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RESEARCH ARTICLE

Solvability for Couple Nonlinear Elliptic Partial Differential Equations

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ABSTRACT

As it famous the solvability of any mathematical problem play an important rule for many investigators whose interested about the numerical solution for such mathematical problems because it gives them the green light about the ability to solve such problems numerically, besides to give the green light for solving numerically many real life problems in various field of sciences; like the physics, medicine, astronomy, chemistry, biology, different branch of engineering and many other fields. For this reason, this work is devoted to studying the solvability (in infinite dimension) of a couple of nonlinear elliptic partial differential equations (CNLEPDEs) with four different types of boundary conditions (BCs): Dirichlet (DBC), Neumann (NBCs), Robin (RBCs), and Mixed BCs (MBCs). The weak formulation (WF) for each problem is determined by the type of each B. The existence and the uniqueness of the solution for the proposed problem (CNLEPDEs) with each type of BCs is proved by employing the Minty-Browder theorem (MBTH), through using the Poincaré-Friedrichs inequality of (PFI) in the case of homogeneous BCs (HBCs) for each type, and through using the generalization of PFI (GPFI) and the trace Operator (TRO) in the case of nonhomogeneous BCs(NHBCs) for each type.

Keywords: Couple nonlinear elliptic partial differential equations, Dirichlet conditions, Mixed boundary conditions, Neumann condition, Robin condition

Introduction

Many real life problems in various and different kinds of sciences like fluid physics, vibrations of solids, quantum mechanics, the flow of fluids, the spread of heat, the structure of molecules, the interactions of photons and electrons, and the radiation of electromagnetic waves, the description for perturbations of the pressure and radius in arterial blood flow¹⁻⁴ are described as mathematical models which are usually represented by ODEs or PDEs.

In addition, differential equations play an important role in modern mathematics and its analysis, qualitative theory, finance, and, more recently, machine learning (ML).^{5,6} Hence, the numerical solution for PDEs became a fundamental task for many investigators to solve different complex mathematical problem⁷ and complex physical phenomena like wave propagation, diffusion, heat transfer, and the interaction of electromagnetic radiation and mechanical waves with the Earth's surface,⁸⁻¹⁰ besides computing the mechanical wave modeling, electromagnetic.¹¹ The study of the solvability of PDEs serves as a cornerstone of the study of their numerical solutions; it plays a significant role in formulating and implementing numerical methods and ensures convergence to the solution. Also, it directly influences the stability of the numerical method and helps avoid unreliable solutions by emphasizing the necessity of well-posed problems.

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As a result, for the above studies, this paper is devoted to studying the solvability (in infinite dimension) of the proposed CNLEPDEs based on MBTH with different types of boundary conditions (BCs), Dirichlet (DBC), Neumann(NBCs), Robin(RBCs), and Mixed BCs(MBCs). The WF of the proposed CNLEPDEs with each of the four types of BCs was found at first. Then secondly, the existence and uniqueness of the solution to each WF is demonstrated by using the MBTH¹² under suitable assumptions with help of the (PFI) in the case of homogenous BCs (HBCs) for each type and through using the generalization PFI (GPFI) and the trace Operator (TRO) in the case of nonhomogeneous BCs(NHBCs) for each type, moreover Cauchy Schwarz Inequality(CSI) utilized in some steps. In fact, such mathematical problems have not been studied before, neither in terms of solvability nor in terms of their solution. Therefore, we found it important to delve into the study of the solvability of such problems. In addition, this study is very important, since it has a great influence on ensuring the solvability of interesting and complex mathematical problems in,¹³ and many other natural problems besides, to give the green light for many investigators to solve such types of problems numerically, like.¹⁴ Finally, although many mathematical techniques were used in the proposed method, we can say it is more suitable than other methods like the variational method, because the variational method requires requirements like symmetry with respect to the bilinear form and the nondegenerate property, which may not be satisfied if we try to use it here.

Basic concepts

This section includes some theorems and lemmas that are of interest to our work.

Definition 1¹⁵: Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$, the Sobolev Space $W^{1,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ whose distributional first-order partial derivatives belong to $L^p(\Omega)$; that is, for all $i = 1, 2, \dots, N$ there exists a function $g_i \in L^p(\Omega)$, such that $\int_{\Omega} u \frac{\partial \theta}{\partial x_i} dx = - \int_{\Omega} g_i \theta dx$, for all $\theta \in C_c^\infty(\Omega)$. The function g_i is called the weak, or distributional, partial derivatives of u with respect to x_i and is denoted $\frac{\partial u}{\partial x_i}$.

Theorem 1 (MBTH)¹²: Let V be a reflexive Banach space, and $A : V \rightarrow V^*$ be a continuous nonlinear map s.t.

$$\langle Ay_1 - Ay_2, y_1 - y_2 \rangle > 0, \forall y_1, y_2 \in V, y_1 \neq y_2 \text{ and } \lim_{\|y\|_{H_0^1(\Omega)} \rightarrow \infty} \frac{\langle Ay, y \rangle}{\|y\|_{H_0^1(\Omega)}} = \infty.$$

Then for every $f \in V^*$, there exists a unique solution $y \in V$ of the equation $Ay = f$.

Theorem 2¹⁶: If x and y are two arbitrary vectors in an inner product space, then $|(x, y)| \leq \|x\| \|y\|$.

Definition 2¹⁷: Let $p \geq 1$ be a real number, a function f defined on a subset Ω of \mathbb{R}^n is said to belong to $L^p(\Omega)$ if f is measurable, and if the integral $\int_{\Omega} |f(x)|^p dx$ exists. If we let $p \rightarrow \infty$, then the space $L^\infty(\Omega)$ of all measurable functions on Ω that are bounded almost everywhere on Ω :

$$L^\infty(\Omega) = \{f : |f(x)| \leq k, \text{ a.e. on } \Omega \text{ for some } k \geq 0\}$$

Remark 1¹⁶: Clearly, for a bounded domain Ω , $L^\infty(\Omega)$ is a subset of $L^p(\Omega)$ for all $p \geq 1$, since any $f \in L^\infty(\Omega)$ satisfies $\int_{\Omega} |f(x)|^p dx \leq \int_{\Omega} k^p dx < \infty$, so that $f \in L^p(\Omega)$ also.

Lemma 1 (PFI)¹⁸: For any Lipschitz bounded domain Ω , there is a real positive constant $c(\Omega) > 0$, which depends only on the domain Ω , s.t.

$$\int_{\Omega} |y|^2 dx \leq c(\Omega) \int_{\Omega} |\nabla y|^2 dx, \forall y \in H_0^1(\Omega).$$

Lemma 2 (GPFI)¹⁸: Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain, and let $\Gamma_1 \subset \Gamma$ be a measurable set such that $|\Gamma_1| > 0$. Then there exists a real constant. $c(\Gamma_1) > 0$, which is independent of $y \in H^1(\Omega)$, such that

$$\|y\|_{H^1(\Omega)}^2 \leq c(\Gamma_1) \left(\int_{\Omega} |\nabla y|^2 dx + \left(\int_{\Gamma_1} y ds \right)^2 \right), \text{ for all } y \in H^1(\Omega).$$

Proposition 1¹⁹: Let $f : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of Carathéodory type, let F be a functional, such that $F(y) = \int_{\Omega} f(x, y(x)) dx$, where Ω is a measurable subset of \mathbb{R}^n , and suppose that

$$\|f(x, y)\| \leq \zeta(x) + \eta(x) \|y\|^\alpha, \quad \forall x \in \Omega, y \in L^p(\Omega \times \mathbb{R}^n),$$

where $\zeta \in L^1(\Omega \times \mathbb{R})$, $\eta \in L^{\frac{p}{p-\alpha}}(\Omega \times \mathbb{R})$, and $\alpha \in [1, P]$, if $P \in [1, \infty)$, and $\eta \equiv 0$, if $P = \infty$. Then F is continuous on $L^p(\Omega \times \mathbb{R}^n)$.

