)$ $(i = 1, 2, \ldots, n)$, the aggregated values satisfy:

$$\mathbb{N}(\mathbb{S})^{-} \leqslant \mathbb{G}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{S}_n)) \leqslant \mathbb{N}(\mathbb{S})^{+}$$

and

$$\mathbb{N}(\mathbb{S})^{-} \leqslant \mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{S}_n)) \leqslant \mathbb{N}(\mathbb{S})^{+}.$$

Here, $\mathbb{N}(\mathbb{S})^{-}$ and $\mathbb{N}(\mathbb{S})^{+}$ are the lower and upper bounds defined previously (refer to Property 2).

Proof: The proof for boundedness follows from the definitions of the geometric aggregation operators and the properties of Einstein operators, similar to Property 2.

Property 6 (Monotonicity): If $\mathbb{N}(\mathbb{S}_i) \leqslant \mathbb{N}(\mathbb{S}_i^*)$ for $i = 1, 2, \ldots, n$, then:

$$\mathbb{G}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{G}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1^*), \ldots, \mathbb{N}(\mathbb{S}_n^*)),$$

And:

$$\mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{G}^{w}_{\mathrm{MV}\mathbb{RN}}(\mathbb{N}(\mathbb{S}_1^*), \ldots, \mathbb{N}(\mathbb{S}_n^*)).$$

Proof: Since $\mathbb{N}(\mathbb{S}_i) \leqslant \mathbb{N}(\mathbb{S}_i^*)$ for all $i = 1, 2, \ldots, n$, it follows that:

$$\mathbb{G}_{MVRN}\left(\mathbb{N}\left(\mathbb{S}_1\right), \mathbb{N}\left(\mathbb{S}_2\right), \ldots, \mathbb{N}\left(\mathbb{S}_n\right)\right) \leqslant \mathbb{G}_{MVRN}\left(\mathbb{N}\left(\mathbb{S}_1\right), \mathbb{N}\left(\mathbb{S}_2\right), \ldots, \mathbb{N}\left(\mathbb{S}_n^*\right)\right),$$

And similarly: $\mathbb{G}^{w}_{MVRN}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n)) \leqslant \mathbb{G}^{w}_{MVRN}(\mathbb{N}(\mathbb{S}_1), \mathbb{N}(\mathbb{S}_2), \ldots, \mathbb{N}(\mathbb{S}_n^*))$.

Definition 12: *Score function and accuracy function*

Let $\mathbb{N}(\mathbb{S}) = \langle (\mathbb{T}_{\mathbb{N}(\mathbb{S})}, \mathbb{I}_{\mathbb{N}(\mathbb{S})}, \mathbb{F}_{\mathbb{N}(\mathbb{S})}), (\mathbb{T}_{\bar{\mathbb{N}}(\mathbb{S})}, \mathbb{I}_{\bar{\mathbb{N}}(\mathbb{S})}, \mathbb{F}_{\bar{\mathbb{N}}(\mathbb{S})}) \rangle$ be an MVRNN. The associated score function s for $\mathbb{N}(\mathbb{S})$ can be defined as:

$$s(\mathbb{N}(\mathbb{S})) = \frac{1}{4}\left(3 + \frac{1}{l_{\underline{\mathbb{T}}}}\sum_{k=1}^{l_{\underline{\mathbb{T}}}} \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})k} + \frac{1}{l_{\overline{\mathbb{T}}}}\sum_{k=1}^{l_{\overline{\mathbb{T}}}} \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})k} - \frac{2}{l_{\underline{\mathbb{I}}}}\sum_{g=1}^{l_{\underline{\mathbb{I}}}} \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})g} - \frac{2}{l_{\overline{\mathbb{I}}}}\sum_{g=1}^{l_{\overline{\mathbb{I}}}} \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})g} - \frac{1}{l_{\underline{\mathbb{F}}}}\sum_{r=1}^{l_{\underline{\mathbb{F}}}} \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})r} - \frac{1}{l_{\overline{\mathbb{F}}}}\sum_{r=1}^{l_{\overline{\mathbb{F}}}} \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})r}\right) \quad (11)$$

The accuracy function a for $\mathbb{N}(\mathbb{S})$ can then be defined as:

$$\begin{aligned} a(\mathbb{N}(\mathbb{S})) = \frac{1}{4}\Bigg(&3 + \frac{1}{l_{\underline{\mathbb{T}}}}\sum_{k=1}^{l_{\underline{\mathbb{T}}}} \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})k} + \frac{1}{l_{\overline{\mathbb{T}}}}\sum_{k=1}^{l_{\overline{\mathbb{T}}}} \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})k} - \frac{2}{l_{\underline{\mathbb{I}}}l_{\underline{\mathbb{T}}}}\sum_{g=1}^{l_{\underline{\mathbb{I}}}}\sum_{k=1}^{l_{\underline{\mathbb{T}}}} \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})g}(1 - \mathbb{T}_{\underline{\mathbb{N}}(\mathbb{S})k}) - \frac{2}{l_{\overline{\mathbb{I}}}l_{\overline{\mathbb{T}}}}\sum_{g=1}^{l_{\overline{\mathbb{I}}}}\sum_{k=1}^{l_{\overline{\mathbb{T}}}} \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})g}(1 - \mathbb{T}_{\overline{\mathbb{N}}(\mathbb{S})k}) \\ &- \frac{1}{l_{\underline{\mathbb{F}}}l_{\underline{\mathbb{I}}}}\sum_{r=1}^{l_{\underline{\mathbb{F}}}}\sum_{g=1}^{l_{\underline{\mathbb{I}}}} \mathbb{F}_{\underline{\mathbb{N}}(\mathbb{S})r}(1 - \mathbb{I}_{\underline{\mathbb{N}}(\mathbb{S})g}) - \frac{1}{l_{\overline{\mathbb{F}}}l_{\overline{\mathbb{I}}}}\sum_{r=1}^{l_{\overline{\mathbb{F}}}}\sum_{g=1}^{l_{\overline{\mathbb{I}}}} \mathbb{F}_{\overline{\mathbb{N}}(\mathbb{S})r}(1 - \mathbb{I}_{\overline{\mathbb{N}}(\mathbb{S})g})\Bigg) \end{aligned} \quad (12)$$

Proposed MCDM methods

In real-world situations, choosing the best option from a set of possibilities often involves complex criteria and inherent uncertainties. Traditional approaches may oversimplify these complexities, leading to suboptimal decisions. The Multi-Valued Rough Neutrosophic Arithmetic Mean Operator ($\mathbb{A}_{MVRN}$), the Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator ($\mathbb{A}^{\mathrm{W}}_{\mathrm{MVRN}}$), the Multi-Valued Rough Neutrosophic Geometric Mean Operator ($\mathbb{G}_{MVRN}$), and the Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator ($\mathbb{G}^{w}_{MVRN}$) are four advanced multi-criteria decision-making methods based on MVRNNs. These methods provide a reliable way to evaluate options under uncertainty.

The process includes forming a decision matrix, aggregating values, computing scores, and ranking alternatives, with differences in weights and aggregation operators. In this context, let the set of criteria be represented by $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_n\}$ and the set of alternatives is represented by $\mathbb{A} = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\}$. Decision-makers assess each alternative $\mathbb{A}_j$ with respect to each criterion $\mathbb{C}_i$, utilizing MVRNNs for evaluation. Tables 1 and 2 show the MVRNN-based relationship and the corresponding decision matrix, respectively. MVRNNs were used to create the decision matrix, which has the following structure:

The suggested MCDM method's procedural flow is shown in a unified framework in Fig. 1, which highlights the similarities and differences between the four aggregation operators.

Table 1. The relation between alternatives and criteria in terms of MVRNNs.

$DM[\mathbb{A}\|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_n] =$	$\mathbb{C}_1$	$\mathbb{C}_2$	$\cdots$	$\mathbb{C}_n$
$\mathbb{A}_1$	$\langle \underline{\mathbb{N}}_{11}, \bar{\mathbb{N}}_{11} \rangle$	$\langle \underline{\mathbb{N}}_{12}, \bar{\mathbb{N}}_{12} \rangle$	$\cdots$	$\langle \underline{\mathbb{N}}_{1n}, \bar{\mathbb{N}}_{1n} \rangle$
$\mathbb{A}_2$	$\langle \underline{\mathbb{N}}_{21}, \bar{\mathbb{N}}_{21} \rangle$	$\langle \underline{\mathbb{N}}_{22}, \bar{\mathbb{N}}_{22} \rangle$	$\cdots$	$\langle \underline{\mathbb{N}}_{2n}, \bar{\mathbb{N}}_{2n} \rangle$
$\mathbb{A}_m$	$\langle \underline{\mathbb{N}}_{m1}, \bar{\mathbb{N}}_{m1} \rangle$	$\langle \underline{\mathbb{N}}_{m2}, \bar{\mathbb{N}}_{m2} \rangle$	$\cdots$	$\langle \underline{\mathbb{N}}_{mn}, \bar{\mathbb{N}}_{mn} \rangle$

Method 1: Multi-valued rough neutrosophic arithmetic mean operator

This method uses the MVRNAMO to aggregate the evaluation values. The procedure is:

1. **Construct a Decision Matrix:** Define the relationship between alternatives $\mathbb{A}_j$ and criteria $\mathbb{C}_i$ using an MVRNN-based decision matrix, see Table 1.
2. **Aggregate Evaluations:** Apply MVRNAMO (using Eq. (3) and Eq. (4)) to compute aggregated total values for each alternative.
3. **Compute Score and Accuracy Functions:** Use Eq. (11) and Eq. (12) to calculate the score and accuracy values for the aggregated alternative.
4. **Rank Alternatives:** Sort alternatives by score values in decreasing order. Accuracy values are used to break ties if needed.

