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Decomposition of Fundamental Function in Finsler Space and Some Tensors by Lie-Derivative in $\mathcal{GBK}$- $5RF_n$

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Abstract:

This paper discuses the decomposition of Fundamental Function of Finsler space by Lie-derivative in generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald. The decomposition of Lie-derivative of some tensors is studied by using a covariant, contravariant and mixed tensor in $\mathcal{GBK}-5RF_n$. The decomposition tensor φ^i_{kh} behaves as fifth-recurrent is proved. Also, new identities are established in generalized $\mathcal{BK}$ − fifth recurrent Finsler space that admits tensors decomposition.

Keywords: Generalized $\mathcal{BK}$-fifth recurrent Finsler space, Lie-derivative L_v , Finsler space $F(x, y)$.

تحليل الدالة الأساسية في فضاء فينسلر وبعض الموترات بواسطة مشتقة لي في فضاء فينسلر المعمم خماسي المعاودة

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الخلاصة:

يتناول هذا البحث تحليل الدالة الأساسية وبعض الموترات في فضاء فينسلر باستخدام مشتقة لي. ويهدف هذا النهج إلى اكتساب فهم أعمق للبنية الهندسية لفضاءات فينسلر وتطوير أدوات جديدة لتحليلها. وتُظهر نتائجنا أن التحليل المقترح يوفر إطاراً قوياً لدراسة خصائص الأجسام الفينسلرية وتفاعلاتها.

يدرس هذا البحث تفكيك الدالة الأساسية لفضاء فينسلر بواسطة مشتقة لي بإستخدام كميات ممتدة متضادة متحدة الإختلاف ومختلطة في فضاء فينسلر المعمم خماسي المعاودة بالنسبة للموتر التقوسي الرابع لكارتان باستخدام مشتقة بروالد. ونثبت علاقات جديدة في هذا الفضاء الذي يسمح بتفكيك الموترات.

الكلمات المفتاحية: فضاء فينسلر المعمم خماسي المعاودة، مشتقة لي، فضاء فينسلر $F(x,y)$.

1. Introduction and Preliminaries

Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for modeling various physical phenomena. Finsler geometry has gained significant attention in recent years due to its applications in various fields, including physics, engineering, and mathematics. This approach aims to provide a deeper understanding of the underlying geometric properties of these spaces and to develop novel analytical tools. Our work builds upon the foundational studies of Finsler geometry [1] and the subsequent investigations into recurrence decompositions by [4,5,12,13,16,20,22]. Specifically, the application of the Lie derivative is explored

, a powerful mathematical tool for analyzing the behavior of geometric objects, to the decomposition process within the $G\mathcal{B}K\text{-}5RF_n$ framework. The $G\mathcal{B}K\text{-}5RF_n$ framework, a specialized mathematical setting for Finsler geometry, provides a suitable environment for our analysis.

By leveraging the Lie derivative and the $G\mathcal{B}K\text{-}5RF_n$ framework, it is aimed to contribute to the ongoing research on Finsler geometry by offering a new perspective on the decomposition of its fundamental elements. The findings have potential implications for various applications of Finsler geometry, such as the study of physical phenomena, the development of new mathematical models, and the advancement of computational methods.

In the following sections, a more detailed overview of our research methodology is provided, present our key findings, and discuss the potential implications of our results. The decompositions of W-curvature tensor studied by opondo **[18]**, Negi **[17]** studied decomposition of Curvature Tensor Manifolds of First order. While the decomposition of normal projective Curvature Tensor fields in Finsler manifolds studied by Bisht and Negi **[10]**. Bidabad and Sepasi **[9]** completed Finsler spaces of constant negative Ricci curvature. The curvature tensor field on D-recurrent Finsler space have been studied by Atashafrouz and Najafi **[8]**. A generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald introduced by AL-Qashbari and Baleedi **[6]**. Various identities of curvature tensor studied by authors **[2, 3,11].**

A Lie - derivative evaluate the rate of change of avector field or a tensor field along the flow of another vector or tensor field. AL-Qashbari and Baleedi **[7]** studied the Lie-derivative in G$\mathcal{B}$K- 5RF$_{\text{n}}$ and established various identities of tensors in Finsler space by Lie-derivative. Gouin **[14]** introduced some remarks on the Lie-derivative. The Lie- derivative of forms and its application was investigated by authors **[15, 19, 21].** An n-dimensional Finsler space F_n endowed with the metric function $F(x, y)$ satisfying the request conditions **[1]**. The metric function F, the metric tensor g_{ij} and the Kronecker delta δ^i_h are given by

(1.1) a) $g_{ij}\, y^i\, y^j = F^2$, b) $g_{ij}\, y^j = y_i$, c) $g_{hj} = \dot{\partial}_h y_j$, d) $y_i\, y^i = F^2$,

e) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 \quad if \quad j = k \ , \\ 0 \quad if \quad j \neq k \ . \end{cases}$ f) $\delta^i_h\, g_{ik} = g_{hk}$,

g) $\delta^i_k\, y^k = y^i$, h) $\delta^j_h = \dot{\partial}_h y^j$ i) $\delta^i_i = n$ and j) $\delta^i_k\, y_i = y_k$

The torsion tensor H^i_{kh} and the deviation tensor H^i_h are defined as follows

(1.2) a) $H^i_{kh} = K^i_{jkh} y^j = R^i_{jkh}\, y^j$, b) $H^i_{kh} y^k = H^i_h$, c) $H^i_k\, y^k = 0$,

d) $H^i_{kh} = \dot{\partial}_k H^i_h$, e) $H^i_{kh} = \frac{1}{3}\,(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$,

f) $\mathcal{B}_m H^i_{kh} + \mathcal{B}_h H^i_{mk} + \mathcal{B}_k H^i_{hm} = 0$, g) $y_i H^i_{kh} = 0$ and h) $y^r \dot{\partial}_j \dot{\partial}_r H^i_{kh} = 0$.

Bianchi identities for the curvature tensor K^i_{jkh} is given by

(1.3) $K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0$.

The deviation tensor R^i_h are given by

(1.4) $R^i_{jkh}\, g^{jk} = R^i_h$.

Berwald's covariant derivative of the vectors y^i and y_i are defined to be identically zero

(1.5) a) $\mathcal{B}_k\, y^i = 0$ and b) $\mathcal{B}_k\, y_i = 0$.

The (h) hv-torsion tensor C_{ijk} possesses the following properties

(1.6) $C_{ijk}\, y^k = C_{kij}\, y^k = C_{jki}\, y^k = 0$.

The unit vector l^i and associative vector l_i in the directions of y^i and y_i are defined by

(1.7) $l^i = \frac{y^i}{F}$.

(1.8) $l_i = \frac{y_i}{F}$.

(1.9) $l^i \, l_i = 1$.