Theorem 3 (TRO)²⁰: Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain, then there exists a linear operator (TRO) $T_0 : H^1(\Omega) \rightarrow L^2(\Gamma)$, s.t.

$T_0u = u|_{\Gamma}$ if $u \in C^\infty(\bar{\Omega})$, and $\|T_0u\|_{L^2(\Gamma)} \leq c_r \|u\|_{H^1(\Omega)}$

For a suitable real positive constant $c_r = c_r(\Omega, d)$; T_0u is called the trace of u on Γ .

Description of the proposed problem

Let $\Omega \subset \mathbb{R}^2$, be a Lipschitz bounded regular domain with boundary $\Gamma = \partial\Omega$, consider the following CNLEPDEs with the HDBC's:

$$A_1y_1 + B_1y_1 + a_{01}(x)y_1 - b(x)y_2 + f_1(x, y_1) = h_1(x), \text{ in } \Omega \tag{1}$$

$$A_2y_2 + B_2y_2 + a_{02}(x)y_2 + b(x)y_1 + f_2(x, y_2) = h_2(x), \text{ in } \Omega \tag{2}$$

$$y_1 = 0, \text{ on } \Gamma \tag{3}$$

$$y_2 = 0, \text{ on } \Gamma \tag{4}$$

where $A_l y_l = -\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} (a_{lij}(x) \frac{\partial y_l}{\partial x_j})$, $B_l y_l = \sum_{i=1}^2 b_{li}(x) \frac{\partial y_l}{\partial x_i}$, $l = 1, 2$, $X = (x_1, x_2) \in \Omega$, $\vec{y} = (y_1, y_2) \in (H_0^2(\Omega))^2$, $a_{lij}, b_{li}, a_{0l}, b \in L^\infty(\Omega)$, $a_{lij} \geq \alpha_{lij} > 0$, $b_{li} \geq \beta_{lij} > 0$, $a_{0l} \geq 0$, $b \geq 0$, and the functions $f_l(x, y_l) \in L^2(\Omega)$, $(h_l(x) \in L^2(\Omega))$ for $l = 1, 2$ are of Carathéodory type on $\Omega \times \mathbb{R}$ (on Ω).

The WF of the proposed system

The WF of Eqs. (1) to (4) is obtained by integrating both sides of Eqs. (1) to (4) after multiplying them by the test function. $v_l \in H_0^1(\Omega) = V, (l = 1, 2)$, resp., and then applying the generalized Green's theorem for the terms which have the derivative of 2nd order, once we get

$$\int_{\Omega} \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i} \frac{\partial v_1}{\partial x_j} dx + \int_{\Omega} \sum_{i=1}^2 b_{1i}(x) \frac{\partial y_1}{\partial x_i} v_1 dx + \int_{\Omega} a_{01}(x)y_1 v_1 dx - \int_{\Omega} b(x)y_2 v_1 dx + \int_{\Omega} f_1(x, y_1)v_1 dx = \int_{\Omega} h_1 v_1 dx, \quad \forall v_1 \in H_0^1(\Omega) \tag{5}$$

and

$$\int_{\Omega} \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i} \frac{\partial v_2}{\partial x_j} dx + \int_{\Omega} \sum_{i=1}^2 b_{2i}(x) \frac{\partial y_2}{\partial x_i} v_2 dx + \int_{\Omega} a_{02}(x)y_2 v_2 dx + \int_{\Omega} b(x)y_1 v_2 dx + \int_{\Omega} f_2(x, y_2)v_2 dx = \int_{\Omega} h_2 v_2 dx, \quad \forall v_2 \in H_0^1(\Omega) \tag{6}$$

Equalities Eq. (5) and Eq. (6) can also be expressed as

$$a_1(y_1, v_1) + b_1(y_1, v_1) + (a_{01}y_1, v_1)_{\Omega} - (by_2, v_1)_{\Omega} + (f_1(x, y_1), v_1)_{\Omega} = (h_1(x), v_1)_{\Omega} \tag{7}$$

$$a_2(y_2, v_2) + b_2(y_2, v_2) + (a_{02}y_2, v_2)_{\Omega} + (by_1, v_2)_{\Omega} + (f_2(x, y_2), v_2)_{\Omega} = (h_2(x), v_2)_{\Omega} \tag{8}$$

Collecting Eq. (7) and Eq. (8)

$$a_1(y_1, v_1) + b_1(y_1, v_1) + (a_{01}y_1, v_1)_{\Omega} - (by_2, v_1)_{\Omega} + (f_1(x, y_1), v_1)_{\Omega} + a_2(y_2, v_2) + b_2(y_2, v_2) + (a_{02}y_2, v_2)_{\Omega} + (by_1, v_2)_{\Omega} + (f_2(x, y_2), v_2)_{\Omega} = (h_1(x), v_1)_{\Omega} + (h_2(x), v_2)_{\Omega} \tag{9}$$

Or

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = \langle \vec{F}(x), \vec{v} \rangle \tag{10}$$

where $\bar{A} : \bar{V} \rightarrow \bar{V}^*$, with $\bar{V} = V \times V$ and \bar{V}^* is the dual of \bar{V} , s.t.

$$\langle \bar{A}(\bar{y}), \bar{v} \rangle = a(\bar{y}, \bar{v}) + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), v_2)_\Omega \quad (11)$$

with

$$a(\bar{y}, \bar{v}) = a_1(y_1, v_1) + b_1(y_1, v_1) + (a_{01}y_1, v_1)_\Omega - (by_2, v_1)_\Omega + a_2(y_2, v_2) + b_2(y_2, v_2) + (a_{02}y_2, v_2)_\Omega + (by_1, v_2)_\Omega \quad (12)$$

$$a_l(y_l, v_l) = \sum_{i,j=1}^2 a_{lij} \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j}, b_l(y_l, v_l) = \sum_{i=1}^2 b_{li}(x) \frac{\partial y_l}{\partial x_i} v_l, l = 1, 2 \quad (13)$$

And $\bar{F}(x) = ((h_1(x), h_2(x)))$, with

$$\langle \bar{F}(x), \bar{v} \rangle = (h_1(x), v_1)_\Omega + (h_2(x), v_2)_\Omega \quad (14)$$

Assumptions 1: The functions $f_l(x, y_l)$ for $l = 1, 2$ are monotone w.r.t. y_l for each $x \in \Omega$ and satisfied

- i) $|f_l(x, y_l)| \leq \phi_l(x) + \bar{c}_l |y_l|, \quad \forall (x, y_l) \in \Omega \times \mathbb{R}$ with $\phi_l \in L^2(\Omega), \bar{c}_l \in \mathbb{R}^+$.
- ii) $f_l(x, 0) = 0, \forall x \in \Omega$.

Main results

Solvability of the CNLEPDEs

The following theorem deals with the solvability of WF (Eq. (10)) with HDBC

Theorem 4: In addition to Assumptions 1, if “at least.” f_1 is strictly monotone, then the WF(10) has a unique couple solution(COS) $\bar{y} = (y_1, y_2) \in (H_0^1(\Omega))^2$.

