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Comparison between Maximum Likelihood and Maximum Entropy Estimation Methods for Reliability of IDAL Distribution

Alaa M. Hamad[1] and Iden H. Alkanani [2]
[1] Department of Mathematics, College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq
[2] Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq
* Corresponding author

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Abstract
IDAL distribution is based on expanding the exponential Weibull distribution by adding a IDAL distribution is based on expanding the exponential Weibull distribution by adding a third parameter, which is a shape parameter, to the exponential-Weibull distribution. This modification was done in order to create models that are more flexible and realistic. The IDAL distribution is characterized by three parameters, which are the scale parameter and two shape parameters. Estimating the reliability for a Noval distribution. The unknown parameters of its distribution have been estimated which have the first of these methods is the Maximum Likelihood Estimate (MLE) method, followed by the Maximum Entropy Estimation (MEE) method. A comparison of the outcomes and results of the applied methods has been carried out through data analysis and computer simulation between the estimating methods based on the applicable indicator mean square error (MSE) to investigate which way is the most effective. Additionally, the data that were observed have been displayed through the use of the MATLAB software package.
Keywords: IDAL Distribution, Estimation methods, Maximum Likelihood, Maximum Entropy.

1. Introduction

Recently, the scientific and mathematical efforts of developing a new probability distribution have been carried out to take benefit of these distributions in several mathematical applications and different life fields. A new family of probability distributions has been introduced in several kinds of literature by adding a new parameter to the basic and original distribution[1-4]. Certain methodologies for constructing new distributions apply classical lifetime distributions, such as exponential, Rayleigh, and Weibull, by incorporating additional parameters to enhance the flexibility of the proposed distribution[5-8].Several researchers have expanded the extent of the Weibull distribution and introduced a mixture of distributions in a newer area of study that has gained significant traction in statistical research articles, particularly in reliability analysis applications[9-13]. The distribution elucidates the methodology for combining Weibull distributions using a mixing Weibull distribution and adding two parameters that denote the proportions of the amalgamation of the two components of Weibull distributions that have been introduced in[14-16]. A novel distribution has been proposed by constructing a log-logistic distribution utilizing the Weibull distribution as a technique for producing composite distributions[17]. The updated model used the ratio of two separate random variables as a novel lifespan mode that was presented in[18] .The mixture of Weibull distributions is a newer area of study that has gained significant traction in statistical research articles, particularly in reliability analysis applications. This research elucidates the methodology for combining Weibull distributions using a mixing parameter and the process of adding two Weibull distributions. which denotes the proportions of the amalgamation of the two components of Weibull distributions[19-20]. The combination of distribution functions is regarded as a method for producing new distributions. They investigated the derivation of a new distribution by constructing a log-logistic distribution from a Weibull distribution[21].

In this article a new lifetime distribution called the IDAL distribution with three parameters has been proposed to provide comparisons of performance. The estimation of the reliability function for a new distribution has been carried out through data analysis and computer simulation between two estimation methods, the maximum likelihood estimation method (MLE) and the maximum entropy estimation (MEE) method, according to the applied indicator. and obtain the best using mean squares error (MSE) and determine the best method.

2. Materials and Methods

In this section, present the principal functions of IDAL distribution after we have constructed our new distribution by adding the shape parameter to the exponential Weibull distribution named IDAL distribution and studied all its properties in the preview research. Therefore, we will discuss the main function and summary of the properties of the new distribution as follows:

2.1. The probability density function (p.d.f.) of IDAL distribution

$$f(x,\alpha,\beta,\lambda) = \left(\frac{\beta}{\alpha} x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x^{\frac{\lambda}{\alpha}-1}\right) e^{-\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \quad \textbf{(1)}$$

2.2. The cumulative distribution function (C.D.F.) of the IDAL Distribution

$$F(x,\alpha,\lambda,B) = 1 - e^{-\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \quad \textbf{(2)}$$

2.3. The reliability function of the IDAL Distribution

$$R(x;\alpha,\lambda,\beta) = 1 - F(x;\alpha,\lambda\beta) \; = e^{-\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \qquad ,x \geqslant 0 \quad \textbf{(3)}$$

2.4. The hazard rate function for IDAL distribution

$$h(x;\alpha,\lambda,\beta) = \frac{f(x;\alpha,\lambda,\beta)}{S(x;\alpha,\lambda,\beta)} \quad , H(x,\alpha,B,\lambda) = \frac{\frac{B}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}e^{-\frac{\lambda}{\alpha}-1}}{1-e^{\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)}} \quad \textbf{(4)}$$

2.5. Properties for IDAL Distribution

In this section, we present the mathematical properties of IDAL Distribution through **Table 1**.

Table 1. Mathematical properties of new IDAL Distribution (by researcher)