The vector y^i is Lie-invariant i.e.,

(1.10) $L_v y^i = 0$.

Let us investigate a generalized fifth-order recurrent Finsler geometry satisfying condition [3]

(1.11)
$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k \, g_{jh}\right)$$
$$- c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k \, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k \, C_{jhl}\right)$$
$$- e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k \, C_{jhq}\right) - 2b_{qlnm} y^r \mathcal{B}_r\left(\delta^i_h C_{jks} - \delta^i_k \, C_{jhs}\right).$$

(1.12)
$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm} H^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right) .$$

(1.13)
$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = a_{sqlnm} H^i_h + b_{sqlnm}\left(\delta^i_h F^2 - y^i y_h\right).$$

(1.14)
$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm} R^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k \, g_{jh}\right)$$
$$-c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k \, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k \, C_{jhl}\right) - e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k \, C_{jhq}\right)$$
$$-2b_{qlnm} y^r \mathcal{B}_r\left(\delta^i_h C_{jks} - \delta^i_k \, C_{jhs}\right) + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (C^i_{jt} H^t_{kh}) - a_{sqlnm}(C^i_{jt} H^t_{kh}) .$$

The tensor product of two tensors, one of type (r_1, s_1) and the other of type (r_2, s_2) is a tensor of type $(r_1 + r_2 , s_1 + s_2)$. Tensor decomposition is the process of decomposing a higher-rank tensor into a sum of simpler tensors, often of lower rank. These simpler tensors can be expressed in terms of products of contravariant and covariant basis vectors. The Cartan's fourth curvature tensor K^i_{jkh} is a mixed tensor of type (1,3) it may be written as a tensor product of a tensor of type (0,1) represented by y_j and a tensor of type (1,2) represented by φ^i_{kh} , or it may be written as a tensor product of a tensor of type (1,0) represented by y^i and a tensor of type (0,3) represented by φ_{jkh} , with the possibilities as

(1.15) a) $K^i_{jkh} = y_j \varphi^i_{kh}$, b) $K^i_{jkh} = y^i \varphi_{jkh}$, c) $K^i_{jkh} = y_k \varphi^i_{jh}$ and d) $K^i_{jkh} = y_h \varphi^i_{jk}$.

It can be represented as the tensor product involving a mixed tensor of rank (1,1), represented by X^i_j, and a covariant tensor of type (0,2), represented by φ_{kh} .

(1.16) a) $K^i_{jkh} = X^i_j \varphi_{kh}$, b) $K^i_{jkh} = X^i_k \varphi_{jh}$ and c) $K^i_{jkh} = X^i_h \varphi_{jk}$.

The Lie-derivative of a general mixed tensor field $T^i_j(x , \dot{x})$ expressed in the form:

(1.17)
$$L_v T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh} \mathcal{B}_m v^i + T^i_{mkh} \mathcal{B}_j v^m + T^i_{jmh} \mathcal{B}_k v^m$$
$$+ T^i_{jkm} \mathcal{B}_h v^m + \dot{\partial}_m T^i_{jkh} \mathcal{B}_r v^m y^r .$$

Taking the Lie-derivative of both sides of [(1.5)j] and using [(1.5)e] when $i \neq j$, we get

(1.18) $L_v y_k = 0$.

By applying (1.17) on the Cartan's fourth curvature tensor K^i_{jkh} and on the h(v)-torsion tensor H^i_{kh} and using [(1.5)a] and [(1.5)h,e], when $r \neq m$, we get

(1.19) $L_v K^i_{jkh} = v^m \mathcal{B}_m K^i_{jkh} - K^m_{jkh} \mathcal{B}_m v^i + K^i_{mkh} \mathcal{B}_j v^m + K^i_{jmh} \mathcal{B}_k v^m + K^i_{jkm} \mathcal{B}_h v^m$, and

(1.20) $L_v H^i_{kh} = v^m \mathcal{B}_m H^i_{kh} - H^m_{kh} \mathcal{B}_m v^i + H^i_{mh} \mathcal{B}_k v^m + H^i_{km} \mathcal{B}_h v^m$, respectively.

Taking the Lie-derivative of both sides of [(1.2) a], we get

(1.21) $L_v H^i_{kh} = L_v K^i_{jkh} y^j$.

Using (1.10) in above equation, we get

(1.22) $L_v H^i_{kh} = y^j L_v K^i_{jkh}$.

Using (1.19), [(1.2)a] and (1.10) in above equation, we get

(1.23) $L_v H^i_{kh} = v^m \mathcal{B}_m H^i_{kh} - H^m_{kh} \mathcal{B}_m v^i + K^i_{mkh} \mathcal{B}_j v^m y^j + H^i_{mh} \mathcal{B}_k v^m + H^i_{km} \mathcal{B}_h v^m$.

In view of (1.23) and (1.20), we get

(1.24) $K^i_{mkh} \mathcal{B}_j v^m y^j = 0$.

Using [(1.5)a] in above equation and since $K^i_{mkh} \neq 0$ and $y^j \neq 0$, we get

(1.25) $\mathcal{B}_j v^m = 0$.

Taking the Lie-derivative of (1.7) and using [(1.1)d], (1.10) and (1.18) in (1.7), we get

(1.26) $L_v l^i = 0$.

The main aim of this paper is to study the decomposition of Fundamental Function of Finsler space and some tensors by Lie-derivative in generalized fifth recurrent Finsler space. By employing the Lie derivative within the newly introduced $G\mathcal{B}K\text{-}5RF_n$ framework, a novel approach for analyzing the geometric structure of Finsler spaces is developed.

2. A Decomposition of Lie-Derivative of Some Tensors by using a Covariant Tensor y_j and Mixed Tensor φ^i_{kh} in $G\mathcal{B}K\text{-}5RF_n$

The Lie-derivative is a fundamental tool in differential geometry and its applications. In this study, a novel decomposition of the Lie-derivative for a specific class of tensors is introduced. The decomposition of Cartan's fourth curvature tensor K^i_{jkh} in the form [(1.15a)] is studied, where y_j it is a non-vanishing covariant tensor and φ^i_{kh} is a decomposition tensor.

Taking $\mathcal{B}$-covariant derivative of fifth order for [(1.15a)], with respect to x^m x^n, x^l , x^q and x^s, then using [(1.5)b], we get

(2.1) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K^i_{jkh} = y_j (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \varphi^i_{kh})$.

Using [(1.15)a], (1.11) and [(1.1)e] in (2.1), we get

(2.2) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\varphi^i_{kh} = a_{sqlnm}\varphi^i_{kh}$.

In conclusion, it can be stated that

Theorem 2.1: In $G\mathcal{B}K - 5RF_n$, the decomposition tensor φ^i_{kh} behaves as fifth-recurrent.