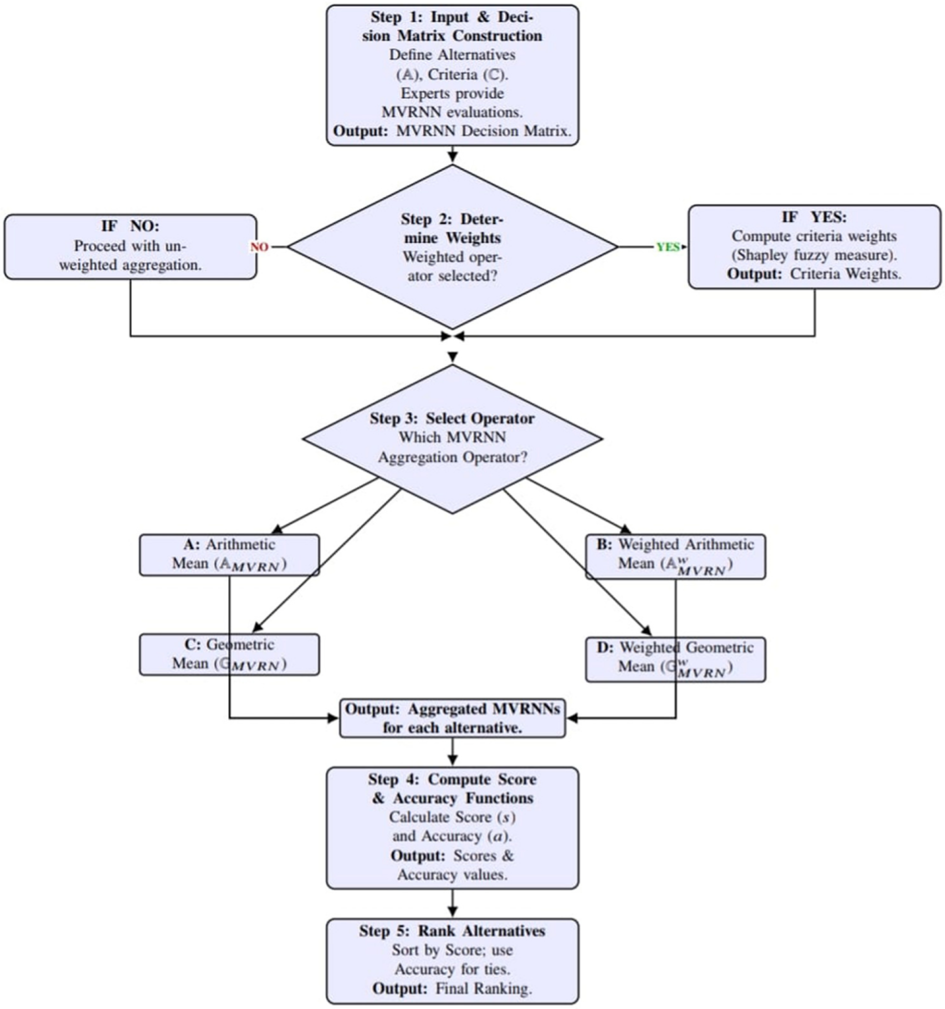

Fig. 1. Unified framework for multi-criteria decision-making using MVRN aggregation operators.

Method 2: Weighted multi-valued rough neutrosophic arithmetic mean operator

This method extends method 1 by incorporating criteria. The procedure is:

1. **Construct the Decision Matrix:** The decision matrix is the same as method 1, representing evaluations as MVRNNs Table 1.
2. **Determine Criteria Weights:** Calculate weights using an appropriate approach, such as the Shapley fuzzy measure (e.g., as defined in Eqs. (1) and (2)).
3. **Compute Weighted Aggregation:** Using the Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator Eqs. (5) and (6) to obtain weighted aggregated values for each alternative.
4. **Compute Score and Accuracy Functions:** Use Eqs. (11) and (12) to calculate the score and accuracy values for the aggregated alternative.
5. **Rank Alternatives:** Sort alternatives by score values in decreasing order. Accuracy values are used to break ties if needed.

Method 3: Multi-valued rough neutrosophic geometric mean operator

This method uses the MVRNGMO to aggregate the evaluation values. The procedure is:

1. **Construct the Decision Matrix:** The decision matrix is constructed using MVRNNs, as described in previous methods Table 1.

2. **Aggregate Evaluations:** Apply MVRNGMO (using Eqs. (7) and (8)) to compute aggregated total values for each alternative.
3. **Compute Score and Accuracy Functions:** Use Eqs. (11) and (12) to calculate the score and accuracy values for the aggregated alternative.
4. **Rank Alternatives:** Sort alternatives by score values in decreasing order. Accuracy values are used to break ties if needed.

Method 4: Weighted multi-valued rough neutrosophic geometric mean operator

This method combines the geometric mean aggregation with criteria weights, using the WMVRNGMO.

1. **Construct the Decision Matrix:** The decision matrix is the same as method 1, representing evaluations as MVRNNs Table 1.
2. **Determine Criteria Weights:** Calculate weights using an appropriate approach, such as the Shapley fuzzy measure (e.g., as defined in Eqs. (1) and (2)).
3. **Compute Weighted Aggregation:** Using the Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator Eqs. (9) and (10) to obtain weighted aggregated values for each alternative.
4. **Compute Score and Accuracy Functions:** Use Eqs. (11) and (12) to calculate the score and accuracy values for the aggregated alternative.
5. **Rank Alternatives:** Sort alternatives by score values in decreasing order. Accuracy values are used to break ties if needed.

Numerical example: Selecting the best hospital for a patient

In real-world decision-making, uncertainty and vagueness often make it difficult to choose the best option. This is especially true in critical situations like hospital selection, where both time and accuracy are vital. Traditional decision-making methods may struggle to deal with such complexity and conflicting criteria.

This study proposes an MVRNNs-based MCDM using aggregation operators and Shapley weights for accurate decision-making under uncertainty.

Hospital selection scenario

In critical situations, a family must quickly choose the most suitable hospital. The decision is based on 3 criteria:

- $\mathbb{C}_1$: **Doctor Availability**
- $\mathbb{C}_2$: **Waiting Time**
- $\mathbb{C}_3$: **Treatment Success Rate**

Four alternatives are considered:

- $\mathbb{A}_1$: **Private Hospital**
- $\mathbb{A}_2$: **Government Hospital**
- $\mathbb{A}_3$: **Multi-Speciality Hospital**
- $\mathbb{A}_4$: **Charity Hospital**

Expert evaluations are analyzed using MVRNN-based aggregation to handle uncertainty and rank the hospitals.

Table 2. The decision matrix based on MVRNNs for hospital selection.