Proof: The proof will be upon using Theorem 1, and it starts by checking the coercivity of the $\langle \bar{A}(\bar{y}), \bar{y} \rangle$ in the LHS of Eq. (10), as

$$\langle \bar{A}(\bar{y}), \bar{y} \rangle = a_1(y_1, y_1) + b_1(y_1, y_1) + (a_{01}y_1, y_1)_\Omega - (by_2, y_1)_\Omega + (f_1(x, y_1), v_1)_\Omega + a_2(y_2, y_2) + b_2(y_2, y_2) + (a_{02}y_2, y_2)_\Omega + (by_1, y_2)_\Omega + (f_2(x, y_2), y_2)_\Omega \quad (15)$$

$$\langle \bar{A}(\bar{y}), \bar{y} \rangle = \int_\Omega \sum_{i,j=1}^2 a_{1ij}(x) \frac{\partial y_1}{\partial x_i} \frac{\partial y_1}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{1i}(x) \frac{\partial y_1}{\partial x_i} y_1 dx + \int_\Omega a_{01}(x) y_1^2 dx + \int_\Omega f_1(x, y_1) y_1 dx + \int_\Omega \sum_{i,j=1}^2 a_{2ij}(x) \frac{\partial y_2}{\partial x_i} \frac{\partial y_2}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{2i}(x) \frac{\partial y_2}{\partial x_i} y_2 dx + \int_\Omega a_{02}(x) y_2^2 dx + \int_\Omega f_2(x, y_2) y_2 dx$$

Using Assumptions 1, with $a_{lij} \geq \alpha_{lij} > 0, (\forall l, i, j = 1, 2)$, and with setting $\frac{\partial y_l}{\partial x_i} y_l = \frac{1}{2} \frac{\partial}{\partial x_i} (y_l^2)$, in the second and the sixth terms, i.e.

$$\langle \bar{A}(\bar{y}), \bar{y} \rangle \geq \alpha_{1ij} \sum_{i=1}^2 \int_\Omega \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx + \sum_{i=1}^2 \int_\Omega b_{1i}(x) \frac{1}{2} \frac{\partial}{\partial x_i} (y_1^2) dx + \int_\Omega a_{01}(x) |y_1|^2 dx + \alpha_{2ij} \sum_{i=1}^2 \int_\Omega \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx + \sum_{i=1}^2 \int_\Omega b_{2i}(x) \frac{1}{2} \frac{\partial}{\partial x_i} (y_2^2) dx + \int_\Omega a_{02}(x) |y_2|^2 dx + \int_\Omega (f_1(x, y_1) - f_1(x, 0))(y_1 - 0) dx + \int_\Omega (f_2(x, y_2) - f_2(x, 0))(y_2 - 0) dx \quad (16)$$

Since f_l is monotone w.r.t. y_l , let $\alpha = \min(\alpha_{li}), \forall l, i, j = 1, 2$, and using integration by parts in the second and fifth terms on the RHS of Eq. (16), to get

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle \geq \alpha \left(\sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx + \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx \right) + \int_{\Omega} \left(a_{01}(x) - \frac{1}{2} \sum_{i=1}^2 \frac{\partial b_{1i}}{\partial x_i} \right) |y_1|^2 dx + \int_{\Omega} \left(a_{02}(x) - \frac{1}{2} \sum_{i=1}^2 \frac{\partial b_{2i}}{\partial x_i} \right) |y_2|^2 dx$$

Choose b_{li} and a_{0l} , for $l, i = 1, 2$, for which the following is held

$$a_{0l}(x) - \frac{1}{2} \sum_{i=1}^2 \frac{\partial b_{li}}{\partial x_i} \geq 0, \quad x \in \bar{\Omega}, \tag{17}$$

which gives

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle \geq \alpha \sum_{i=1}^2 \|\Delta y_i\|_{L^2(\Omega)}^2 \tag{18}$$

Then, by virtue of Lemma 1, to get

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle \geq \frac{\alpha}{C_*} \sum_{i=1}^2 \|y_i\|_{L^2(\Omega)}^2 \tag{19}$$

Summing Eq. (18) and Eq. (19), and letting $C_0 = \min\left(\alpha, \frac{\alpha}{C_*}\right)$, one has

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle \geq C_0 \sum_{i=1}^2 (\|y_i\|_{L^2(\Omega)}^2 + \|\Delta y_i\|_{L^2(\Omega)}^2) = C_0 \sum_{i=1}^2 \|y_i\|_{H_0^1}^2.$$

Hence

$$\bar{A}(\vec{y}), \vec{y} \geq C_0 \| \vec{y} \|_{(H_0^1(\Omega))^2}^2 \tag{20}$$

$$\Rightarrow \lim_{\| \vec{y} \|_{(H_0^1(\Omega))^2} \rightarrow \infty} \frac{\langle \bar{A}(\vec{y}), \vec{y} \rangle}{\| \vec{y} \|_{(H_0^1(\Omega))^2}^2} \rightarrow \infty$$

This means that \bar{A} is coercive.

Also, since $a_{ij}(x), b_{li}(x), a_{0l}(x), b(x) \in L^\infty(\Omega), (\forall l, i = 1, 2)$, it flows

$$\left| \int_{\Omega} \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j} dx \right| \leq \sum_{i,j=1}^2 \mu_l \left| \int_{\Omega} \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j} dx \right|, \quad \text{with } \mu_l = \max_{x \in \bar{\Omega}} |a_{ij}(x)| \tag{21}$$

$$\left| \int_{\Omega} \sum_{i=1}^2 b_{li}(x) \frac{\partial y_l}{\partial x_i} v_l dx \right| \leq \sum_{i=1}^2 \bar{\mu}_l \left| \int_{\Omega} \frac{\partial y_l}{\partial x_i} v_l dx \right|, \quad \text{with } \bar{\mu}_l = \max_{x \in \bar{\Omega}} |b_{li}(x)| \tag{22}$$

$$\left| \int_{\Omega} a_{0l}(x) y_l v_l dx \right| \leq \bar{\bar{\mu}}_l \left| \int_{\Omega} y_l v_l dx \right|, \quad \text{with } \bar{\bar{\mu}}_l = \max_{x \in \bar{\Omega}} |a_{0l}(x)| \tag{23}$$

$$\left| \int_{\Omega} b(x) y_{3-l} v_l dx \right| \leq \bar{\bar{\bar{\mu}}}_l \left| \int_{\Omega} y_{3-l} v_l dx \right|, \quad \text{with } \bar{\bar{\bar{\mu}}}_l = \max_{x \in \bar{\Omega}} |b(x)| \tag{24}$$

Then, by inequalities Eqs. (21) to (24)

$$|a(\vec{y}, \vec{v})| \leq \sum_{l=1}^2 \sum_{i,j=1}^2 \mu_l \left| \int_{\Omega} \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j} dx \right| + \sum_{l=1}^2 \sum_{i,j=1}^2 \bar{\mu}_l \left| \int_{\Omega} \frac{\partial y_l}{\partial x_i} v_l dx \right| \\ + \sum_{l=1}^2 \bar{\bar{\mu}}_l \left| \int_{\Omega} y_l v_l dx \right| + \sum_{l=1}^2 \bar{\bar{\bar{\mu}}}_l \left| \int_{\Omega} y_{3-l} v_l dx \right|$$

And by Theorem 2, one gets

$$|a(\vec{y}, \vec{v})| \leq \mu \left\{ \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{\partial v_1}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} + \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{\partial v_2}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} \right. \\ \left. + \sum_{i=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \sum_{i=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \right. \\ \left. + \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \right. \\ \left. + \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \right\}, \mu = \max \{ \mu_l, \bar{\mu}_l, \bar{\bar{\mu}}_l, \bar{\bar{\bar{\mu}}}_l \}, \\ \forall l, i = 1, 2$$

$$|a(\vec{y}, \vec{v})| \leq \mu \left(\left\{ \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} + \sum_{j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_1}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial v_1}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \right\} \right. \\ \left. + \left\{ \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} + \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_2}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} + \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial v_2}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \right\} \right. \\ \left. + \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \right) \quad (25)$$