Using [(1.15)a] in the right side of (1.11), we get

(2.3) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm}y_j\varphi^i_{kh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\right)$$
$$- c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\right)$$
$$- e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\right) - 2b_{qlnm}\, y^r\mathcal{B}_r\left(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\right) \; .$$

Transvecting (2.3) by y^j, using [(1.1)b,d], [(1.2)a], [(1.5)a] and (1.6), we get

(2.4) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm}F^2\,\varphi^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right)$.

Using [(1.1)e] in (2.4), we get

(2.5) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm}F^2\varphi^i_{kh}$.

Therefore, it can be concluded that

Theorem 2.2: In $G\mathcal{B}K - 5RF_n$, Berwald's covariant derivative of the fifth order for the torsion tensor H^i_{kh} decomposes according to (2.5).

By taking the Lie-derivative of both sides of the equation (2.5), yields

(2.6) $L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}\right) = F^2 L_v(a_{sqlnm}\varphi^i_{kh}) + a_{sqlnm}\varphi^i_{kh}L_vF^2$.

Using (1.12) and [(1.1)e] in (2.6) , we get

$$L_vF^2 = \frac{1}{a_{sqlnm}\,\varphi^i_{kh}}\,[\,H^i_{kh}\,L_v(a_{sqlnm}) + a_{sqlnm}L_v\,H^i_{kh}$$
$$-F^2(\varphi^i_{kh}L_v a_{sqlnm} + a_{sqlnm}L_v\varphi^i_{kh})\,] \, .$$

Using (1.17) and (1.25) in above equation, we get

(2.7) $$L_vF^2 = \frac{v^m}{a_{sqlnm}\,\varphi^i_{kh}}\left[\,H^i_{kh}\,\mathcal{B}_m\,a_{sqlnm} + a_{sqlnm}\mathcal{B}_m H^i_{kh}\right.$$
$$\left. - F^2(\varphi^i_{kh}\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m\varphi^i_{kh})\,\right] .$$

Consequently, it can be concluded that

Theorem 2.3: In $G\mathcal{B}K - 5RF_n$,the Lie-derivative for the quadrature fundamental function of Finsler space decomposes according to (2.7).

Transvecting (2.5) by y^k, using [(1.5)a] and [(1.2)b], we get

(2.8) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = a_{sqlnm}F^2 y^k\varphi^i_{kh}$.

Subtracting (2.8) from (2.5), we get

(2.9) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(H^i_{kh} - H^i_h) = a_{sqlnm}F^2(1 - y^k)\,\varphi^i_{kh}$.

Using [(1.2)d] in (2.9), we get

(2.10) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(\dot{\partial}_k H^i_h - H^i_h) = a_{sqlnm}F^2(1-y^k)\varphi^i_{kh}$.

Which can be written as

(2.11) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\dot{\partial}_k H^i_h = a_{sqlnm}F^2(1-y^k)\,\varphi^i_{kh} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h$.

Using (2.8) in (2.11), we get

(2.12) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\dot{\partial}_k H^i_h = a_{sqlnm}F^2\varphi^i_{kh}$.

In view of (2.5) and (2.12), we get

(2.13) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\dot{\partial}_k H^i_h = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm}F^2\varphi^i_{kh}$.

Thus, it can be concluded that

Corollary 2.1: In $G\mathcal{B}K-5RF_n$, Berwald's covariant derivatives of the fifth order for the torsion tensor H^i_{kh} and for the partial derivative with respect to y^k for the deviation tensor H^i_h have the same decomposition.

In view of (2.8) and (2.12), we get

(2.14) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = y^k\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\dot{\partial}_k H^i_h = y^k a_{sqlnm}\,F^2\varphi^i_{kh}$.

Thus, it can be concluded that

Corollary 2.2: In $G\mathcal{B}K-5RF_n$, Berwald's covariant derivatives of the fifth order for deviation tensor H^i_h and for the partial derivative with respect to y^k for the deviation tensor H^i_h are decomposing in the same direction.

Upon taking the Lie-derivative of both sides of the equation (2.8), we get

(2.15) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h) = F^2y^kL_v(a_{sqlnm}\varphi^i_{kh}) + a_{sqlnm}\varphi^i_{kh}L_v(F^2y^k)$.

Using [(1.1)e], (1.13) and (1.10) in (2.15), we get

$$L_vF^2 = \frac{1}{a_{sqlnm}\varphi^i_{kh}y^k}\left[L_v(a_{sqlnm}H^i_h) - L_v[b_{sqlnm}(y^iy_h)] - F^2y^kL_v(a_{sqlnm}\varphi^i_{kh})\right].$$

Using (1.10), (1.17), (1,18) and (1.25) in above equation, we get

(2.16) $$L_vF^2 = \frac{v^m}{a_{sqlnm}\varphi^i_{kh}y^k}\Big[H^i_h\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m H^i_h - y^iy_h\mathcal{B}_m b_{sqlnm} - F^2y^k(\varphi^i_{kh}\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m\varphi^i_{kh})\Big].$$

In conclusion, it can be stated that

Theorem 2.4: In $G\mathcal{B}K-5RF_n$, the Lie-derivative for the quadrature fundamental function of Finsler space decomposes according to (2.16).

Using (1.12) and [(1.1)e] in (2.5), we get

(2.17) $H^i_{kh} = F^2\varphi^i_{kh}$.

it can be summarized that

Theorem 2.5: In $G\mathcal{B}K-5RF_n$,the h(v)- torsion tensor H^i_{kh} decomposes according to (2.17).

By taking the Lie-derivative of both sides of the equation (2.17), we obtain

$$(2.18)\qquad L_vH^i_{kh} = (L_vF^2)\varphi^i_{kh} + F^2\left(L_v\varphi^i_{kh}\right).$$

Using (2.7), (1.17) and (1.25) in (2.18), we get

$$(2.19)\qquad L_vH^i_{kh} = \frac{v^m}{a_{sqlnm}}\left[H^i_{kh}\,\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m H^i_{kh}\right.$$

$$\left. -F^2(\varphi^i_{kh}\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m\varphi^i_{kh})\right] + F^2 v^m\mathcal{B}_m\varphi^i_{kh}\,.$$

Consequently, it can be concluded that

Theorem 2.6: In $G\mathcal{B}K-5RF_n$,the Lie-derivative for the h(v)-torsion tensor H^i_{kh} decomposes according to (2.19).