$DM[\mathbb{A} \mid \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$
$\mathbb{A}_1$	$\langle(\{0.7\}, \{0.3\}, \{0, 0.2\})$ $(\{0.9, 1\}, \{0.1, 0.2\}, \{0\})\rangle$	$\langle(\{0.4\}, \{0.6\}, \{0.5\})$ $(\{0.7\}, \{0.3\}, \{0, 0.2\})\rangle$	$\langle(\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\})$ $(\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})\rangle$
$\mathbb{A}_2$	$\langle(\{0.4\}, \{0.6\}, \{0.3\})$ $(\{0.8, 1\}, \{0.1, 0.2\}, \{0\})\rangle$	$\langle(\{0.7\}, \{0.3\}, \{0.2\})$ $(\{0.8, 0.9\}, \{0.1\}, \{0\})\rangle$	$\langle(\{0.3\}, \{0.7\}, \{0.8\})$ $(\{0.7\}, \{0.3\}, \{0.1\})\rangle$
$\mathbb{A}_3$	$\langle(\{0.9, 1\}, \{0.2\}, \{0\})$ $(\{0.8, 1\}, \{0, 0.2\}, \{0\})\rangle$	$\langle(\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\})$ $(\{.6\}, \{0, 0.4\}, \{0, 0.2\})\rangle$	$\langle(\{0.4\}, \{0.6\}, \{0, 0.2\})$ $(\{0.6, 0.8\}, \{0, 0.2\},\{0,0.3\})\rangle$
$\mathbb{A}_4$	$\langle(\{0.6, 0.7\}, \{0.3\}, \{0.1\})$ $(\{0.8, 1\}, \{0.2\}, \{0\})\rangle$	$\langle(\{0.3\}, \{0.7\}, \{0.8, 1\})$ $(\{0.6, 0.7\}, \{0.2, 0.3\},\{0.1\})\rangle$	$\langle(\{0, 0.2\}, \{0.8\}, \{0.9\})$ $(\{0.5, 0.8\}, \{0.2\}, \{0.3\})\rangle$

Proposed MCDM methodology application

This section applies the proposed MCDM framework to the hospital selection problem using four MVRNN aggregations. Each operator processes the decision matrix to produce a reliable ranking. The procedure for all operators follows a similar path, including aggregation, score calculation, and ranking.

The individual fuzzy measures for the criteria (i.e., $\mu(e_1) = 0.4, \mu(e_2) = 0.25, \mu(e_3) = 0.45$) were initially determined using a pairwise comparison method based on expert judgments. Subsequently, these measures were utilized to compute the Shapley fuzzy measures, as detailed in the following steps.

Fuzzy measure of attributes and criteria weights

The individual fuzzy measures for the criteria are given as:

$$\mu\left(e_1\right) = 0.4,\ \ \mu\left(e_2\right) = 0.25,\ \ \mu\left(e_3\right) = 0.45$$

The value of λ is calculated using the following equation:

$$\lambda + 1 = \left(1 + 0.4\lambda\right)\left(1 + 0.25\lambda\right)\left(1 + 0.45\lambda\right)$$

Solving this equation gives:

$$\lambda = -0.2622$$

The fuzzy measure of attributes is obtained:

$$\mu(e_1, e_2) = \frac{1}{-0.2622}((1 - 0.2622 \cdot (0.4))(1 - 0.2622 \cdot (0.25)) - 1) = 0.6238$$

Similarly, for other combinations:

$$\mu\left(e_1, e_3\right) = 0.8028,\ \ \mu\left(e_2, e_3\right) = 0.6705,\ \ \mu\left(e_1, e_2, e_3\right) = 1$$

After that, the Shapley fuzzy measures (SFM) can be obtained from Eq. (2) as follows:

$$\begin{aligned} w(\mu, \alpha) &= \frac{(3-0-1)!}{3!}\left(\mu\left(e_1\right) - \mu(\phi)\right) + \frac{(3-1-1)!}{3!}\left(\mu\left(e_1, e_2\right) - \mu\left(e_2\right)\right) + \frac{(3-1-1)!}{3!}\left(\mu\left(e_1, e_3\right) - \mu\left(e_3\right)\right) \\ &\quad + \frac{(3-2-1)!}{3!}(\mu(e_1, e_2, e_3) - \mu(e_2, e_3)) \\ &= \frac{1}{3}(0.4 - 0) + \frac{1}{6}(0.6238 - 0.25) + \frac{1}{6}(0.8025 - 0.45) + \frac{1}{3}(1 - 0.6705) = 0.364 \end{aligned}$$

Similarly,

$$w_2\left(\mu, \alpha\right) = 0.223,\ \ w_3\left(\mu, \alpha\right) = 0.413$$

Thus, the final criteria weights are:

$$w_1 = 0.364,\ \ w_2 = 0.223,\ \ w_3 = 0.413.$$

Method 1: Multi-valued rough neutrosophic arithmetic mean operator

This method applies the MVRNAMO to aggregate the initial MVRNN evaluations.

Step 1: Construct the MVRNN-based decision matrix.
The decision matrix, which defines the relationship between alternatives and criteria, is established using MVRNNs as presented in Table 2.

Step 2: Aggregate the decision matrix using the MVRN Arithmetic Mean Operator. We apply the aggregation operator to each alternative. For alternative $\mathbb{A}_1$:

Let $\mathbb{N}_{11} = \langle (\{0.7\}, \{0.3\}, \{0, 0.2\}), (\{0.9, 1\}, \{0.1, 0.2\}, \{0\}) \rangle$
Let $\mathbb{N}_{12} = \langle (\{0.4\}, \{0.6\}, \{0.5\}), (\{0.7\}, \{0.3\}, \{0, 0.2\}) \rangle$
Let $\mathbb{N}_{13} = \langle (\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\}), (\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\}) \rangle$

The aggregated MVRNN for $\mathbb{A}_{MVRN}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}_{MVRN}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle \tfrac{1}{3}(\underline{\mathbb{N}}_{11} \oplus \underline{\mathbb{N}}_{12} \oplus \underline{\mathbb{N}}_{13}), \tfrac{1}{3}(\overline{\mathbb{N}}_{11} \oplus \overline{\mathbb{N}}_{12} \oplus \overline{\mathbb{N}}_{13}) \right\rangle$$

$$= \left\langle \begin{array}{l} \tfrac{1}{3}((\{0.7\}, \{0.3\}, \{0, 0.2\}) \oplus (\{0.4\}, \{0.6\}, \{0.5\}) \oplus (\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\})), \\ \tfrac{1}{3}((\{0.9, 1\}, \{0.1, 0.2\}, \{0\}) \oplus (\{0.7\}, \{0.3\}, \{0, 0.2\}) \oplus (\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})) \end{array} \right\rangle$$

$$= \langle (\{0.5465, 0.6621\}, \{0.3379, 0.3884\}, \{0, 0.2221\}), (\{0.7662, 1\}, \{0.1836, 0.2904\}, \{0\}) \rangle.$$

For alternative $\mathbb{A}_2$:

Let $\mathbb{N}_{21} = \langle (\{0.4\}, \{0.6\}, \{0.3\}), (\{0.8, 1\}, \{0.1, 0.2\}, \{0\}) \rangle$
Let $\mathbb{N}_{22} = \langle (\{0.7\}, \{0.3\}, \{0.2\}), (\{0.8, 0.9\}, \{0.1\}, \{0\}) \rangle$
Let $\mathbb{N}_{23} = \langle (\{0.3\}, \{0.7\}, \{0.8, 1\}), (\{0.7\}, \{0.3\}, \{0.1\}) \rangle$

The aggregated MVRNN for $\mathbb{A}_{MVRN}[\mathbb{A}_2|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}_{MVRN}[\mathbb{A}_2|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle \tfrac{1}{3}(\underline{\mathbb{N}}_{21} \oplus \underline{\mathbb{N}}_{22} \oplus \underline{\mathbb{N}}_{23}), \tfrac{1}{3}(\overline{\mathbb{N}}_{21} \oplus \overline{\mathbb{N}}_{22} \oplus \overline{\mathbb{N}}_{23}) \right\rangle$$

$$= \left\langle \begin{array}{l} \tfrac{1}{3}((\{0.4\}, \{0.6\}, \{0.3\}) \oplus (\{0.7\}, \{0.3\}, \{0.2\}) \oplus (\{0.3\}, \{0.7\}, \{0.8, 1\})), \\ \tfrac{1}{3}((\{0.8, 1\}, \{0.1, 0.2\}, \{0\}) \oplus (\{0.8, 0.9\}, \{0.1\}, \{0\}) \oplus (\{0.7\}, \{0.3\}, \{0.1\})) \end{array} \right\rangle$$

$$= \langle (\{0.4881\}, \{0.5120\}, \{0.3813, 0.4248\}), (\{0.7705, 1\}, \{0.1460, 0.1678\}, \{0\}) \rangle.$$

For alternative $\mathbb{A}_3$:

Let $\mathbb{N}_{31} = \langle (\{0.9, 1\}, \{0.2\}, \{0\}), (\{0.8, 1\}, \{0, 0.2\}, \{0\}) \rangle$
Let $\mathbb{N}_{32} = \langle (\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\}), (\{0.6\}, \{0, 0.4\}, \{0, 0.2\}) \rangle$
Let $\mathbb{N}_{33} = \langle (\{0.4\}, \{0.6\}, \{0, 0.2\}), (\{0.6, 0.8\}, \{0, 0.2\}, \{0, 0.3\}) \rangle$