The RHS of 25 can be rewritten as

$$|a(\vec{y}, \vec{v})| \leq 4\mu \left\{ \int_{\Omega} |y_1|^2 dx + \sum_{j=1}^2 \int_{\Omega} \left| \frac{\partial y_1}{\partial x_j} \right|^2 dx \right\}^{\frac{1}{2}} \times \left\{ \int_{\Omega} |v_1|^2 dx + \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial v_1}{\partial x_i} \right|^2 dx \right\}^{\frac{1}{2}} \\ + 4\mu \left\{ \int_{\Omega} |y_2|^2 dx + \sum_{j=1}^2 \int_{\Omega} \left| \frac{\partial y_2}{\partial x_j} \right|^2 dx \right\}^{\frac{1}{2}} \times \left\{ \int_{\Omega} |v_2|^2 dx + \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial v_2}{\partial x_i} \right|^2 dx \right\}^{\frac{1}{2}} \\ + \mu \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \mu \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}}$$

$$|a(\vec{y}, \vec{v})| \leq 4\mu \left[\|y_1\|_{H_0^1(\Omega)} \|v_1\|_{H_0^1(\Omega)} + \|y_2\|_{H_0^1(\Omega)} \|v_2\|_{H_0^1(\Omega)} \right] + \mu \|y_2\|_{H_0^1(\Omega)} \|v_1\|_{H_0^1(\Omega)} \\ + \mu \|y_1\|_{H_0^1(\Omega)} \|v_2\|_{H_0^1(\Omega)}$$

So, by letting $\rho = 10\mu$ one concludes that

$$|a(\vec{y}, \vec{v})| \leq \rho \|\vec{y}\|_{(H_0^1(\Omega))^2} \|\vec{v}\|_{(H_0^1(\Omega))^2} \tag{26}$$

Besides this result, let $v_l \in C(\bar{\Omega})$ (for $l = 1, 2$), then v_l is measurable w.r.t. x . Let $F_l(y_l) = \int_{\Omega} \bar{f}_l(x, y_l(x)) dx$, where $\bar{f}_l(x, y_l) = f_l(x, y_l)v_l$, then $\bar{f}_l : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ with $y_l \in L^2(\Omega)$ is of Carathéodory type, and

$$\begin{aligned} \|\bar{f}_l(x, y_l(x))\| &\leq \phi_l(x)|v_l| + \bar{c}_l|y_l||v_l|, \text{ with } \phi_l \in L^1(\Omega), \bar{c}_l \in L^\infty(\Omega) \\ &\leq \frac{1}{2}(\phi_l^2 + (1 + \bar{c}_l)|v_l|^2) + \bar{c}_l\|y_l\|^2, \end{aligned} \tag{27}$$

where $\tilde{c}_l = \frac{1}{2}\bar{c}_l$, with $\frac{1}{2}(\phi_l^2 + (1 + \bar{c}_l)|v_l|^2) \in L^1(\Omega)$, $\tilde{c}_l \in L^\infty(\Omega)$.

By utilizing Proposition 1, the continuity of $\int_{\Omega} \bar{f}_l(x, y_l(x))dx$ w.r.t. y_l (for $l = 1, 2$), is obtained.

From Eq. (26) and Eq. (27), the continuity of $\vec{y} \mapsto \langle \bar{A}(\vec{y}), \vec{v} \rangle$ w.r.t. \vec{y} , is obtained.

It is clear that, $F(\vec{v})$ in the RHS of Eq. (10) is a linear functional on $(H_0^1(\Omega))^2$, it remains to show that it is bounded, so by the CSI

$$\begin{aligned} |F(\vec{v})| &= \sum_{l=1}^2 [|(h_l, v_l)_{\Omega}|] \\ &\leq \sum_{l=1}^2 [\|h_l(x)\|_{L^2(\Omega)} \|v_l\|_{L^2(\Omega)}] \\ &\leq \sum_{l=1}^2 [\gamma_l \|v_l\|_{L^2(\Omega)}], \|h_l(x)\|_{L^2(\Omega)} \leq \gamma_l, \gamma = \max(\gamma_l), \forall l = 1, 2 \\ &\leq \gamma \|\vec{v}\|_{(L^2(\Omega))^2} \leq \gamma \|\vec{v}\|_{(H_0^1(\Omega))^2}. \end{aligned} \tag{28}$$

Since f_1 is strictly monotone, let $\vec{v} = \vec{y} - \vec{y}$, then from 11 one has

$$\langle \bar{A}(\vec{y}), \vec{y} - \vec{y} \rangle = a(\vec{y}, \vec{y} - \vec{y}) + (f_1(x, y_1), y_1 - \bar{y}_1)_{\Omega} + (f_2(x, y_2), y_2 - \bar{y}_2)_{\Omega} \tag{29}$$

$$\langle \bar{A}(\vec{y}), \vec{y} - \vec{y} \rangle = a(\vec{y}, \vec{y} - \vec{y}) + (f_1(x, \bar{y}_1), y_1 - \bar{y}_1)_{\Omega} + (f_2(x, \bar{y}_2), y_2 - \bar{y}_2)_{\Omega} \tag{30}$$

Subtracting 30 from 29, to get

$$\langle \bar{A}(\vec{y}) - \bar{A}(\vec{y}), \vec{y} - \vec{y} \rangle > c\|\vec{y} - \vec{y}\|_{(H_0^1(\Omega))^2}^2 > 0, \text{ for } \vec{y} \neq \vec{y}, \text{ and } c \in \mathbb{R}^+. \tag{31}$$

So, from the above steps, all the hypotheses of Theorem 1 are obtained, hence there exists a unique COS $\vec{y} = (y_1, y_2) \in (H_0^1(\Omega))^2$ of the WF (10).

Corollary 1: Consider the proposed CNLEPDEs (Eqs. (1) and (2)), but with the following nonhomogeneous Dirichlet BCs(NHDBC)

$$y_1 = g_1, \text{ on } \Gamma \tag{32}$$

$$y_2 = g_2, \text{ on } \Gamma. \tag{33}$$

Has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$, where $\vec{g} = (g_1, g_2) \in (L^2(\Gamma))^2$.

Proof: The WF, which results from Eqs. (1) and (2) with the NHNBCs Eqs. (32) and (33) is similar to the WF (10), with one difference, that the RHS in this case becomes

$$F(\vec{v}) = \sum_{l=1}^2 \left[\int_{\Omega} h_l(x)v_l dx + \int_{\Gamma} g_l \frac{\partial v_l}{\partial n_l} d\gamma \right], \quad \forall v_l \in H^1(\Omega), l = 1, 2. \tag{34}$$

While the LHS of Eq. (34) is $\langle \bar{A}(\vec{y}), \vec{v} \rangle$ which is defined in Eq. (11).

Now, to apply the MBTH, and since the proof of the coercivity and the boundedness of $\langle \bar{A}(\vec{y}), \vec{v} \rangle$ “in the LHS of Eq. (34) are similar to those that were proved in Theorem 4. So it remains only, to prove the RHS of Eq. (34) is bounded, then by Theorem 2 and Theorem 3, one has

$$\begin{aligned} |F(\vec{v})| &\leq \sum_{i=1}^2 \left[|(h_i, v_i)_\Omega| + \left| \left(g_i, \frac{\partial v_i}{\partial n_i} \right)_\Gamma \right| \right] \leq \sum_{i=1}^2 \left[\|h_i(x)\|_{L^2(\Omega)} \|v_i\|_{L^2(\Omega)} + \|g_i(x)\|_{L^2(\Gamma)} \left\| \frac{\partial v_i}{\partial n_i} \right\|_{L^2(\Gamma)} \right] \\ &\leq \sum_{i=1}^2 [\mu_i \|v_i\|_{H^1(\Omega)} + \bar{\mu}_i \bar{\mu}_i \|v_i\|_{H^1(\Omega)}], \text{ where } \|h_i\|_{L^2(\Omega)} \leq \mu_i, \|g_i\|_{L^2(\Gamma)} \leq \bar{\mu}_i \\ &\leq \check{C} \|\vec{v}\|_{(H^1(\Omega))^2}, \text{ where } \check{C} = 2 \max(\mu_1, \bar{\mu}_1, \bar{\mu}_1) \end{aligned} \quad (35)$$

Hence, by Theorem 1, the WF(35) has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$.