In view of (2.17), we have

$$(2.20)\qquad F = \frac{1}{\sqrt{\varphi^i_{kh}}}\left[\sqrt{H^i_{kh}}\right].$$

Taking the Lie-derivative of both sides of (2.20), we get

$$L_vF = \left[L_v\left(\frac{1}{\sqrt{\varphi^i_{kh}}}\right)\right]\left[\sqrt{H^i_{kh}}\right] + \frac{1}{\sqrt{\varphi^i_{kh}}}\left[L_v\sqrt{H^i_{kh}}\right].$$

Using (1.17) and (1.25) in above equation, we get

$$L_vF = \frac{1}{2\sqrt{\varphi^i_{kh}H^i_{kh}}}\left[v^m\mathcal{B}_mH^i_{kh} - \frac{H^i_{kh}\,v^m\mathcal{B}_m\,\varphi^i_{kh}}{\varphi^i_{kh}}\right].$$

Since the torsion tensor H^i_{kh} is is skew-symmetric in its two lower indices h and k , then above equation can be written as

$$L_vF = \frac{v^m}{2\sqrt{\varphi^i_{kh}H^i_{kh}}}\left[\frac{H^i_{hk}\,\mathcal{B}_m\varphi^i_{kh}}{\varphi^i_{kh}} + \mathcal{B}_mH^i_{kh}\right].$$

If the torsion tensor H^i_{kh} and decomposition tensor φ^i_{kh} behaves as recurrent, then above equation can be written as

$$(2.21)\qquad L_vF = \frac{v^m\lambda_m}{2\sqrt{\varphi^i_{kh}H^i_{kh}}}\left[H^i_{hk} + H^i_{kh}\right]$$, where λ_m is non-zero covariant vector field.

it can be concluded that

Theorem 2.7: In $G\mathcal{B}K-5RF_n$, the Lie-derivative for the fundamental function of Finsler space decomposes according to (2.21) if the torsion tensor H^i_{kh} and decomposition tensor φ^i_{kh} behaves as recurrent.

Using [(1.1)e] in (2.3), we get

(2.22) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = (a_{sqlnm} y_j)\, \varphi^i_{kh}$.

In conclusion, it can be stated that

Theorem 2.8: In $G\mathcal{B}K - 5RF_n$, Berwald's covariant derivative of the fifth order for the Cartan's fourth curvature tensor K^i_{jkh} decomposes according to (2.22).

Using [(1.15) a] in (1.3), we get

(2.23) $y_j\, \varphi^i_{kh} + y_h\, \varphi^i_{jk} + y_k\, \varphi^i_{hj} = 0$.

Transvecting (2.23) by $(a_{sqlnm}\, y^j)$, using [(1.1)d], we get

(2.24) $a_{sqlnm}\, F^2 \varphi^i_{kh} + a_{sqlnm}\, y_h\, y^j \varphi^i_{jk} + a_{sqlnm}\, y_k\, y^j \varphi^i_{hj} = 0$.

Using (2.22) and (2.5) in (2.24), we get

(2.25) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} + y^j \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{hjk} + y^j \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{khj} = 0$.

Which can be written as

(2.26) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = -\, y^j \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left(K^i_{hjk} + K^i_{khj} \right)$.

In conclusion, it can be stated that

Theorem 2.9: In $G\mathcal{B}K - 5RF_n$, that admits tensors decomposition, the Berwald's covariant derivatives of the fifth order for the torsion tensor H^i_{kh} and for the tensor $(K^i_{hjk} + K^i_{khj})$ are in the opposite directions.

Using [(1.2)e] in (2.5), we get

(2.27) $\varphi^i_{kh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k \right)$.

If and only if

(2.28) $F^2 = \dfrac{1}{3a_{sqlnm}}$.

Hence, it can be concluded that

Theorem 2.10: In $G\mathcal{B}K - 5RF_n$, the decomposition tensor φ^i_{kh} is giving by (2.27) if and only if the quadrature fundamental function of Finsler space is giving by (2.28).

Using (2.17) in [(1.2)f], we get

(2.29) $F^2 (\mathcal{B}_m \varphi^i_{kh} + \mathcal{B}_h \varphi^i_{mk} + \mathcal{B}_k \varphi^i_{hm}) = 0$.

Since the fundamental function of Finsler space is bigger than zero, so (2.29) can be written as

(2.30) $(\mathcal{B}_m \varphi^i_{kh} + \mathcal{B}_h \varphi^i_{mk} + \mathcal{B}_k \varphi^i_{hm}) = 0$.

Hence, it can be concluded that

Theorem 2.11: In $G\mathcal{B}K - 5RF_n$, that admits tensors decomposition, we have identity (2.30).

By taking the Lie-derivative of both sides of the equation (2.22), we get

(2.31) $\quad L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = \varphi^i_{kh} L_v(a_{sqlnm} y_j) + a_{sqlnm} y_j (L_v \varphi^i_{kh})$.

Using [(1.1)e] and (1.11) in (2.31), we get

$$L_v K^i_{jkh} = \frac{1}{a_{sqlnm}}\left[\, \varphi^i_{kh} L_v(a_{sqlnm} y_j) + a_{sqlnm} y_j (L_v \varphi^i_{kh}) - (L_v a_{sqlnm}) K^i_{jkh} \,\right].$$

Using (1.17), (1.18) and (1.25) in above equation, we get

(2.32) $\quad L_v K^i_{jkh} = \frac{v^m}{a_{sqlnm}}\left[\, \mathcal{B}_m a_{sqlnm} (\varphi^i_{kh} y_j - K^i_{jkh}) + a_{sqlnm} y_j \mathcal{B}_m \varphi^i_{kh} \right]$.

In conclusion, it can be concluded that

Theorem 2.12: In $G\mathcal{B}K - 5RF_n$, the Lie-derivative for the Cartan's fourth curvature tensor K^i_{jkh} decomposes according to (2.32).

3. A Decomposition of Lie-derivative of Some Tensors by Using A Contravariant Tensor y^i and a Covariant Tensor φ_{jkh} in $G\mathcal{B}K$-$5RF_n$

This study explores a novel method for decomposing the Lie-derivative of specific tensors within the context of $G\mathcal{B}K\text{-}5\mathrm{RF}_n$ geometry. Our approach leverages the properties of a contravariant tensor y^i and a covariant tensor φ_{jkh} to provide a more refined analysis of these derivatives. The decomposition of Cartan's fourth curvature tensor K^i_{jkh} in the form [(1.15)b] is studied, where y^i is a non-zero contravariant tensor and φ_{jkh} is a decomposition. tensor.

Using [(1.15)b] in the right side of (1.11), we get

(3.1) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} y^i \varphi_{jkh} + b_{sqlnm}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$

$$- c_{sqlnm}(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - d_{sqlnm}(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})$$

$$- e_{sqlnm}(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}) - 2b_{qlnm}\, y^r \mathcal{B}_r(\delta^i_h C_{jks} - \delta^i_k C_{jhs}) \ .$$

Transvecting (3.1) by y^j, using [(1.1)b], [(1.2)a], [(1.5)a] and (1.6), we get

(3.2) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = (a_{sqlnm}\, y^i y^j)\, \varphi_{jkh} + b_{sqlnm}(\delta^i_h y_k - \delta^i_k y_h)$.