The aggregated MVRNN for $\mathbb{A}_{MVRN}[\mathbb{A}_3|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}_{MVRN}[\mathbb{A}_3|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle \tfrac{1}{3}(\underline{\mathbb{N}}_{31} \oplus \underline{\mathbb{N}}_{32} \oplus \underline{\mathbb{N}}_{33}), \tfrac{1}{3}(\overline{\mathbb{N}}_{31} \oplus \overline{\mathbb{N}}_{32} \oplus \overline{\mathbb{N}}_{33}) \right\rangle$$

$$= \left\langle \begin{array}{l} \tfrac{1}{3}((\{0.9, 1\}, \{0.2\}, \{0\}) \oplus (\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\}) \oplus (\{0.4\}, \{0.6\}, \{0, 0.2\})), \\ \tfrac{1}{3}((\{0.8, 1\}, \{0, 0.2\}, \{0\}) \oplus (\{0.6\}, \{0, 0.4\}, \{0, 0.2\}) \oplus (\{0.6, 0.8\}, \{0, 0.2\}, \{0, 0.3\})) \end{array} \right\rangle$$

$$= \langle (\{0.6724, 1\}, \{0.2968, 0.3379\}, \{0\}), (\{0.6796, 1\}, \{0, 0.2542\}, \{0\}) \rangle.$$

For alternative $\mathbb{A}_4$:

Let $\mathbb{N}_{41} = \langle (\{0.6, 0.7\}, \{0.3\}, \{0.1\}), (\{0.8, 1\}, \{0.2\}, \{0\}) \rangle$
Let $\mathbb{N}_{42} = \langle (\{0.3\}, \{0.7\}, \{0.8, 1\}), (\{0.6, 0.7\}, \{0.2, 0.3\}, \{0.1\}) \rangle$
Let $\mathbb{N}_{43} = \langle (\{0, 0.2\}, \{0.8\}, \{0.9\}), (\{0.5, 0.8\}, \{0.2\}, \{0.3\}) \rangle$

The aggregated MVRNN for $\mathbb{A}_{MVRN}[\mathbb{A}_4|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}_{\mathrm{MVRN}}[\mathbb{A}_4|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle \tfrac{1}{3}(\underline{\mathbb{N}}_{41} \oplus \underline{\mathbb{N}}_{42} \oplus \underline{\mathbb{N}}_{43}), \tfrac{1}{3}(\overline{\mathbb{N}}_{41} \oplus \overline{\mathbb{N}}_{42} \oplus \overline{\mathbb{N}}_{43}) \right\rangle$$

$$= \left\langle \begin{array}{l} \tfrac{1}{3}((\{0.6, 0.7\}, \{0.3\}, \{0.1\}) \oplus (\{0.3\}, \{0.7\}, \{0.8, 1\}) \oplus (\{0, 0.2\}, \{0.8\}, \{0.9\})), \\ \tfrac{1}{3}((\{0.8, 1\}, \{0.2\}, \{0\}) \oplus (\{0.6, 0.7\}, \{0.2, 0.3\}, \{0.1\}) \oplus (\{0.5, 0.8\}, \{0.2\}, \{0.3\})) \end{array} \right\rangle$$

$$= \langle (\{0.3223, 0.4300\}, \{0.570\}, \{0.4688, 0.5191\}), (\{0.6529, 1\}, \{0.2, 0.2295\}, \{0\}) \rangle.$$

Step 3: Compute score and accuracy metrics. Using Eqs. (11) and (12), we calculate each alternative's score (s) and accuracy (a) functions for the aggregated MVRNNs:

$$s(\mathbb{A}_1) = 0.7940, \quad s(\mathbb{A}_2) = 0.6581, \quad s(\mathbb{A}_3) = 0.9468, \quad s(\mathbb{A}_4) = 0.4274.$$

Since all the score values (SV) are distinct, in this case, there is no need to compute the accuracy values (AV).

Step 4: Rank alternatives.
The alternatives are ranked based on their scores in descending order as follows:

$$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4.$$

Therefore, $\mathbb{A}_3$ (Multi-Speciality Hospital) is identified as the best choice.

Method 2: Weighted multi-valued rough neutrosophic arithmetic mean operator

This method extends Method 1 by incorporating criteria weights to reflect their relative importance, using the WMVRNAMO.

Step 1: Construct the decision matrix.
The decision matrix is constructed similarly to Method 1, representing evaluations as MVRNNs Table 2.

Step 2: Determine the criteria weights.
The criteria weights (w_1, w_2, w_3) are determined using the Shapley fuzzy measure, as detailed in the "Fuzzy Measure of Attributes and Criteria Weights" subsection. The weights assigned to the criteria are as follows:
$w_1 = 0.364,\ w_2 = 0.223,\ w_3 = 0.413.$

Step 3: Calculation of weighted aggregation values.
Using Eq. (6), weighted aggregation values are computed for each alternative, showing their MVRNNs and corresponding weights.

For alternative $\mathbb{A}_1$:

Let $\mathbb{N}_{11} = \langle(\{0.7\}, \{0.3\}, \{0, 0.2\}), (\{0.9, 1\}, \{0.1, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{12} = \langle(\{0.4\}, \{0.6\}, \{0.5\}), (\{0.7\}, \{0.3\}, \{0, 0.2\})\rangle$
Let $\mathbb{N}_{13} = \langle(\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\}), (\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})\rangle$

The aggregated MVRNN for $\mathbb{A}^{w}{}_{MVRN}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}^{w}{}_{\mathrm{MVRN}}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle w_1\underline{\mathbb{N}}_{11} \oplus w_2\underline{\mathbb{N}}_{12} \oplus w_3\underline{\mathbb{N}}_{13},\quad w_1\overline{\mathbb{N}}_{11} \oplus w_2\overline{\mathbb{N}}_{12} \oplus w_3\overline{\mathbb{N}}_{13}\right\rangle$$

$$= \left\langle \begin{matrix} (0.364(\{0.7\}, \{0.3\}, \{0, 0.2\}) \oplus 0.223(\{0.4\}, \{0.6\}, \{0.5\}) \oplus 0.413(\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\})), \\ (0.364(\{0.9, 1\}, \{0.1, 0.2\}, \{0\}) \oplus 0.223(\{0.7\}, \{0.3\}, \{0, 0.2\}) \oplus 0.413(\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})) \end{matrix} \right\rangle$$

$$= \langle(\{0.5629, 0.6982\}, \{0.3017, 0.3540\}, \{0, 0.1888\}), (\{0.7680, 1\}, \{0.1716, 0.2320\}, \{0\})\rangle.$$

For alternative $\mathbb{A}_2$:

Let $\mathbb{N}_{21} = \langle(\{0.4\}, \{0.6\}, \{0.3\}), (\{0.8, 1\}, \{0.1, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{22} = \langle(\{0.7\}, \{0.3\}, \{0.2\}), (\{0.8, 0.9\}, \{0.1\}, \{0\})\rangle$
Let $\mathbb{N}_{23} = \langle(\{0.3\}, \{0.7\}, \{0.8, 1\}), (\{0.7\}, \{0.3\}, \{0.1\})\rangle$

The aggregated MVRNN for $\mathbb{A}^{w}{}_{MVRN}[\mathbb{A}_2|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}^{w}{}_{\mathrm{MVRN}}[\mathbb{A}_2|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle w_1\underline{\mathbb{N}}_{21} \oplus w_2\underline{\mathbb{N}}_{22} \oplus w_3\underline{\mathbb{N}}_{23}, w_1\overline{\mathbb{N}}_{21} \oplus w_2\overline{\mathbb{N}}_{22} \oplus w_3\overline{\mathbb{N}}_{23}\right\rangle$$

$$= \left\langle \begin{matrix} (0.364(\{0.4\}, \{0.6\}, \{0.3\}) \oplus 0.223(\{0.7\}, \{0.3\}, \{0.2\}) \oplus 0.413(\{0.3\}, \{0.7\}, \{0.8, 1\})), \\ (0.364(\{0.8, 1\}, \{0.1, 0.2\}, \{0\}) \oplus 0.223(\{0.8, 0.9\}, \{0.1\}, \{0\}) \oplus 0.413(\{0.7\}, \{0.3\}, \{0.1\})) \end{matrix} \right\rangle$$

$$= \langle(\{0.4426\}, \{0.5574\}, \{0.4321, 0.4915\}), (\{0.7628, 1\}, \{0.1596, 0.2045\}, \{0\})\rangle.$$

For alternative $\mathbb{A}_3$:

Let $\mathbb{N}_{31} = \langle(\{0.9, 1\}, \{0.2\}, \{0\}), (\{0.8, 1\}, \{0, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{32} = \langle(\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\}), (\{.6\}, \{0, 0.4\}, \{0, 0.2\})\rangle$
Let $\mathbb{N}_{33} = \langle(\{0.4\}, \{0.6\}, \{0, 0.2\}), (\{0.6, 0.8\}, \{0, 0.2\}, \{0, 0.3\})\rangle$