Solvability of the CNLEPDEs with the NBCs

In this section, the CNLEPDEs with Neumann BCs (NBCs) are studied.

Theorem 5: Consider the proposed CNLEPDEs (Eqs. (1) and (2)), but with the following nonhomogeneous NBCs (NHNBCs)

$$\frac{\partial y_1}{\partial n_1} = \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i} \cos(n_1, x_j) = g_1, \text{ on } \Gamma \quad (36)$$

$$\frac{\partial y_2}{\partial n_2} = \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i} \cos(n_2, x_j) = g_2, \text{ on } \Gamma \quad (37)$$

Has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$, with $\vec{g} = (g_1, g_2) \in (L^2(\Gamma))^2$, n_ℓ (for $\ell = 1, 2$) is an outer normal vector on Γ and (n_ℓ, x_j) refers to the angle between the x_j – axis and n_ℓ .

Proof: The WF, which results from of Eqs. (1) and (2) with the NHNBCs (Eqs. (36) and (37)) is similar to the WF (10) with one different, that the RHS in this case becomes

$$F(\vec{v}) = \sum_{l=1}^2 [f_\Omega h_l(x) v_l dx + f_\Gamma g_l v_l d\gamma], \forall v_l \in H^1(\Omega), l = 1, 2. \quad (38)$$

While the LHS of Eq. (38) is

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), v_2)_\Omega \quad (39)$$

Now, to apply the MBTH, and since the proof of the coercivity and the boundedness of $\langle \bar{A}(\vec{y}), \vec{v} \rangle$ “in the LHS of Eq. (38) are similar to those that were proved in Theorem 4. So it remains only to prove the RHS of Eq. (38) is bounded, then by Theorem 2 and Theorem 3, one has

$$\begin{aligned} |F(\vec{v})| &\leq \sum_{i=1}^2 [|(h_i, v_i)_\Omega| + |(g_i, v_i)_\Gamma|] \leq \sum_{i=1}^2 [\|h_i(x)\|_{L^2(\Omega)} \|v_i\|_{L^2(\Omega)} + \|g_i(x)\|_{L^2(\Gamma)} \|v_i\|_{L^2(\Gamma)}] \\ &\leq \sum_{i=1}^2 [\mu_i \|v_i\|_{H^1(\Omega)} + \bar{\mu}_i \bar{\mu}_i \|v_i\|_{H^1(\Omega)}], \text{ where } \|h_\ell(x)\|_{L^2(\Omega)} \leq \mu_\ell, \|g_\ell(x)\|_{L^2(\Gamma)} \leq \bar{\mu}_\ell \\ &\leq \check{C} \|\vec{v}\|_{(H^1(\Omega))^2}, \text{ where } \check{C} = 2 \max(\mu_\ell, \bar{\mu}_\ell, \bar{\mu}_\ell). \end{aligned} \quad (40)$$

Hence, by Theorem 1, the WF(38) has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$.

Remark 2: It is noted that; if the NBCs in Eqs. (36) and (37) are homogeneous, then the resulting WF in this case will be similar to those obtained in the proof of Theorem 5, i.e.

$$\langle \bar{A}(\vec{v}), \vec{v} \rangle = F(\vec{v}) \quad (41)$$

where

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_{\Omega} + (f_2(x, y_2), v_2)_{\Omega} \tag{42}$$

and

$$F(\vec{v}) = \sum_{l=1}^2 \int_{\Omega} h_l(x) v_l dx. \tag{43}$$

And hence the existence of a unique $\vec{y} \in (H_0^1(\Omega))^2$ is obtained directly from [Theorem 5](#).

Solvability of the CNLEPDEs with the RBCs

Theorem 6: Consider the proposed system in [Eqs. \(1\) and \(2\)](#), but with the following nonhomogeneous Robin BCs (NHRBCs):

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i} \cos(n_1, x_j) = g_1, \quad \text{on } \Gamma, \tag{44}$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i} \cos(n_2, x_j) = g_2, \quad \text{on } \Gamma. \tag{45}$$

Has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$, with $\vec{g} = (g_1, g_2) \in (L^2(\Gamma))^2$, $\alpha_1, \alpha_2 > 0$ and n_ℓ , (for $\ell = 1, 2$) are as defined in [Theorem 5](#).

Proof: The WF of the CNLEPDES ([Eqs. \(1\) and \(2\)](#)) with the NHRBCs is obtained by multiplying both sides of [Eqs. \(1\) and \(2\)](#) and [Eqs. \(44\) and \(45\)](#) by the test function $v_l \in H^1(\Omega)$, ($l = 1, 2$), resp., then applying the generalized Green’s theorem for the terms which have the 2nd order derivatives, once get:

$$\begin{aligned} \int_{\Omega} \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i} \frac{\partial v_1}{\partial x_j} dx + \int_{\Gamma} \alpha_1 y_1 v_1 d\gamma + \int_{\Omega} \sum_{i=1}^2 b_{1i}(x) \frac{\partial y_1}{\partial x_i} v_1 dx + \int_{\Omega} a_{01}(x) y_1 v_1 dx \\ - \int_{\Omega} b(x) y_2 v_1 dx + \int_{\Omega} f_1(x, y_1) y_1 dx = \int_{\Omega} h_1 v_1 dx + \int_{\Gamma} g_1 v_1 d\gamma, \quad \forall v_1 \in H^1(\Omega) \end{aligned} \tag{46}$$

$$\begin{aligned} \int_{\Omega} \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i} \frac{\partial v_2}{\partial x_j} dx + \int_{\Gamma} \alpha_2 y_2 v_2 d\gamma + \int_{\Omega} \sum_{i=1}^2 b_{2i}(x) \frac{\partial y_2}{\partial x_i} v_2 dx + \int_{\Omega} a_{02}(x) y_2 v_2 dx \\ + \int_{\Omega} b(x) y_1 v_2 dx + \int_{\Omega} f_2(x, y_2) y_2 dx = \int_{\Omega} h_2 v_2 dx + \int_{\Gamma} g_2 v_2 d\gamma, \quad \forall v_2 \in H^1(\Omega) \end{aligned} \tag{47}$$

[Eq. \(46\)](#) and [Eq. \(47\)](#) can also be expressed as

$$\begin{aligned} a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_{\Gamma} + b_1(y_1, v_1) + (a_{01} y_1, v_1)_{\Omega} - (b y_2, v_1)_{\Omega} + (f_1(x, y_1), v_1)_{\Omega} \\ = (h_1, v_1)_{\Omega} + (g_1, v_1)_{\Gamma}, \end{aligned} \tag{48}$$

and

$$\begin{aligned} a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_{\Gamma} + b_2(y_2, v_2) + (a_{02} y_2, v_2)_{\Omega} + (b y_1, v_2)_{\Omega} + (f_2(x, y_2), v_1)_{\Omega} \\ = (h_2, v_2)_{\Omega} + (g_2, v_2)_{\Gamma}. \end{aligned} \tag{49}$$

Adding [Eq. \(48\)](#) and [Eq. \(49\)](#) to obtain

$$a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_{\Omega} + (f_2(x, y_2), v_1)_{\Omega} = (h_1, v_1)_{\Omega} + (g_1, v_1)_{\Gamma} + (h_2, v_2)_{\Omega} + (g_2, v_2)_{\Gamma}. \tag{50}$$