Using [(1.1)e] in (3.2), we get

(3.3) $\quad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm}\, y^i y^j\, \varphi_{jkh}$.

From the previous steps, the following theorem can be concluded that

Theorem 3.1: In $G\mathcal{B}K - 5RF_n$, Berwald's covariant derivative of the fifth order for the h(v)-torsion tensor H^i_{kh} decomposes according to (3.3).

Taking the Lie-derivative of both sides of (3.3), we get

(3.4) $\quad L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = y^i y^j L_v(a_{sqlnm}\, \varphi_{jkh}) + a_{sqlnm}\, \varphi_{jkh} L_v(y^i y^j)$.

Using (1.12), [(1.1)e] and (1.10) in (3.4), we get

$$L_v(a_{sqlnm}H^i_{kh}) - y^i y^j L_v(a_{sqlnm}\,\varphi_{jkh}) = 0\ .$$

Using (1.17) and (1.25) in above equation, we get

(3.5) $\mathcal{B}_m a_{sqlnm}\big(H^i_{kh} - y^i y^j\,\varphi_{jkh}\big) + a_{sqlnm}\big(\mathcal{B}_m H^i_{kh} - y^i y^j \mathcal{B}_m\,\varphi_{jkh}\big) = 0\ .$

Therefore, the proof of theorem is completed, it can be said that

Theorem 3.2: In $G\mathcal{B}K - 5RF_n$, that admits tensors decomposition, we have identity (3.5).

Transvecting (3.2) by y_i, using [(1.1)d], [(1.2)g] and [(1.5)b], we get

(3.6) $(a_{sqlnm}F^2 y^j)\,\varphi_{jkh} + b_{sqlnm}\,y_i\,\big(\delta^i_h y_k - \delta^i_k y_h\big) = 0\,.$

Using [(1.1)b] in (3.6), we get

(3.7) $(a_{sqlnm}F^2)\,\varphi_{jkh} + b_{sqlnm}\,g_{ij}\,\big(\delta^i_h y_k - \delta^i_k y_h\big) = 0\,.$

Which can be written as

(3.8) $\varphi_{jkh} = \big(\delta^i_k y_h - \delta^i_h y_k\big)\,.$

If and only if

(3.9) $F^2 = \alpha_{sqlnm}\,g_{ij}$, where $\alpha_{sqlnm} = \dfrac{b_{sqlnm}}{a_{sqlnm}}$,is non-zero covariant tensor of fifth order.

So, the proof of theorem is completed, it can be said that

Theorem 3.3: In $G\mathcal{B}K - 5RF_n$, the decomposition tensor φ_{jkh} is giving by (3.8) if and only if the quadrature fundamental function of Finsler space is giving by (3.9).

Using (2.5) and [(1.1)d] in (3.3), we get

(3.10) $y_i\varphi^i_{kh} = y^j\varphi_{jkh}\ .$

In conclusion the proof of theorem is completed, it can be concluded that

Theorem 3.4: In $G\mathcal{B}K - 5RF_n$, the admits tensors decomposition, we have the identity (3.10).

Transvecting (3.2) by y^k, using [(1.1)d], [(1.1)g], [(1.2)b] and [(1.5)a], we get

(3.11) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = (a_{sqlnm}y^i y^j y^k)\,\varphi_{jkh} + b_{sqlnm}\big(\delta^i_h F^2 - y^i y_h\big)\ .$

Taking the Lie- derivative of both sides of (3.11), we get

(3.12) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h) = a_{sqlnm}\,\varphi_{jkh}L_v(y^i y^j y^k) + y^i y^j y^k(L_v a_{sqlnm}\,\varphi_{jkh})$

$\qquad + L_v[b_{sqlnm}\big(\delta^i_h\,F^2 - y^i y_h\big)]\,.$

Using (1.13) and (1.10) in (3.12), we get

$$L_v(a_{sqlnm}H^i_h) - y^i y^j y^k\big(L_v a_{sqlnm}\,\varphi_{jkh}\big) = 0\ .$$

Using (1.17) and (1.25) in above equation, we get

(3.13) $\mathcal{B}_m a_{sqlnm}\big(H^i_h - y^i y^j y^k\,\varphi_{jkh}\big) + a_{sqlnm}(\mathcal{B}_m H^i_h - y^i y^j y^k \mathcal{B}_m\,\varphi_{jkh}) = 0\ .$

Thus, it can be concluded that

Theorem 3.5: In $G\mathcal{B}K-5RF_n$,the admits tensors decomposition, we have the identity (3.13).

In view of (1.12) and (3.2), we get

(3.14) $H^i_{kh}=y^iy^j\varphi_{jkh}$.

Also, in view of (3.11) and (1.13), we get

(3.15) $H^i_h=y^iy^jy^k\ \varphi_{jkh}$.

Thus, the proof of theorem is completed, it can be concluded that

Corollary 3.1: In $G\mathcal{B}K-5RF_n$, the torsion tensor H^i_{kh} and the deviation tensor H^i_h decompose according to (3.14) and (3.15), respectively.

Using [(1.2)d] in (3.14), we get

(3.16) $\dot{\partial}_kH^i_h=y^iy^j\varphi_{jkh}$.

Using (3.15) in (3.16), we get

(3.17) $\dot{\partial}_k\left(y^iy^jy^k\ \varphi_{jkh}\right)=(y^iy^j\varphi_{jkh})$.

In conclusion the proof of theorem is completed, it can be determined that

Theorem 3.6: In $G\mathcal{B}K-5RF_n$, the partial derivative with respect to y^k for the tensor $\left(y^iy^jy^k\ \varphi_{jkh}\right)$ is $(y^iy^j\varphi_{jkh})$.

Using (3.14) in [(1.2)a], we get

(3.18) $R^i_{jkh}=y^i\varphi_{jkh}$.

Transvecting (3.18) by y_i , using [(1.1)d], we get

(3.19) $y_iR^i_{jkh}=F^2\varphi_{jkh}$.