The aggregated MVRNN for $\mathbb{A}^{w}{}_{MVRN}[\mathbb{A}_3|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}^{w}{}_{\mathrm{MVRN}}[\mathbb{A}_3|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle w_1\underline{\mathbb{N}}_{31} \oplus w_2\underline{\mathbb{N}}_{32} \oplus w_3\underline{\mathbb{N}}_{33}, w_1\overline{\mathbb{N}}_{31} \oplus w_2\overline{\mathbb{N}}_{32} \oplus w_3\overline{\mathbb{N}}_{33}\right\rangle$$

$$= \left\langle \begin{array}{l} (0.364(\{0.9, 1\}, \{0.2\}, \{0\}) \oplus 0.223(\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\}) \oplus 0.413(\{0.4\}, \{0.6\}, \{0, 0.2\})), \\ (0.364(\{0.8, 1\}, \{0, 0.2\}, \{0\}) \oplus 0.223(\{.6\}, \{0, 0.4\}, \{0, 0.2\}) \oplus 0.413(\{0.6, 0.8\}, \{0, 0.2\}, \{0, 0.3\})) \end{array} \right\rangle$$

$$= \langle(\{0.6823, 1\}, \{0.3250, 0.3541\}, \{0\}), (\{0.6862, 1\}, \{0, 0.2350\}, \{0\})\rangle.$$

For alternative $\mathbb{A}_4$:

Let $\mathbb{N}_{41} = \langle(\{0.6, 0.7\}, \{0.3\}, \{0.1\}), (\{0.8, 1\}, \{0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{42} = \langle(\{0.3\}, \{0.7\}, \{0.8, 1\}), (\{0.6, 0.7\}, \{0.2, 0.3\}, \{0.1\})\rangle$
Let $\mathbb{N}_{43} = \langle(\{0, 0.2\}, \{0.8\}, \{0.9\}), (\{0.5, 0.8\}, \{0.2\}, \{0.3\})\rangle$

The aggregated MVRNN for $\mathbb{A}^{w}{}_{MVRN}[\mathbb{A}_4|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{A}^{w}{}_{\mathrm{MVRN}}[\mathbb{A}_4|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle w_1\underline{\mathbb{N}}_{41} \oplus w_2\underline{\mathbb{N}}_{42} \oplus w_3\underline{\mathbb{N}}_{43}, w_1\overline{\mathbb{N}}_{41} \oplus w_2\overline{\mathbb{N}}_{42} \oplus w_3\overline{\mathbb{N}}_{43}\right\rangle$$

$$= \left\langle \begin{array}{l} (0.364(\{0.6, 0.7\}, \{0.3\}, \{0.1\}) \oplus 0.223(\{0.3\}, \{0.7\}, \{0.8, 1\}) \oplus 0.413(\{0, 0.2\}, \{0.8\}, \{0.9\})), \\ (0.364(\{0.8, 1\}, \{0.2\}, \{0\}) \oplus 0.223(\{0.6, 0.7\}, \{0.2, 0.3\}, \{0.1\}) \oplus 0.413(\{0.5, 0.8\}, \{0.2\}, \{0.3\})) \end{array} \right\rangle$$

$$= \langle(\{0.3107, 0.4369\}, \{0.5631\}, \{0.4471, 0.4866\}), (\{0.6535, 1\}, \{0.2, 0.2194\}, \{0\})\rangle.$$

Step 4: Compute score and accuracy metrics. Using Eqs. (11) and (12), we calculate each alternative's score (s) and accuracy (a) functions for the aggregated MVRNNs:

$$s(\mathbb{A}_1) = 0.8402, \quad s(\mathbb{A}_2) = 0.5958, \quad s(\mathbb{A}_3) = 0.9426, \quad s(\mathbb{A}_4) = 0.5470.$$

Step 5: Rank alternatives.
The alternatives are ranked based on their scores in descending order as follows:

$$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_4 \succ \mathbb{A}_2.$$

Therefore, $\mathbb{A}_3$ (Multi-Speciality Hospital) is identified as the best choice.

Method 3: Multi-valued rough neutrosophic geometric mean operator

This method employs the MVRNGMO for aggregation, offering an alternative perspective to the arithmetic mean.

Step 1: Construct the decision matrix.
The decision matrix is constructed using MVRNNs, as described in previous methods Table 2.
Step 2: Aggregate the decision matrix.

We apply the aggregation operator to each alternative.
For alternative $\mathbb{A}_1$:

Let $\mathbb{N}_{11} = \langle(\{0.7\}, \{0.3\}, \{0, 0.2\}), (\{0.9, 1\}, \{0.1, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{12} = \langle(\{0.4\}, \{0.6\}, \{0.5\}), (\{0.7\}, \{0.3\}, \{0, 0.2\})\rangle$
Let $\mathbb{N}_{13} = \langle(\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\}), (\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})\rangle$

The aggregated MVRNN for $\mathbb{G}_{MVRN}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{G}_{\mathrm{MVRN}}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle (\underline{\mathbb{N}}_{11} \otimes \underline{\mathbb{N}}_{12} \otimes \underline{\mathbb{N}}_{13})^{\frac{1}{3}}, (\overline{\mathbb{N}}_{11} \otimes \overline{\mathbb{N}}_{12} \otimes \overline{\mathbb{N}}_{13})^{\frac{1}{3}} \right\rangle$$

$$=\left\langle\begin{array}{l}((\{0.7\},\{0.3\},\{0,0.2\})\otimes(\{0.4\},\{0.6\},\{0.5\})\otimes(\{0.5,0.8\},\{0.2,0.3\},\{0.1\}))^{\frac{1}{3}},\\ ((\{0.9,1\},\{0.1,0.2\},\{0\})\otimes(\{0.7\},\{0.3\},\{0,0.2\})\otimes(\{0.6,0.9\},\{0.2,0.4\},\{0.2\}))^{\frac{1}{3}}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}(\{0.5244,0.6185\},\{0.3815,0.4115\},\{0.2132,0.2767\}),\\ (\{0.7291,0.8642\},\{0.2014,0.3022\},\{0.0675,0.1343\})\end{array}\right\rangle.$$

For alternative $\mathbb{A}_2$:

Let $\mathbb{N}_{21}=\langle(\{0.4\},\{0.6\},\{0.3\}),(\{0.8,1\},\{0.1,0.2\},\{0\})\rangle$
Let $\mathbb{N}_{22}=\langle(\{0.7\},\{0.3\},\{0.2\}),(\{0.8,0.9\},\{0.1\},\{0\})\rangle$
Let $\mathbb{N}_{23}=\langle(\{0.3\},\{0.7\},\{0.8,1\}),(\{0.7\},\{0.3\},\{0.1\})\rangle$

The aggregated MVRNN for $\mathbb{G}_{MVRN}[\mathbb{A}_2|\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3]$ is calculated as:

$$\mathbb{G}_{MVRN}[\mathbb{A}_2|\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3]=\left\langle(\underline{\mathbb{N}}_{21}\otimes\underline{\mathbb{N}}_{22}\otimes\underline{\mathbb{N}}_{23})^{\frac{1}{3}},(\overline{\mathbb{N}}_{21}\otimes\overline{\mathbb{N}}_{22}\otimes\overline{\mathbb{N}}_{23})^{\frac{1}{3}}\right\rangle$$

$$=\left\langle\begin{array}{l}((\{0.4\},\{0.6\},\{0.3\})\otimes(\{0.7\},\{0.3\},\{0.2\})\otimes(\{0.3\},\{0.7\},\{0.8,1\}))^{\frac{1}{3}},\\ ((\{0.8,1\},\{0.1,0.2\},\{0\})\otimes(\{0.8,0.9\},\{0.1\},\{0\})\otimes(\{0.7\},\{0.3\},\{0.1\}))^{\frac{1}{3}}\end{array}\right\rangle$$

$$=\langle(\{0.4465\},\{0.5534\},\{0.4907,1\}),(\{0.7661,0.8642\},\{0.1669,0.2014\},\{0.0334\})\rangle.$$

For alternative $\mathbb{A}_3$:

Let $\mathbb{N}_{31}=\langle(\{0.9,1\},\{0.2\},\{0\}),(\{0.8,1\},\{0,0.2\},\{0\})\rangle$
Let $\mathbb{N}_{32}=\langle(\{0.5,0.7\},\{0.2,0.3\},\{0,0.1\}),(\{.6\},\{0,0.4\},\{0,0.2\})\rangle$
Let $\mathbb{N}_{33}=\langle(\{0.4\},\{0.6\},\{0,0.2\}),(\{0.6,0.8\},\{0,0.2\},\{0,0.3\})\rangle$