Or

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = F(\vec{v}) \quad (51)$$

where

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), v_1)_\Omega, \quad (52)$$

with

$$a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_\Gamma + b_1(y_1, v_1) + (a_{01} y_1, v_1)_\Omega - (b y_2, v_1)_\Omega + a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_\Gamma + b_2(y_2, v_2) + (a_{02} y_2, v_2)_\Omega + (b y_1, v_2)_\Omega, \quad (53)$$

$$a_l(y_l, v_l) = \sum_{i,j=1}^2 a_{lij} \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j}, \quad (54)$$

$$b_l(y_l, v_l) = \sum_{i=1}^2 b_{li}(x) \frac{\partial y_l}{\partial x_i} v_l, \quad l = 1, 2 \quad (55)$$

$$F(\vec{v}) = \sum_{l=1}^2 [f_\Omega h_l(x) v_l dx + f_\Gamma g_l v_l d\gamma], \quad \forall v_l \in H^1(\Omega), \quad l = 1, 2. \quad (56)$$

and

$$(\alpha_l y_l, v_l)_\Gamma = \int_\Gamma \alpha_l y_l v_l d\gamma, \quad \text{for } l = 1, 2. \quad (57)$$

Now, the proof of the coercivity and boundedness of $\langle \bar{A}(\vec{y}), \vec{v} \rangle$ in LHS of Eq. (48), are as follows

$$\begin{aligned} \langle \bar{A}(\vec{y}), \vec{y} \rangle &= \int_\Omega \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i} \frac{\partial y_1}{\partial x_j} dx + \int_\Gamma \alpha_1 y_1^2 d\gamma + \int_\Omega \sum_{i=1}^2 b_{1i}(x) \frac{\partial y_1}{\partial x_i} y_1 dx \\ &+ \int_\Omega a_{01}(x) y_1^2 dx - \int_\Omega b(x) y_2 y_1 dx + \int_\Omega f_1(x, y_1) y_1 dx \\ &+ \int_\Omega \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i} \frac{\partial y_2}{\partial x_j} dx + \int_\Gamma \alpha_2 y_2^2 d\gamma + \int_\Omega \sum_{i=1}^2 b_{2i}(x) \frac{\partial y_2}{\partial x_i} y_2 dx \\ &+ \int_\Omega a_{02}(x) y_2^2 dx + \int_\Omega b(x) y_1 y_2 dx + \int_\Omega f_2(x, y_2) y_2 dx \end{aligned} \quad (58)$$

Using Assumptions 1, setting $\frac{\partial y_l}{\partial x_i} y_l = \frac{1}{2} \frac{\partial}{\partial x_i} (y_l^2)$, $l = 1, 2$, for the third and ninth terms, then the above equality becomes

$$\begin{aligned} \langle \bar{A}(\vec{y}), \vec{y} \rangle &\geq \hat{c}_1 \sum_{i=1}^2 \int_\Omega \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx + \hat{c}_2 \int_\Gamma |y_1|^2 d\gamma + \sum_{i=1}^2 \int_\Omega b_{1i}(x) \frac{1}{2} \frac{\partial}{\partial x_i} (y_1^2) dx \\ &+ \int_\Omega a_{01}(x) |y_1|^2 dx + \hat{c}_3 \sum_{i=1}^2 \int_\Omega \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx + \hat{c}_4 \int_\Gamma |y_2|^2 d\gamma \\ &+ \sum_{i=1}^2 \int_\Omega b_{2i}(x) \frac{1}{2} \frac{\partial}{\partial x_i} (y_2^2) dx + \int_\Omega a_{02}(x) |y_2|^2 dx \\ &+ \int_\Omega (f_1(x, y_1) - f_1(x, 0))(y_1 - 0) dx + \int_\Omega (f_2(x, y_2) - f_2(x, 0))(y_2 - 0) dx. \end{aligned} \quad (59)$$

Since, f_l is monotone w.r.t. y_l and by using integration by parts on the third and the seventh terms in the RHS of Eq. (59), to obtain

$$\begin{aligned} \langle \bar{A}(\vec{y}), \vec{y} \rangle &\geq \hat{c}_1 \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx + \hat{c}_2 \int_{\Gamma} |y_1|^2 d\gamma + \sum_{i=1}^2 \int_{\Gamma} \frac{b_{1i}(x)}{2} |y_1|^2 d\gamma \\ &+ \int_{\Omega} \left(a_{01}(x) - \frac{1}{2} \sum_{i=1}^2 \frac{\partial b_{1i}}{\partial x_i} \right) |y_1|^2 dx + \hat{c}_3 \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx + \hat{c}_4 \int_{\Gamma} |y_2|^2 d\gamma \\ &+ \sum_{i=1}^2 \int_{\Gamma} \frac{b_{2i}(x)}{2} |y_2|^2 d\gamma + \int_{\Omega} \left(a_{02}(x) - \frac{1}{2} \sum_{i=1}^2 \frac{\partial b_{2i}}{\partial x_i} \right) |y_2|^2 dx, \hat{c}_l \in \mathbb{R}^+ \text{ for } l = 1, 2. \end{aligned} \tag{60}$$

Using Eq. (17) in Eq. (60), and that $b_{li} \geq 0$ for $l, i = 1, 2$, then Eq. (54) becomes

$$\begin{aligned} \langle \bar{A}(\vec{y}), \vec{y} \rangle &\geq \hat{c}_1 \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx + \hat{c}_3 \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx + \hat{c}_2 \int_{\Gamma} |y_1|^2 d\gamma + \hat{c}_4 \int_{\Gamma} |y_2|^2 d\gamma, \\ &= C_0 \left(\sum_{i=1}^2 \int_{\Omega} |\nabla y_i|^2 dx + \sum_{i=1}^2 \int_{\Gamma} |y_i|^2 d\gamma \right) \end{aligned} \tag{61}$$

Where $\tau = \min(\hat{c}_2, \hat{c}_4)$, $C_0 = \min(\hat{c}_1, \hat{c}_3, \tau)$.

Now, let $0 < \Gamma_1 \subset \Gamma$ be a measurable set, then by Lemma 2, there exist real constants $\sigma_1(\Gamma_1), \sigma_2(\Gamma_1) > 0$, which are independent of $\vec{y} \in (H^1(\Omega))^2$, s.t.

$$\begin{aligned} \langle \bar{A}(\vec{y}), \vec{y} \rangle &\geq C_0 \sum_{i=1}^2 \sigma_i \left(\int_{\Omega} |\nabla y_i|^2 dx + \left(\int_{\Gamma_1} y_i d\gamma \right)^2 \right) \\ &\geq \sigma_3 \sum_{i=1}^2 \left(\int_{\Omega} |\nabla y_i|^2 dx + \left(\int_{\Gamma_1} y_i d\gamma \right)^2 \right) \geq \sigma_3 \|\vec{y}\|_{(H^1(\Omega))^2}^2. \end{aligned} \tag{62}$$

where $\sigma_i = \min(1 + \frac{1}{\sigma_i(\Gamma_1)})$, $i = 1, 2$, and $\sigma_3 = C_0 \min(\sigma_1, \sigma_2)$.