Which can be written as

(3.20) $F=\frac{1}{\sqrt{\varphi_{jkh}}}\left[\sqrt{y_iR^i_{jkh}}\ \right].$

By taking the Lie-derivative of both sides of the equation (3.20), we get

$$L_vF=\left[L_v\left(\frac{1}{\sqrt{\varphi_{jkh}}}\right)\right]\left[\sqrt{y_iR^i_{jkh}}\right]+\frac{1}{\sqrt{\varphi_{jkh}}}\left[L_v\sqrt{y_iR^i_{jkh}}\ \right].$$

Using (1.17), (1,18) and (1.25) in above equation, we get

$$L_vF=\frac{v^m\,y_i}{2\sqrt{\varphi_{jkh}\,y_i\,R^i_{jkh}}}\left[\mathcal{B}_mR^i_{jkh}-\frac{R^i_{jkh}\mathcal{B}_m\varphi_{jkh}}{\varphi_{jkh}}\right]\ .$$

Since the Cartan's third curvature tensor R^i_{jkh} , is skew-symmetric it their last two lower indices, then above equation can be written as

$$L_vF=\frac{v^m\,y_i}{2\sqrt{\varphi_{jkh}\,y_iR^i_{jkh}}}\left[\mathcal{B}_mR^i_{jkh}+\frac{R^i_{jhk}\mathcal{B}_m\varphi_{jkh}}{\varphi_{jkh}}\right]\ .$$

If the Cartan's third curvature tensor R^i_{jkh} and decomposition tensor φ_{jkh} behaves as recurrent, then above equation can be written as

$$(3.21)\qquad L_v F = \frac{v^m \lambda_m\, y_i}{2\sqrt{\varphi_{jkh}\, y_i R^i_{jkh}}}\left[\, R^i_{jhk} + R^i_{jkh}\right] \ .$$

Thus, it can be concluded that

Theorem 3.7: In $GBK-5RF_n$, the Lie-derivative for the fundamental function of Finsler space decomposes according to (3.21) if the Cartan's third curvature tensor R^i_{jkh} and decomposition tensor φ_{jkh} behaves as recurrent.

Using [(1.2)a] in (2.5), we get

$$(3.22)\qquad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh}y^j) = (a_{sqlnm}F^2)\ \varphi^i_{kh} \ .$$

Using [(1.1)d] and [(1.5)a] in (3.22), we get

$$(3.23)\qquad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = (a_{sqlnm}y_j)\ \varphi^i_{kh} \, .$$

Taking the Lie-derivative of both sides of (3.23), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = \varphi^i_{kh} L_v(a_{sqlnm}y_j) + (a_{sqlnm}y_j)\ L_v\varphi^i_{kh} \, .$$

Using (1.17), (1,18) and (1.25) in above equation, we get

$$(3.24)\qquad L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = y_j v^m(\varphi^i_{kh}\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m\varphi^i_{kh}) \ .$$

In conclusion the proof of theorem is completed, it can be concluded that

Theorem 3.8: In $GBK-5RF_n$, the Lie-derivative of Berwald's covariant derivative of the fifth order for the Cartan's third curvature tensor R^i_{jkh} decomposes according to (3.24).

Using [(1.1)e] in (1.14) ,we get

$$(3.25)\qquad \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm}R^i_{jkh} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh}) - a_{sqlnm}(C^i_{jt}H^t_{kh}) \ .$$

Taking the Lie-derivative of both sides of (3.25), we get

$$(3.26)\qquad L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = R^i_{jkh}(L_v a_{sqlnm}) + a_{sqlnm}(L_v R^i_{jkh}\,)$$
$$+ L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh})] - L_v[\, a_{sqlnm}(C^i_{jt}H^t_{kh})] \ .$$

In view of (3.26) and (3.24), we get

$$L_v R^i_{jkh} = \frac{1}{a_{sqlnm}}\left[\, y_j v^m(\varphi^i_{kh}\mathcal{B}_m a_{sqlnm} + a_{sqlnm}\mathcal{B}_m\varphi^i_{kh}) - R^i_{jkh}(L_v a_{sqlnm})\right.$$
$$\left. - L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh})] + L_v[\, a_{sqlnm}(C^i_{jt}H^t_{kh})]\right] \ .$$

Using (1.17), (1,18) and (1.25) in above equation, we get

$$(3.27)\qquad L_v R^i_{jkh} = \frac{v^m}{a_{sqlnm}}\left[\mathcal{B}_m a_{sqlnm}\left(\varphi^i_{kh}y_j - R^i_{jkh}\right) + a_{sqlnm}y_j\mathcal{B}_m\varphi^i_{kh}\right] ,$$

if and only if the tensor $(C^i_{jt}H^t_{kh})$ behaves as fifth recurrent.

The proof of theorem is completed, it can be concluded that

Theorem 3.9: In $G\mathcal{B}K-5RF_n$, the Lie-derivative for the Cartan's third curvature tensor R^i_{jkh} decomposes according to (3.27) if and only if the tensor $(C^i_{jt}H^t_{kh})$ behaves as fifth recurrent.

Using [(1.15)b] and [(1.٥)a] in (2.22), we get

(3.28) $y^i\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\varphi_{jkh}=(a_{sqlnm}y_j)\varphi^i_{kh}$.

Using (1.7) in (3.28), we get

(3.29) $(l^iF)\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\varphi_{jkh}=(a_{sqlnm}y_j)\ \varphi^i_{kh}$.

Transvecting (3.29) by l_i , we get

(3.30) $(l_il^iF)\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\varphi_{jkh}=l_i(a_{sqlnm}y_j)\varphi^i_{kh}$.

Using (1.9) in (3.30), we get

(3.31) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\varphi_{jkh}=\varphi^i_{kh}$.

If and only if

(3.32) $F=l_i(a_{sqlnm}y_j)$.

In conclusion, it can be determined that

Theorem 3.10: In $G\mathcal{B}K-5RF_n$, Berwald's covariant derivative of the fifth order for the decomposition tensor φ_{jkh} is equal the decomposition tensor φ^i_{kh} if and only if the fundamental function of Finsler space is giving by (3.32).

Using [(1.1)e] in (3.1), we get

(3.33) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mK^i_{jkh}=(a_{sqlnm}y^i)\ \varphi_{jkh}$.

The proof of theorem is completed, it can be concluded that

Theorem 3.11: In $G\mathcal{B}K-5RF_n$, Berwald's covariant derivative of the fifth order for the Cartan's fourth curvature tensor K^i_{jkh} decomposes according to (3.33).

Taking the Lie-derivative of both sides of (3.33), we get

(3.34) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mK^i_{jkh})=\varphi_{jkh}L_v(a_{sqlnm}y^i)+a_{sqlnm}y^i(L_v\varphi_{jkh})$.

Using [(1.1)e] and (1.11) in (3.34), we get

$$L_vK^i_{jkh}=\frac{1}{a_{sqlnm}}\left[\varphi_{jkh}L_v(a_{sqlnm}y^i)+a_{sqlnm}y^i(L_v\varphi_{jkh})-(L_va_{sqlnm})K^i_{jkh}\right].$$

Using (1.17), (1,10) and (1.25) in above equation, we get

(3.35) $L_vK^i_{jkh}=\frac{v^m}{a_{sqlnm}}\left[\mathcal{B}_ma_{sqlnm}(\varphi_{jkh}\ y^i-K^i_{jkh})+a_{sqlnm}y^i(\mathcal{B}_m\varphi_{jkh})\right]$.

The proof of theorem is completed, it can be determined that

Theorem 3.12: In $G\mathcal{B}K-5RF_n$, the Lie-derivative for the Cartan's fourth curvature tensor K^i_{jkh} decomposes according to (3.35).