The aggregated MVRNN for $\mathbb{G}_{MVRN}[\mathbb{A}_3|\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3]$ is calculated as:

$$\mathbb{G}_{\mathrm{MVRN}}[\mathbb{A}_3|\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3]=\left\langle(\underline{\mathbb{N}}_{31}\otimes\underline{\mathbb{N}}_{32}\otimes\underline{\mathbb{N}}_{33})^{\frac{1}{3}},(\overline{\mathbb{N}}_{31}\otimes\overline{\mathbb{N}}_{32}\otimes\overline{\mathbb{N}}_{33})^{\frac{1}{3}}\right\rangle$$

$$=\left\langle\begin{array}{l}((\{0.9,1\},\{0.2\},\{0\})\otimes(\{0.5,0.7\},\{0.2,0.3\},\{0,0.1\})\otimes(\{0.4\},\{0.6\},\{0,0.2\}))^{\frac{1}{3}},\\ ((\{0.8,1\},\{0,0.2\},\{0\})\otimes(\{.6\},\{0,0.4\},\{0,0.2\})\otimes(\{0.6,0.8\},\{0,0.2\},\{0,0.3\}))^{\frac{1}{3}}\end{array}\right\rangle$$

$$=\langle(\{0.5801,0.6777\},\{0.3506,0.3815\},\{0,0.1007\}),(\{0.6636,0.7941\},\{0,0.2648\},\{0,0.1691\})\rangle.$$

For alternative $\mathbb{A}_4$:

Let $\mathbb{N}_{41}=\langle(\{0.6,0.7\},\{0.3\},\{0.1\}),(\{0.8,1\},\{0.2\},\{0\})\rangle$
Let $\mathbb{N}_{42}=\langle(\{0.3\},\{0.7\},\{0.8,1\}),(\{0.6,0.7\},\{0.2,0.3\},\{0.1\})\rangle$
Let $\mathbb{N}_{43}=\langle(\{0,0.2\},\{0.8\},\{0.9\}),(\{0.5,0.8\},\{0.2\},\{0.3\})\rangle$

The aggregated MVRNN for $\mathbb{G}_{MVRN}[\mathbb{A}_4|\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3]$ is calculated as:

$$\mathbb{G}_{\mathrm{MVRN}}[\mathbb{A}_4|\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3]=\left\langle(\underline{\mathbb{N}}_{41}\otimes\underline{\mathbb{N}}_{42}\otimes\underline{\mathbb{N}}_{43})^{\frac{1}{3}},(\overline{\mathbb{N}}_{41}\otimes\overline{\mathbb{N}}_{42}\otimes\overline{\mathbb{N}}_{43})^{\frac{1}{3}}\right\rangle$$

$$=\left\langle\begin{array}{l}((\{0.6,0.7\},\{0.3\},\{0.1\})\otimes(\{0.3\},\{0.7\},\{0.8,1\})\otimes(\{0,0.2\},\{0.8\},\{0.9\}))^{\frac{1}{3}},\\ ((\{0.8,1\},\{0.2\},\{0\})\otimes(\{0.6,0.7\},\{0.2,0.3\},\{0.1\})\otimes(\{0.5,0.8\},\{0.2\},\{0.3\}))^{\frac{1}{3}}\end{array}\right\rangle$$

$$=\langle(\{0,0.3598\},\{0.6402\},\{0.7116,1\}),(\{0.6270,0.8309\},\{0.2,0.2339\},\{1\})\rangle.$$

Step 3: Compute score and accuracy metrics. Using Eqs. (11) and (12), we calculate each alternative's score (s) and accuracy (a) functions for the aggregated MVRNNs:

$$s(\mathbb{A}_1)=0.6814,\quad s(\mathbb{A}_2)=0.5020,\quad s(\mathbb{A}_3)=0.8065,\quad s(\mathbb{A}_4)=0.0847.$$

Step 4: Rank alternatives. The alternatives are ranked based on their scores in descending order as follows:

$$\mathbb{A}_3\succ\mathbb{A}_1\succ\mathbb{A}_2\succ\mathbb{A}_4.$$

Therefore, $\mathbb{A}_3$ (Multi-Speciality Hospital) is identified as the best choice.

Method 4: Weighted multi-valued rough neutrosophic geometric mean operator

This method combines the geometric mean aggregation with criteria weights, using the WMVRNGMO.

Step 1: Construct the decision matrix.
The MVRNN-based decision matrix is constructed as in previous methods Table 2.

Step 2: Determine criteria weights.
The criteria weights (w_1, w_2, w_3) are determined using the Shapley fuzzy measure, as detailed in the "Fuzzy Measure of Attributes and Criteria Weights" subsection. The calculated weights are: $w_1 = 0.364,\ w_2 = 0.223,\ w_3 = 0.413$

Step 3: Calculate weighted aggregation values.
By utilizing Eq. (10), the aggregation values are computed for each alternative. For clarity, the aggregation for each alternative is shown with its constituent MVRNNs and their respective weights.

For alternative $\mathbb{A}_1$:

Let $\mathbb{N}_{11} = \langle(\{0.7\}, \{0.3\}, \{0, 0.2\}), (\{0.9, 1\}, \{0.1, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{12} = \langle(\{0.4\}, \{0.6\}, \{0.5\}), (\{0.7\}, \{0.3\}, \{0, 0.2\})\rangle$
Let $\mathbb{N}_{13} = \langle(\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\}), (\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})\rangle$

The aggregated MVRNN for $\mathbb{G}^{w}{}_{MVRN}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathrm{G}^{w}{}_{\mathrm{MVRN}}[\mathbb{A}_1|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle (\underline{\mathbb{N}}_{11})^{w_1} \otimes (\underline{\mathbb{N}}_{12})^{w_2} \otimes (\underline{\mathbb{N}}_{13})^{w_3}, (\overline{\mathbb{N}}_{11})^{w_1} \otimes (\overline{\mathbb{N}}_{12})^{w_2} \otimes (\overline{\mathbb{N}}_{13})^{w_3} \right\rangle$$

$$= \left\langle \begin{matrix} ((\{0.7\}, \{0.3\}, \{0, 0.2\})^{0.364} \otimes (\{0.4\}, \{0.6\}, \{0.5\})^{0.223} \otimes (\{0.5, 0.8\}, \{0.2, 0.3\}, \{0.1\})^{0.413}), \\ ((\{0.9, 1\}, \{0.1, 0.2\}, \{0\})^{0.364} \otimes (\{0.7\}, \{0.3\}, \{0, 0.2\})^{0.223} \otimes (\{0.6, 0.9\}, \{0.2, 0.4\}, \{0.2\})^{0.413}) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} (\{0.5425, 0.6628\}, \{0.3372, 0.3757\}, \{0.1625, 0.2334\}), \\ (\{0.7267, 0.89\}, \{0.1871, 0.3075\}, \{0.0835, 0.1282\}) \end{matrix} \right\rangle.$$

For alternative $\mathbb{A}_2$:

Let $\mathbb{N}_{21} = \langle(\{0.4\}, \{0.6\}, \{0.3\}), (\{0.8, 1\}, \{0.1, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{22} = \langle(\{0.7\}, \{0.3\}, \{0.2\}), (\{0.8, 0.9\}, \{0.1\}, \{0\})\rangle$
Let $\mathbb{N}_{23} = \langle(\{0.3\}, \{0.7\}, \{0.8, 1\}), (\{0.7\}, \{0.3\}, \{0.1\})\rangle$

The aggregated MVRNN for $\mathbb{G}^{w}{}_{MVRN}[\mathbb{A}_2|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{G}^{w}{}_{\mathrm{MVRN}}[\mathbb{A}_2|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle (\underline{\mathbb{N}}_{21})^{w_1} \otimes (\underline{\mathbb{N}}_{22})^{w_2} \otimes (\underline{\mathbb{N}}_{23})^{w_3}, (\overline{\mathbb{N}}_{21})^{w_1} \otimes (\overline{\mathbb{N}}_{22})^{w_2} \otimes (\overline{\mathbb{N}}_{23})^{w_3} \right\rangle$$

$$= \left\langle \begin{matrix} ((\{0.4\}, \{0.6\}, \{0.3\})^{0.364} \otimes (\{0.7\}, \{0.3\}, \{0.2\})^{0.223} \otimes (\{0.3\}, \{0.7\}, \{0.8, 1\})^{0.413}), \\ ((\{0.8, 1\}, \{0.1, 0.2\}, \{0\})^{0.364} \otimes (\{0.8, 0.9\}, \{0.1\}, \{0\})^{0.223} \otimes (\{0.7\}, \{0.3\}, \{0.1\})^{0.413}) \end{matrix} \right\rangle$$

$$= \langle(\{0.4088\}, \{0.5912\}, \{0.5452, 1\}), (\{0.7581, 0.8509\}, \{0.1846, 0.2203\}, \{0.0414\})\rangle.$$

For alternative $\mathbb{A}_3$:

Let $\mathbb{N}_{31} = \langle(\{0.9, 1\}, \{0.2\}, \{0\}), (\{0.8, 1\}, \{0, 0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{32} = \langle(\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\}), (\{.6\}, \{0, 0.4\}, \{0, 0.2\})\rangle$
Let $\mathbb{N}_{33} = \langle(\{0.4\}, \{0.6\}, \{0, 0.2\}), (\{0.6, 0.8\}, \{0, 0.2\}, \{0, 0.3\})\rangle$

The aggregated MVRNN for $\mathbb{G}^{w}{}_{MVRN}[\mathbb{A}_3|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{G}^{w}{}_{\mathrm{MVRN}}\,[\mathbb{A}_3|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3] = \left\langle (\underline{\mathbb{N}}_{31})^{w_1} \otimes (\underline{\mathbb{N}}_{32})^{w_2} \otimes (\underline{\mathbb{N}}_{33})^{w_3}, (\overline{\mathbb{N}}_{31})^{w_1} \otimes (\overline{\mathbb{N}}_{32})^{w_2} \otimes (\overline{\mathbb{N}}_{33})^{w_3} \right\rangle$$

$$= \left\langle \begin{matrix} ((\{0.9, 1\}, \{0.2\}, \{0\})^{0.364} \otimes (\{0.5, 0.7\}, \{0.2, 0.3\}, \{0, 0.1\})^{0.223} \otimes (\{0.4\}, \{0.6\}, \{0, 0.2\})^{0.413}), \\ ((\{0.8, 1\}, \{0, 0.2\}, \{0\})^{0.364} \otimes (\{.6\}, \{0, 0.4\}, \{0, 0.2\})^{0.223} \otimes (\{0.6, 0.8\}, \{0, 0.2\}, \{0, 0.3\})^{0.413}) \end{matrix} \right\rangle$$

$$= \langle(\{0.5819, 0.6589\}, \{0.3845, 0.4046\}, \{0, 0.1057\}), (\{0.6697, 0.8236\}, \{0, 0.2468\}, \{0, 0.1714\})\rangle.$$

For alternative $\mathbb{A}_4$:

Let $\mathbb{N}_{41} = \langle(\{0.6, 0.7\}, \{0.3\}, \{0.1\}), (\{0.8, 1\}, \{0.2\}, \{0\})\rangle$
Let $\mathbb{N}_{42} = \langle(\{0.3\}, \{0.7\}, \{0.8, 1\}), (\{0.6, 0.7\}, \{0.2, 0.3\}, \{0.1\})\rangle$
Let $\mathbb{N}_{43} = \langle(\{0, 0.2\}, \{0.8\}, \{0.9\}), (\{0.5, 0.8\}, \{0.2\}, \{0.3\})\rangle$

The aggregated MVRNN for $\mathbb{G}^{w}{}_{MVRN}[\mathbb{A}_4|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3]$ is calculated as:

$$\mathbb{G}^{w}{}_{\mathrm{MVRN}}\left[\mathbb{A}_4|\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3\right] = \left\langle (\underline{\mathbb{N}}_{41})^{w_1} \otimes (\underline{\mathbb{N}}_{42})^{w_2} \otimes (\underline{\mathbb{N}}_{43})^{w_3}, (\overline{\mathbb{N}}_{41})^{w_1} \otimes (\overline{\mathbb{N}}_{42})^{w_2} \otimes (\overline{\mathbb{N}}_{43})^{w_3} \right\rangle$$

$$= \left\langle \begin{array}{l} ((\{0.6, 0.7\}, \{0.3\}, \{0.1\})^{0.364} \otimes (\{0.3\}, \{0.7\}, \{0.8, 1\})^{0.223} \otimes (\{0, 0.2\}, \{0.8\}, \{0.9\})^{0.413}), \\ ((\{0.8, 1\}, \{0.2\}, \{0\})^{0.364} \otimes (\{0.6, 0.7\}, \{0.2, 0.3\}, \{0.1\})^{0.223} \otimes (\{0.5, 0.8\}, \{0.2\}, \{0.3\})^{0.413}) \end{array} \right\rangle$$

$$= \langle(\{0.00, 0.3595\}, \{0.6410\}, \{0.7112, 1\}), (\{0.6246, 0.8484\}, \{0.2, 0.2228\}, \{0.1491\})\rangle.$$

Step 4: Compute score and accuracy metrics. Using Eqs. (11) and (12), we calculate each alternative's score (s) and accuracy (a) functions for the aggregated MVRNNs:

$$s(\mathbb{A}_1) = 0.7249, \quad s(\mathbb{A}_2) = 0.4530, \quad s(\mathbb{A}_3) = 0.7982, \quad s(\mathbb{A}_4) = 0.3017.$$

Step 5: Rank alternatives. The alternatives are ranked based on their scores in descending order as follows:

$$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4.$$

Therefore, $\mathbb{A}_3$ (Multi-Speciality Hospital) is identified as the best choice.

Results and discussion

The ranking results of four hospitals ($\mathbb{A}_1$ to $\mathbb{A}_4$), using MVRNNs-based methods are presented. Table 3 shows the scores and ranks for each method.

The Multi-Speciality Hospital consistently attains the highest ranking across all methods, followed by private, government, and charity hospitals. The consistent results show the method's effectiveness and stability under uncertainty. The rankings are presented in Fig. 2, a bar chart created using MATLAB.

Table 3. The ranking results of four MCDM approaches.

Alternative	Method 1	Rank	Method 2	Rank	Method 3	Rank	Method 4	Rank
$\mathbb{A}_1$	0.7940	II	0.8402	II	0.6814	II	0.7249	II
$\mathbb{A}_2$	0.6581	III	0.5958	III	0.5020	III	0.4530	III
$\mathbb{A}_3$	0.9468	I	0.9426	I	0.8065	I	0.7982	I
$\mathbb{A}_4$	0.4274	IV	0.5470	IV	0.0847	IV	0.3017	IV

Discussion on ranking consistency

All four aggregation operators (MVRNAMO, WMVRNAMO, MVRNGMO, WMVRNGMO) produce the same ranking:

$$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4.$$

This consistency is due to the strong performance of the Multi-Speciality Hospital ($\mathbb{A}_3$) across all criteria, making the results robust to different aggregation methods. Although weighted operators use Shapley-based weights, they don't change the final ranking in this case. This shows that when performance differences are large, results remain stable, while in closer cases, the choice of operator and weights may influence the ranking.

Software used for calculations

All numerical computations, from MVRNN aggregation to score and accuracy calculations, and the final ranking, were performed using MATLAB. This software offered a robust platform for executing intricate mathematical operations and confirming the outcomes.

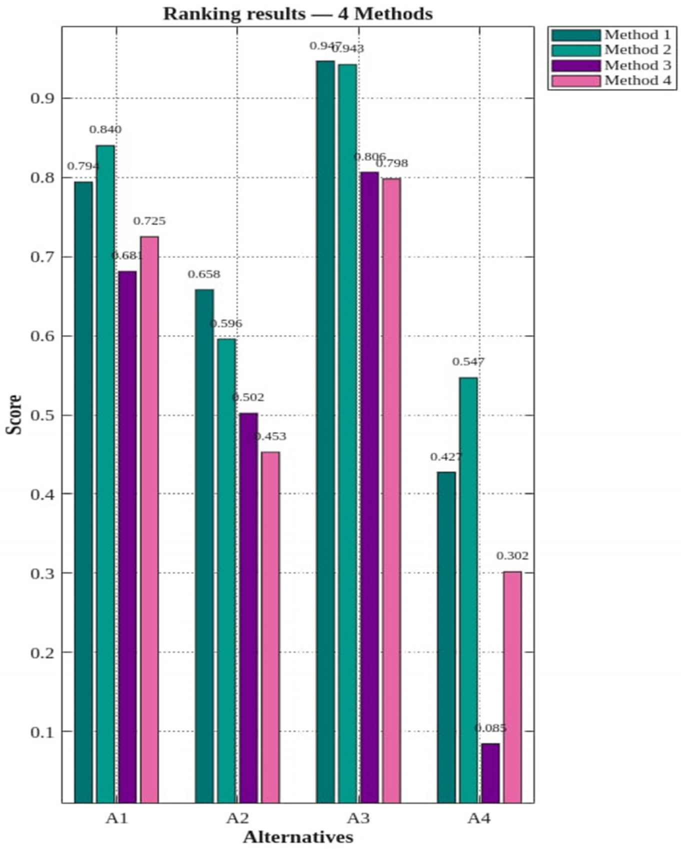

Fig. 2. MATLAB-generated bar chart showing the rankings of the hospital alternatives.