On the other hand, since $a_{li}(x), b_{li}(x), a_{0l}(x), b(x) \in L^\infty(\Omega)$, ($\forall l, i = 1, 2$), and that

$$\left| \int_{\Gamma} \sum_{l=1}^2 \alpha_l y_l v_l d\gamma \right| \leq \sum_{l=1}^2 \lambda_l \left| \int_{\Gamma} y_l v_l d\gamma \right|, \lambda_l = \max_{l=1,2} |\alpha_l|. \tag{63}$$

Using Eq. (57), besides Eqs. (21) to (24), to obtain

$$\begin{aligned} \left| \langle \bar{A}(\vec{y}), \vec{y} \rangle \right| &\leq \sum_{i,j=1}^2 \mu_1 \left| \int_{\Omega} \frac{\partial y_1}{\partial x_i} \frac{\partial v_1}{\partial x_j} dx \right| + \sum_{l=1}^2 \lambda_l \left| \int_{\Gamma} y_l v_l d\gamma \right| + \sum_{i,j=1}^2 \mu_2 \left| \int_{\Omega} \frac{\partial y_2}{\partial x_i} \frac{\partial v_2}{\partial x_j} dx \right| \\ &+ \sum_{i=1}^2 \bar{\mu}_1 \left| \int_{\Omega} \frac{\partial y_1}{\partial x_i} v_1 dx \right| + \sum_{i=1}^2 \bar{\mu}_2 \left| \int_{\Omega} \frac{\partial y_2}{\partial x_i} v_2 dx \right| \\ &+ \sum_{l=1}^2 \bar{\bar{\mu}}_l \left| \int_{\Omega} y_l v_l dx \right| + \sum_{l=1}^2 \bar{\bar{\bar{\mu}}}_l \left| \int_{\Omega} y_{3-l} v_l dx \right| \end{aligned} \tag{64}$$

Using Theorem 2 for each term in the RHS of the above inequality, Eq. (64), to get that

$$\begin{aligned} \left| \langle \bar{A}(\vec{y}), \vec{y} \rangle \right| &\leq \mu \left\{ \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{\partial v_1}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Gamma} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Gamma} |v_1|^2 dx \right)^{\frac{1}{2}} \right. \\ &+ \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{\partial v_2}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Gamma} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Gamma} |v_2|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_1}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \sum_{i=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_2}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \\
& + \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \\
& + \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \Big\} \tag{65}
\end{aligned}$$

where $\mu = \max\{\mu_l, \lambda_l, \bar{\mu}_l, \bar{\bar{\mu}}_l\}, \forall l, i = 1, 2$.

Using [Theorem 3](#) for the second and the fourth term in [Eq. \(65\)](#), and then rewriting the resulting inequality in another form, it becomes

$$\begin{aligned}
|\langle \bar{A}(\vec{y}), \vec{y} \rangle| & \leq \mu \left\{ \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} + \sum_{j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_1}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial v_1}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \right\} \\
& + \left\{ \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} + \sum_{j=1}^2 \left(\int_{\Omega} \left| \frac{\partial y_2}{\partial x_j} \right|^2 dx \right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} + \sum_{i,j=1}^2 \left(\int_{\Omega} \left| \frac{\partial v_2}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}} \right\} \\
& + \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}} \\
& + \bar{c}_1 \left(\int_{\Omega} |y_1|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_1|^2 dx \right)^{\frac{1}{2}} + \bar{c}_2 \left(\int_{\Omega} |y_2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_2|^2 dx \right)^{\frac{1}{2}}, \tag{66}
\end{aligned}$$

$$\begin{aligned}
|\langle \bar{A}(\vec{y}), \vec{y} \rangle| & \leq \mu_3 \left[\|y_1\|_{H^1(\Omega)} \|v_1\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)} \|v_1\|_{H^1(\Omega)} \right. \\
& \left. + \|y_1\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} + \bar{c}_1 \|y_1\|_{H^1(\Omega)} \|v_1\|_{H^1(\Omega)} + \bar{c}_2 \|y_2\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} \right], \tag{67}
\end{aligned}$$

where $\mu_3 = \mu \max(1, \max_{1 \leq l \leq 2} (1 + \bar{c}_l))$,

$$|\langle \bar{A}(\vec{y}), \vec{y} \rangle| \leq \rho \|\vec{y}\|_{(H^1(\Omega))^2} \|\vec{v}\|_{(H^1(\Omega))^2}, \quad \text{where } \rho = 6\mu_3 \tag{68}$$

From [Eq. \(55\)](#) and [Eq. \(27\)](#), the continuity of $\vec{y} \mapsto \langle \bar{A}(\vec{y}), \vec{v} \rangle$ w.r.t. \vec{y} , is obtained.

In this case, it is also clear that $F(\vec{v})$ in the RHS of [Eq. \(66\)](#) is a linear functional on $(H^1(\Omega))^2$, it remains to show it is bounded, since $F(\vec{v})$ in this case, it is exactly the same term that is in the RHS of [Eq. \(38\)](#), and it was proved to be bounded, by [Theorem 1](#), the WF (51) has a unique COS $\vec{y} \in (H^1(\Omega))^2$.

Corollary 2: Consider the CNLEPDEs ([Eqs. \(1\)](#) and [\(2\)](#)), but with the following homogenous Robin BCs (HRBCs)

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i} \cos(n_1, x_j) = 0, \quad \text{on } \Gamma \tag{69}$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i} \cos(n_2, x_j) = 0, \quad \text{on } \Gamma. \tag{70}$$

Has a unique COS $\vec{y} = (y_1, y_2) \in (H_0^1(\Omega))^2$, with α_1, α_2 and n_ℓ , (for $\ell = 1, 2$) are as defined in [Theorem 6](#).

Proof: The WF of the CNLEPDES ([Eqs. \(1\)](#) and [\(2\)](#)) and ([Eqs. \(69\)](#) and [\(70\)](#)) is

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = F(\vec{v}) \tag{71}$$

where

$$\bar{A}(\vec{y}), \vec{v} = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), v_1)_\Omega, \text{ with} \tag{72}$$

$$F(\vec{v}) = \sum_{\ell=1}^2 \left[\int_{\Omega} h_{\ell}(x)v_{\ell} dx \right], \quad \forall v_{\ell} \in H^1(\Omega), \ell = 1, 2. \tag{73}$$

It is clear that the resulting WF (71) in this case is similar to the WF in [Theorem 6](#). Hence, the existence of a unique COS $\vec{y} \in (H_0^1(\Omega))^2$ In this case, it will be obtained directly after applying [Theorem 6](#).

Solvability of the CNLEPDEs with the MBCs

Theorem 7: Consider the CNLEPDEs ([Eqs. \(1\) and \(2\)](#)), but with the following Mixed BCs (MBCs),

$$y_l = g_{lD} \text{ on } \Gamma_D, \quad l = 1, 2 \tag{74}$$

$$\frac{\partial y_l}{\partial n_l} = g_{lN} \text{ on } \Gamma_N, \quad l = 1, 2 \tag{75}$$

Has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$, where the boundary Γ of Ω is split into two nonempty, disjoint open sets Γ_D and Γ_N , with $\vec{g}_i = (g_{iD}, g_{iN}) \in L^2(\Gamma_D) \times L^2(\Gamma_N)$. Here n_l is the unit outer vector normal on the boundary Γ_N , $\Gamma = \Gamma_D \cup \Gamma_N$, and $\Gamma_D \cap \Gamma_N = \emptyset$.