4. A Decomposition of Lie-derivative for Some Tensors by Using a Mixed Tensor X^i_j and A Covariant Tensor φ_{kh} in $G\mathcal{B}K$- $5RF_n$

This paper presents a novel decomposition of the Lie derivative for specific tensors, utilizing a mixed tensor. We provide a detailed analysis of this decomposition and explore its implications for some tensors. We study the decomposition of Cartan's fourth curvature tensor K^i_{jkh} in the form [(1.16)a], where X^i_j it is a non-vanishing mixed tensor and φ_{kh} is a decomposition tensor.

Using ([(1.16) a] and [(1.1)e] in right side of (1.11), we get

(4.1) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = (\, a_{sqlnm} X^i_j \,)\, \varphi_{kh}$.

In conclusion, it can be stated that

Theorem 4. 1: In $G\mathcal{B}K-5RF_n$, Berwald's covariant derivative of the fifth order for the Cartan's fourth curvature tensor K^i_{jkh} decomposes according to (4.1).

In view of (4.1) and (2.22), we get

(4.2) $X^i_j\, \varphi_{kh} = y_j \varphi^i_{kh}$.

In conclusion, it can be stated that

Corollary 4.1: In $G\mathcal{B}K-5RF_n$,that admits tensors decomposition, we have the identity (4.2).

Taking the Lie-derivative of both sides of (4.1), we get

(4.3) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = \varphi_{kh} L_v(a_{sqlnm} X^i_j) + a_{sqlnm} X^i_j (L_v\, \varphi_{kh})$.

Using (1.11) and [(1.1)e] in (4.3), we get

$$L_v K^i_{jkh} = \frac{1}{a_{sqlnm}}\left[\, \varphi_{kh} L_v(a_{sqlnm} X^i_j) + a_{sqlnm} X^i_j (L_v\, \varphi_{kh}) - (L_v a_{sqlnm}) K^i_{jkh} \right].$$

Using (1.17) and (1.25) in above equation, we get

(4.4) $L_v K^i_{jkh} = \frac{v^m}{a_{sqlnm}}[\mathcal{B}_m a_{sqlnm}(\, \varphi_{kh} X^i_j - K^i_{jkh}\,) + a_{sqlnm}(\varphi_{kh}\mathcal{B}_m X^i_j + X^i_j \mathcal{B}_m \varphi_{kh})]$.

In conclusion, it can be stated that

Theorem 4.2: In $G\mathcal{B}K-5RF_n$,the Lie-derivative for the Cartan's fourth curvature tensor K^i_{jkh} decomposes according to (4.4).

Transvecting (4.2) by y^j, using [(1.1)d], we get

(4.5) $F^2 \varphi^i_{kh} = y^j X^i_j\, \varphi_{kh}$.

Which can be written as

(4.6) $\varphi^i_{kh} = y^j \varphi_{kh}$.

If and only if

(4.7) $F^2 = X^i_j$.

In conclusion, it can be stated that

Theorem 4.3: In $G\mathcal{B}K - 5RF_n$, the decomposition tensor φ^i_{kh} is giving by (4.6) if and only if the quadrature fundamental function of Finsler space is equal the mixed tensor X^i_j .

Transvecting (3.10) by y^i, using [(1.1)d], we get

(4.8) $F^2\varphi^i_{kh} = y^i y^j \varphi_{jkh}$.

Using [Theorem 4.3] in (4.8), we get

(4.9) $X^i_j \, \varphi_{kh} = y^i \varphi_{jkh}$.

Transvecting (4. 9) by y_i , using [(1.1)d], we get

(4.10) $y_i X^i_j \, \varphi_{kh} = F^2 \varphi_{jkh}$.

Which can be written as

(4.11) $\varphi_{jkh} = y_i \, \varphi_{kh}$.

If and only if $\left[F^2 = X^i_j \right]$.

In conclusion, it can be stated that

Theorem 4.4: In $G\mathcal{B}K - 5RF_n$, the decomposition tensor φ_{jkh} is giving by (4.11) if and only if the quadrature fundamental function of Finsler space is equal the mixed tensor X^i_j .

Using (1.7) in (3.14), we get

(4.12) $H^i_{kh} = l^i l^j \, F^2 \varphi_{jkh}$.

Using [Theorem 4.4] in (4.12), we get

(4.13) $H^i_{kh} = l^i l^j \, y_i \, X^i_j \varphi_{kh}$.

Using (1.7) in (3.15), we get

(4.14) $H^i_h = l^i l^j \, F^2 y^k \, \varphi_{jkh}$.

Using [Theorem 4.4] in (4.14), we get

(4.15) $H^i_h = l^i l^j \, y_i \, X^i_j y^k \varphi_{kh}$.

In conclusion, it can be stated that

Theorem 4.5: In $G\mathcal{B}K - 5RF_n$, the torsion tensor H^i_{kh} and the deviation tensor H^i_h decompose according to (4.13) and (4.15), respectively.

By taking the Lie-derivative of both sides of the equation (4.13), we get

$$L_v H^i_{kh} = y_i\, X^i_j \varphi_{kh} L_v(l^i l^j) + l^i l^j \varphi_{kh} L_v\big(y_i\, X^i_j\big) + l^i l^j y_i\, X^i_j L_v(\varphi_{kh})\,.$$

Using (1.17), (1,18), (1.25) and (1.26) in above equation, we get

$$L_v H^i_{kh} = l^i l^j \varphi_{kh} y_i\, v^m \mathcal{B}_m X^i_j + l^i l^j y_i\, X^i_j v^m \mathcal{B}_m \varphi_{kh}\,.$$

Using [(1.1)d] and (1.7) in above equation, we get

(4.16) $L_v H^i_{kh} = y^j v^m(\varphi_{kh} \mathcal{B}_m X^i_j + X^i_j \mathcal{B}_m \varphi_{kh})\ .$

In conclusion, it can be stated that

Theorem 4.6: In $G\mathcal{B}K - 5RF_n$, the Lie-derivative for the torsion tensor H^i_{kh} decomposes according to (4.16).

By taking the Lie-derivative of both sides of the equation (4.15), we get

$$L_v H^i_h = y_i\, X^i_j y^k \varphi_{kh} L_v(l^i l^j) + l^i l^j y^k \varphi_{kh} L_v\big(y_i\, X^i_j\big) + l^i l^j y_i\, X^i_j L_v(y^k \varphi_{kh})\,.$$

Using (1.10), (1.17), (1,18), (1.25) and (1.26) in above equation, we get

$$L_v H^i_h = l^i l^j y^k \varphi_{kh} y_i\, v^m \mathcal{B}_m X^i_j + l^i l^j y_i\, X^i_j y^k v^m \mathcal{B}_m \varphi_{kh}\,.$$

Using [(1.1)d] and (1.7) in above equation, we get

(4.17) $L_v H^i_h = y^j y^k v^m(\varphi_{kh} \mathcal{B}_m X^i_j + X^i_j \mathcal{B}_m \varphi_{kh})\,.$

In conclusion, it can be stated that

Theorem 4.7: In $G\mathcal{B}K - 5RF_n$, Lie-derivative for the deviation tensor H^i_h decomposes according to (4.17).