Comparative analysis

To rigorously validate the effectiveness and robustness of our proposed MVRNN-based aggregation operators, we conduct a comprehensive comparative analysis with several prominent MCDM methods from the existing literature. Specifically, we compare our framework against:

1. An Aczel-Alsina aggregation-based outranking method for single-valued neutrosophic numbers (SVNNs) by Senapati et al.,[57] including SVNAAWA, SVNAAOWA, and SVNAAHA operators.
2. Single-Valued Neutrosophic Dombi Weighted Averaging Aggregation Operators (SVNDWAA and SVND-WGA) by Chen and Ye.[14]
3. Dual Generalized Single-Valued Neutrosophic Bonferroni Mean (DGSNNWBM) and Dual Hesitant Single-Valued Neutrosophic Weighted Geometric Bonferroni Mean (DHSVNNWGBM) Operators by Wang et al.[58]
4. Neutrosophic Hesitant Fuzzy Heronian Mean Aggregation Operators (NHFIGWHM and NHFIGGWHM) by Liu and Zhang.[59]

Data transformation and method application

To facilitate direct comparison, our initial MVRNN decision matrix Table 2 is transformed into formats compatible with each comparative method. The transformation process is as follows:

1. MVRNN to MVNN Conversion: First, the MVRNNs are converted into MVNNs using the accumulated geometric operator as proposed by Martina DJ, Deepa G.[2] This involves combining the lower and upper approximations of each MVRNN element into a single set of multi-values.

2. MVNN to SVNN Conversion: For Senapati's,[57] Chen and Ye's,[14] and Wang et al.'s[58] methods (all of which operate on SVNNs), these MVNNs are further converted into SVNNs by taking the mean of their multi-values within each truth, indeterminacy, and falsity component.
3. MVNN to Neutrosophic Hesitant Fuzzy Set (NHFS) Conversion: For Liu and Zhang's[59] methods, these MVNNs are converted into NHFSs.

The criteria weights ($w_1 = 0.364, w_2 = 0.223, w_3 = 0.413$) derived from the Shapley fuzzy measure are consistently applied where applicable.

Comparative ranking results

The final scores and ranking of alternatives obtained from applying the comparative methods, alongside our proposed MVRNN-based framework, are summarized in Table 4.

Table 4. Comparative ranking results of proposed and existing methods.

Method	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$	Ranking
SVNAAWA Operator[57]	0.8159	0.7615	0.8746	0.7298	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
SVNAAOWA Operator[57]	0.7588	0.7120	0.8725	0.6609	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
SVNDWAA Operator[14]	0.8243	0.7765	0.9008	0.7556	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
SVNDWGA Operator[14]	0.7642	0.6971	0.8093	0.5401	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
DGSVNNWBM Operator[58]	0.7653	0.7039	0.8112	0.6015	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
DGSVNNWGBM Operator[58]	0.8020	0.7492	0.8364	0.6773	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
NHFIGWHM Operator[59]	0.8175	0.7522	0.8682	0.7257	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
NHFIGGWHM Operator[59]	0.6988	0.6707	0.7565	0.6353	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
(WMVRNAMO) Proposed Operator	0.8402	0.5958	0.9426	0.5470	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$
(WMVRNGMO) Proposed Operator	0.7249	0.4530	0.7982	0.3017	$\mathbb{A}_3 \succ \mathbb{A}_1 \succ \mathbb{A}_2 \succ \mathbb{A}_4$

Discussion

The comparative analysis reveals a remarkable consistency in the final ranking of alternatives across all investigated methods, both our proposed MVRNN-based framework and the four prominent existing approaches (Senapati's SVNAAWA,[57] Chen and Ye's SVNDWAA,[14] Wang et al.'s DGSNNWBM,[58] and Liu and Zhang's NHFIGWHM.[59]) In every instance, Alternative $\mathbb{A}_3$ (Multi-Speciality Hospital) consistently emerged as the top-ranked choice, followed by $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_4$.

In spite of this consistency, the MVRN-based framework presents distinct benefits when modeling the intricacies of real-world scenarios. Unlike SVNN-based methodologies, which tend to simplify expert assessments, the proposed framework inherently accommodates multi-valued information, rough approximations, and the complete range of truth, indeterminacy, and falsity. The MVRNN structure, by directly incorporating multiple, potentially conflicting expert evaluations, facilitates a more detailed and realistic evaluation. Although the ultimate ranking remained unchanged in this instance, the framework offers a comprehensive and theoretically sound approach, adept at managing vagueness and uncertainty while yielding more profound insights into the decision-making process.

Conclusion

This paper proposes a new MCDM framework based on MVRNNs to handle decision-making problems with uncertainty, vagueness, and inconsistency. Four new aggregation operators (MVRNAMO, WMVRNAMO, MVRNGMO, WMVRNGMO) are introduced with proven properties such as idempotency, boundedness, and monotonicity. The Shapley fuzzy measure is used for effective criteria weighting, and new score and accuracy functions are defined for ranking alternatives. A hospital selection case study demonstrates the effectiveness of the model, where the Multi-Speciality Hospital consistently ranks as the best alternative, confirming the stability and reliability of the proposed approach.

Advantages, limitations, and future work

The proposed framework effectively handles different types of uncertainty, such as vagueness, indeterminacy, and rough information, while providing flexible and consistent rankings through various aggregation operators.

The use of Shapley fuzzy measures enhances the accurate evaluation of the importance of criteria. However, the framework may face challenges with large datasets and situations involving highly conflicting expert opinions. Future work can focus on improving computational efficiency, applying the method to diverse real-world problems such as energy, engineering, and environmental management, and developing automated and real-time decision-making tools.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images that are not ours have been included with the necessary permission for republication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Vellore Institute of Technology, Vellore, India.

Authors' contribution statement

K.G. and G.D. were responsible for the design and implementation of the research. K.G. performed drafting the manuscript, conducted the analysis, acquired data, interpreted the results, and contributed to the writing of the manuscript. G.D. did the revision and interpretation. All authors read and agreed upon the published version of the manuscript.

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Appendix A. List of abbreviations

- MCDM: Multi-Criteria Decision-Making
- MVRNNs: Multi-Valued Rough Neutrosophic Numbers
- MVRNAMO: Multi-Valued Rough Neutrosophic Arithmetic Mean Operator
- WMVRNAMO: Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator
- MVRNGMO: Multi-Valued Rough Neutrosophic Geometric Mean Operator
- WMVRNGMO: Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator
- $\mathbb{A}_{\mathbb{MVRN}}$: Multi-Valued Rough Neutrosophic Arithmetic Mean Operator (in equations)
- $\mathbb{A}^{w}_{\mathbb{MVRN}}$: Weighted Multi-Valued Rough Neutrosophic Arithmetic Mean Operator (in equations)
- $\mathbb{G}_{\mathbb{MVRN}}$: Multi-Valued Rough Neutrosophic Geometric Mean Operator (in equations)
- $\mathbb{G}^{w}_{\mathbb{MVRN}}$: Weighted Multi-Valued Rough Neutrosophic Geometric Mean Operator (in equations)
- HCNS: Hesitant Complex Neutrosophic Sets
- NS: Neutrosophic Set
- SVNNs: Single-Valued Neutrosophic Numbers
- IVNNs: Interval-valued Neutrosophic Numbers
- MVNNs: Multi-valued Neutrosophic Numbers
- MVRNNs: Multi-valued Rough Neutrosophic Numbers

نهج محايد تقريبي متعدد القيم لاتخاذ القرارات متعددة المعايير: تطبيق في اختيار المستشفيات

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الخلاصة

تُوصَف مشكلاتُ اتخاذ القرار غالبًا بالتعقيد وعدم اليقين، مما يطرح تحدياتٍ كبيرة أمام الأفراد أو المنظمات. وغالبًا ما تعجز طرق التجميع التقليدية عنِ التعامل مع البيانات غير الدقيقة والمتناقضة وغير الواضحة التي تميّز هذه الحالات. ولمعالجة هذه الإشكاليات، تقترح هذه الدراسة طريقةً جديدة لاتخاذ القرار متعدد المعايير بالاعتماد على الأعداد النيوتروسوفية الخشنة متعددة القيم. في البداية، نقدّم القوانين التشغيلية الأساسية للأعداد النيوتروسوفية الخشنة متعددة القيم، استنادًا إلى معياري t-norm وt-conorm. بعد ذلك، نطوّر أربعة معاملات تجميع جديدة قائمة على هذه الأعداد، وهي: معامل المتوسط الحسابي للأعداد النيوتروسوفية الخشنة متعددة القيم، ومعامل المتوسط الحسابي الموزون لها، ومعامل المتوسط الهندسي لها، ومعامل المتوسط الهندسي الموزون.

الكلمات المفتاحية: معاملات التجميع، اختيار المستشفيات، اتخاذ القرار متعدد المعايير، الأعداد النيوتروسوفية الخشنة متعددة القيم، المجموعة النيوتروسوفية الخشنة، مقياس شابلي الضبابي.