Proof: The WF, which is obtained from [Eqs. \(1\) to \(3\)](#) with the NHNBCs [Eqs. \(74\) and \(75\)](#) is similar to the WF in [Eq. \(51\)](#), i.e

$$\langle \bar{A}(\vec{y}), \vec{y} \rangle = F(\vec{v}) \tag{76}$$

With

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, y_1), v_1)_\Omega + (f_2(x, y_2), v_1)_\Omega, \tag{77}$$

and that

$$F(\vec{v}) = \sum_{l=1}^2 \left[\int_{\Omega} h_l(x)v_l dx + \int_{\Gamma_D} g_{lD} \frac{\partial v_l}{\partial n_l} d\gamma_D + \int_{\Gamma_N} g_{lN} v_l d\gamma_N \right], \quad \forall v_l \in H^1(\Omega) \tag{78}$$

Now, to apply the MBTH and since the proof of the coercivity and the boundedness of $\langle \bar{A}(\vec{y}), \vec{v} \rangle$ “in the LHS of [Eq. \(76\)](#) are similar to those that were proved in [Theorem 4](#). So it remains only, to prove the RHS of [Eq. \(76\)](#), which is given by [Eq. \(78\)](#) is bounded, then by [Theorem 2](#) and [Theorem 3](#), one has

$$\begin{aligned} |F(\vec{v})| &\leq \sum_{\ell=1}^2 \left[|(h_{\ell}, v_{\ell})_{\Omega}| + \left| \left(g_{\ell}, \frac{\partial v_{\ell}}{\partial n_{\ell}} \right)_{\Gamma_D} \right| + |(g_{\ell N}, v_{\ell})_{\Gamma_N}| \right] \\ &\leq \sum_{\ell=1}^2 \left[\|h_{\ell}\|_{L^2(\Omega)} \|v_{\ell}\|_{L^2(\Omega)} + \|g_{\ell D}(\mathbf{x})\|_{L^2(\Gamma_D)} \left\| \frac{\partial v_{\ell}}{\partial n_{\ell}} \right\|_{L^2(\Gamma_D)} + \|g_{\ell N}\|_{L^2(\Gamma_N)} \|v_{\ell}\|_{L^2(\Gamma_N)} \right] \\ &\leq \sum_{\ell=1}^2 \left[\mu_{\ell} \|v_{\ell}\|_{H^1(\Omega)} + \bar{\mu}_{\ell} \left\| \frac{\partial v_{\ell}}{\partial n_{\ell}} \right\|_{L^2(\Gamma_D)} + \bar{\bar{\mu}}_{\ell} \|v_{\ell}\|_{L^2(\Gamma_D)} \right] \\ &\leq \sum_{\ell=1}^2 \left[\mu_{\ell} \|v_{\ell}\|_{H^1(\Omega)} + c_{\ell} \bar{\mu}_{\ell} \|v_{\ell}\|_{H^1(\Omega)} + \bar{c}_{\ell} \bar{\bar{\mu}}_{\ell} \|v_{\ell}\|_{H^1(\Omega)} \right] \\ &\leq \check{C} \|\vec{v}\|_{(H^1(\Omega))^2} \end{aligned} \tag{79}$$

where $\check{C} = 3 \max(\mu_l, c_l \bar{\mu}_l, \bar{c}_l \bar{\mu}_l)$.

Hence, by [Theorem 1](#), the WF(76) has a unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$.

Example

Consider the following CNLEPDEs with the HDBC:

$$-\frac{\partial}{\partial x_1} \left[(x_1^2 + 1) \frac{\partial y_1}{\partial x_1} \right] - \frac{\partial}{\partial x_2} \left[(x_2^2 + 1) \frac{\partial y_1}{\partial x_2} \right] + x_1^2 \frac{\partial y_1}{\partial x_1} + x_2^2 \frac{\partial y_1}{\partial x_2} + (x_2^2 + 1) y_1 - (x_1^4 + x_2^2 + 1) y_2 + f_1(\vec{x}, y_1)$$

$$= h_1(x), \text{ in } \Omega$$

$$-\frac{\partial}{\partial x_1} \left[(x_2^2 + 1) \frac{\partial y_2}{\partial x_1} \right] - \frac{\partial}{\partial x_2} \left[(x_1^2 + 1) \frac{\partial y_2}{\partial x_2} \right] + x_2^2 \frac{\partial y_2}{\partial x_1} + x_1^2 \frac{\partial y_2}{\partial x_2} + (x_1^2 + 1) y_2 + (x_1^4 + x_2^2 + 1) y_1 + f_2(\vec{x}, y_2)$$

$$= h_2(x), \text{ in } \Omega$$

$$y_1(\vec{x}) = 0, \text{ on } \Gamma$$

$$y_2(\vec{x}) = 0, \text{ on } \Gamma$$

where $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$, be a Lipschitz bounded domain with boundary $\Gamma = \partial\Omega$, $f_l(\vec{x}, y_l) = 0.5y_l|y_l| \in L^2(\Omega)$, and $h_l(x) \in L^2(\Omega)$ for $l = 1, 2$ are of Carathéodory type on $\Omega \times \mathbb{R}$ (on Ω).

According to [Theorem 4](#), the above problem has a unique solution in Ω .

Results and discussion

In this paper, the solvability (in infinite dimensions) of the proposed CNLEPDEs based on MBTH with different types of boundary conditions (BCs) (Dirichlet (DBC), Neumann (NBC), Robin (RBC), and Mixed BCs (MBC)) was studied. The WF of the proposed CNLEPDEs with each of the four types of BCs was found at first by utilizing the generalization theorem of Green. Secondly and because the proposed model of PDEs (the CNLEPDEs) is nonlinear therefore it was a need for using the MBTH to prove the existence of unique solution for each WF under suitable assumptions either with help of the PFI in the cases where the BCs were homogenous, or with the help of the generalization PFI (GPFI) besides to the TOR where the BCs were nonhomogeneous. The results for this study are very important because they have not been studied before; in addition, they give the green light for investigators to solve this kind of problem numerically.

Conclusions

The MBTH is employed successfully to prove the existence of a unique COS (in infinite dimension) of WF for the CNLEPDEs with each one of the four different types of BCs (DBC, NBC, RBC, and MBC). The PFI ([Lemma 1](#)) was used successfully in the proof of the coercivity of the bilinear form where the BCS is homogeneous, while it fails to apply in case that the BCS became nonhomogeneous, in this case and to avoid this problem the generalized PFI (GPFI" [Lemma2](#)") was used instead of it, beside to use for the TOR ([Theorem 3](#)) successfully to proof the coercivity.

Authors' declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Mustansiriyah.

Author's contributions statement

J. A. A. and A. A. M. contributed to the design and implementation of the research, the proof of the theorems, and the writing of the manuscript.

Data availability

The datasets generated during and analyzed during the current study are available from the corresponding author on reasonable request.

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قابلية حل المعادلات التفاضلية الجزئية الإهليلجية غير الخطية المزدوجة

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الخلاصة

كما هو معروف فإن قابلية حل أي مشكلة رياضية تلعب قاعدة مهمة للعديد من الباحثين المهتمين بالحل العددي لمثل هذه المشاكل الرياضية لأنه يمنحهم الضوء الأخضر حول القدرة على حل مثل هذه المشاكل عددياً، بالإضافة إلى إعطاء الضوء الأخضر لحل العديد من مشاكل الحياة الواقعية عددياً في مختلف مجالات العلوم؛ مثل الفيزياء والطب والفلك والكيمياء والأحياء وفروع الهندسة المختلفة والعديد من المجالات الأخرى. ولهذا السبب، كرس هذا العمل لدراسة قابلية الحل (في البعد اللانهائي) لمعادلات التفاضل الجزئي الإهليلجية غير الخطية المزدوجة (CNLEPDEs) مع أربعة أنواع مختلفة من الشروط الحدودية (BCs)، دي ريتشلت (DBC) نيومان (NBCs)، روبن (RBCs) و BCs المختلطة (MBCs). تم إيجاد الصيغة الضعيفة (WF) لكل مسألة بناءً على نوع كل نوع من أنواع BC. وتم إثبات وجود وتفرد حل المسألة المقترحة (CNLEPDEs) لكل نوع من أنواع BC باستخدام نظرية مينتي-براوذر (MBTH)، وذلك باستخدام متباينة بوانكاريه-فريدريش (PFI) في حالة BCs المتجانسة (HBCs) لكل نوع، وباستخدام معامل التعميم PFI (GPFI) ومشغل التتبع (TRO) في حالة BCs غير المتجانسة (NHBCs) لكل نوع.

الكلمات المفتاحية: زوج معادلات تفاضلية جزئية إهليلجية غير خطية، شروط دريشليت، شروط الحدود المختلطة، شرط نيومان، شرط روبن.