Transvecting (3.18) by y_i, using [(1.1)d], we get

(4.18) $y_i\, R^i_{jkh} = F^2 \varphi_{jkh}\,.$

Using [Theorem 4.4] in (4.18), we get

(4.19) $y_i\, R^i_{jkh} = y_i\, X^i_j\, \varphi_{kh}\,.$

Since y_i is a non-zero covariant tensor, then (4.20), can be written as

(4.20) $R^i_{jkh} = X^i_j \varphi_{kh}\,.$

In conclusion, it can be stated that

Theorem 4.8: In $G\mathcal{B}K - 5RF_n$, the Cartan's third curvature tensor R^i_{jkh} decomposes according to (4.20).

Using (4.20) in the right side of (3.27), we get

(4.21) $L_v R^i_{jkh} = \frac{v^m}{a_{sqlnm}}\big[\mathcal{B}_m a_{sqlnm}\big(\varphi^i_{kh} y_j - X^i_j \varphi_{kh}\big) + a_{sqlnm} y_j \mathcal{B}_m \varphi^i_{kh}\big]\ .$

In conclusion, it can be stated that

Theorem 4.9: In $\mathcal{GBK}-5RF_n$, the Lie-derivative for the Cartan's third curvature tensor R^i_{jkh} decomposes according to (4.21).

Taking the partial derivative with respect to y^r of both sides of (3.16), we get

(4.22) $\dot{\partial}_r \dot{\partial}_k H^i_h = \dot{\partial}_r y^i y^j \varphi_{jkh}$.

Using [(1.2)d] in (4.22), we get

(4.23) $\dot{\partial}_r H^i_{kh} = \dot{\partial}_r y^i y^j \varphi_{jkh}$.

Taking the partial derivative with respect to y^j of both sides of (4.23), we get

(4.24) $\dot{\partial}_j \dot{\partial}_r H^i_{kh} = \dot{\partial}_j \dot{\partial}_r y^i y^j \varphi_{jkh}$.

Transvecting (4.24) by y^r, using [(1.2)h], we get

(4.25) $y^r \dot{\partial}_j \dot{\partial}_r y^i y^j \varphi_{jkh} = 0$. Since $y^r \neq 0$, so above equation can be written as

(4.26) $\dot{\partial}_j \dot{\partial}_r y^i y^j \varphi_{jkh} = 0$.

Transvecting above equation by y_i , using [(1.1)d,h,i], we get

(4.27) $\dot{\partial}_r F^2 \varphi_{jkh} = 0$.

Using [Theorem 4.4] in above equation, we get

(4.28) $\dot{\partial}_r (X^i_j y_i \, \varphi_{kh}) = 0$.

In conclusion, it can be stated that

Theorem 4.10: In $\mathcal{GBK}-5RF_n$, the partial derivative with respect to y^r for the tensor $(X^i_j y_i \, \varphi_{kh})$ is vanishing.

Using [(1.16)a] in (1.3), we get

(4.29) $X^i_j \varphi_{kh} + X^i_h \varphi_{jk} + X^i_k \varphi_{hj} = 0$.

Transvecting (4.29) by $(y^j y^h y^k)$, using [Theorem 4.3], we get

(4.30) $F^2 (y^h y^k \varphi^i_{kh} + y^j y^k \varphi^i_{jk} + y^j y^h \, \varphi^i_{hj}) = 0$.

Since the fundamental function of Finsler space is bigger than zero,so (4.30) can be written as

(4.31) $y^h y^k \varphi^i_{kh} + y^j y^k \varphi^i_{jk} + y^j y^h \, \varphi^i_{hj} = 0$.

Thus, it can be stated that

Theorem 4.11: In $\mathcal{GBK}-5RF_n$, that admits tensors decomposition, we have identity (4.31).

In view of (4. 31) and (2.23), we get

(4.31) $y^h y^k = y_j$.

In conclusion, it can be stated that

Theorem 4.12: In $\mathcal{GBK}-5RF_n$, that admits tensors decomposition, the product of different two contravariant vectors is a covariant vector.

Transvecting (4.31) by y^j and using [(1.1)d], we get

(4.32) $F=\sqrt{y^h y^k y^j}$.

In conclusion, it can be stated that

Theorem 4.13: In $\mathcal{GBK}-5RF_n$, that admits tensors decomposition, the fundamental function of Finsler space is giving by (4.32).

Taking the partial derivative with respect to y^h of (4.31) and using [(1.1)c,h,i], we get

(4.33) $g_j = n\, y^k$.

In conclusion, it can be stated that

Theorem 4.14: In $\mathcal{GBK}-5RF_n$, that admits tensors decomposition, the metric tensor g_j is giving by (4.33).

Transvecting (4.33) by y_k and using [(1.1)d], we get

(4.34) $F=\sqrt{\frac{g_j\, y_k}{n}}$.

In conclusion, it can be stated that

Theorem 4.15: In $\mathcal{GBK}-5RF_n$, that admits tensors decomposition, the fundamental function of Finsler space is giving by (4.34).

5. A Illustrative Example

Let’s assume we have a vector field representing the velocity of a rivers flow moving in a specific direction. If we have another vector field representing a force acting on the river (for example, a ship), then the Lie-derivative of the first field along the second field represents the rate of change of the rivers flow velocity due to this force.

If the force acts in the opposite direction of the flow, then the Lie-derivative will be negative, meaning the flow velocity will decrease. If the force acts in the same direction as the flow, then the Lie- derivative will be positive, meaning the flow velocity will increase.

For example, Let’s assume a large tree trunk falls in to the river during the flow. Then the tensors will undergo decomposition into tensors of a lower order. For instance, the tensor of the fifth recurrent during decomposition maybe will return to the first, second, third or fourth recurrent.

6. Conclusions

In this paper the decomposition of Fundamental Function of Finsler space and some tensors by Lie-derivative in generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald has been introduced. Also, different identities and several theorems have been discussed under the decomposition. Based on the results of this study, we propose the following recommendations for future research:

1. Investigate the relationship between the proposed decomposition and other decomposition techniques in Finsler geometry, such as the decomposition of the curvature tensor.
2. Develop a computational algorithm to implement the decomposition procedure and apply it to specific examples of Finsler spaces.
3. Explore the potential applications of this decomposition in the study of Finslerian gravity theories, particularly in the context of modified gravity models.
4. Examine the role of the GBK framework in other areas of Finsler geometry, such as Finsler information geometry.

7. References

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