Term	Definition	Formal
Moment origin	$E(x^r)$	$\sum_{n=0}^{\infty} \frac{(-B)^n}{n^n}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n+\alpha r+1}{\lambda}\right) + \Gamma\left(\frac{n+\alpha r}{\lambda}+1\right)\right]$
Mean	$E\,(x)$	$\sum_{n=0}^{\infty} \frac{(-B)^n}{n^n}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n+\alpha+1}{\lambda}\right) + \Gamma\left(\frac{n+\alpha}{\lambda}+1\right)\right] = \frac{\beta}{\lambda}\Gamma\left(\frac{n+\alpha+1}{\lambda}\right)$
Variance	$\mathrm{Var}(x)$	$\sum_{n=0}^{\infty} \frac{(-B)^n}{n^n}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n+\alpha 2+1}{\lambda}\right) + \Gamma\left(\frac{n+\alpha 2}{\lambda}+1\right)\right] - \left[\sum_{n=0}^{\infty} \frac{(-B)^n}{n^n}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n+\alpha+1}{\lambda}\right) + \Gamma\left(\frac{n+\alpha}{\lambda}+1\right)\right]\right]^2$
Moment generating function	$\mu_x(t)$	$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{(-\beta)^m}{m!}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n\alpha+m+1}{\lambda}\right) + \Gamma\left(\frac{n\alpha+m}{\lambda}+1\right)\right]$
Median	$F(x)$	$t = (-B \mp \sqrt{}(B\text{^}2 + 4ln\,2))/2$
Quantile Function	$F^{-1}(u)$	$Bx^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}} - \ln(1-u) = 0$
Coefficient of Skewness	c. S	$\frac{\sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n+3\alpha+1}{\lambda}\right)+\Gamma\left(\frac{n+3\alpha+\lambda}{\lambda}\right)\right]}{\sum_{n=0}^{\infty} \frac{-(B)^n}{n!}\left[\frac{\beta}{\lambda}\Gamma\left(\frac{n+2\alpha+1}{\lambda}\right)+\Gamma\left(\frac{n+2\alpha+\lambda}{\lambda}\right)\right]^{\frac{3}{2}}}$

2.6. Estimation Methods for IDAL Distribution

In this section, some estimation methods for IDAL distribution such as the Maximum Likelihood Estimation (MLE) and Maximum Entropy Estimation (MEE) methods, will be used. These methods are employed to predict the best performance for the available system as follows.

2.6.1. Maximum Likelihood Estimation Method (MLE)

The maximum likelihood method has a lot of important properties in comparison with other methods, so statisticians almost prefer this method in many statistical applications[22-23].

Let $(x_1, x_2, \ldots\ldots, x_n)$; be random variable samples for sizes n. From **Equation 1** the probability density function for IDAL distribution will be

$$f(x;\lambda,\beta,\alpha)=\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)e^{-\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)}$$

$$Lf(x;\lambda,\beta,\alpha)=\prod_{I=1}^{n}\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)e^{-\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \quad \textbf{(5)}$$

The natural log- likelihood function is:

$$lnLf(x;\lambda,\beta,\alpha)=\sum_{i=1}^{n}ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)-\sum_{i=1}^{n}\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right) \quad \textbf{(6)}$$

Taking the derivative for the obtained *Ln* function with respect to parameters (α), (β) and (λ), respectively, and equate to zero, then

$$\frac{\partial \ln L}{\partial \alpha}=\sum_{i=1}^{n} -\frac{\frac{\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}-\frac{\beta}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i-\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i-\frac{\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i}{\frac{1}{\alpha^2}\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{1}{\alpha}-1}\right)}+\sum_{i=1}^{n}\frac{1}{\alpha^2}\left(\beta x_i^{\frac{1}{\alpha}}+\lambda x_i^{\frac{1}{\alpha}}\right) \quad \textbf{(7)}$$

$$\frac{\partial \ln L}{\partial \beta}=\sum_{i=1}^{n}\frac{\frac{1}{\alpha}x^{\frac{1}{\alpha}-1}}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)}-\sum_{i=1}^{n}x_i^{\frac{1}{\alpha}} \quad \textbf{(8)}$$

$$=\sum_{i=1}^{n}\frac{x^{\frac{1}{2}-1}}{\left(Bx_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)}-\sum_{i=1}^{n}x_i^{\frac{1}{\alpha}} \quad \textbf{(9)}$$

$$\frac{\partial \ln L}{\partial \lambda}=\sum_{i=1}^{n}\frac{\frac{1}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{1}{\alpha}-1}\right)}-\sum_{i=1}^{n}\frac{1}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln xi \quad \textbf{(10)}$$

$$\begin{bmatrix}\hat{\alpha}_{k+1}\\ \hat{B}_{k+1}\\ \hat{\lambda}_{k+1}\end{bmatrix}=\begin{bmatrix}\hat{\alpha}_{k}\\ \hat{B}_{k}\\ \lambda_{k}\end{bmatrix}-J^{-1}\begin{bmatrix}\frac{\partial \ln L}{\partial \alpha}\\ \frac{\partial \ln L}{\partial B}\\ \frac{\partial \ln L}{\partial \lambda}\end{bmatrix} \text{Where } J=\begin{bmatrix}\frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha\,\partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha\,\partial \lambda}\\ \frac{\partial^2 \ln L}{\partial \beta\,\partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta\,\partial \lambda}\\ \frac{\partial^2 \ln L}{\partial \lambda\,\partial \alpha} & \frac{\partial^2 \ln L}{\partial \lambda\,\partial \beta} & \frac{\partial^2 \ln L}{\partial \lambda^2}\end{bmatrix}$$

$$\frac{\partial^2 \ln L}{\partial B^2}=\sum_{i=1}^{n}\frac{-\frac{1}{\alpha}x_i^{\frac{1}{\alpha}-1}\left(\frac{1}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\right)}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2}=\sum_{i=1}^{n}\frac{-\left(x_i^{\frac{1}{\alpha}-1}\right)^2}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2} \quad \textbf{(11)}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2}=\sum_{i=1}^{n}\frac{\frac{2\beta}{\alpha^3}x_i^{\frac{1+\lambda}{\alpha}-2}\ln x_i+\frac{\lambda\beta}{\alpha^4}x_i^{\frac{1+\lambda}{\alpha}-2}(\ln x_i)^2-\frac{1}{\alpha^2}x_i^{\frac{2\lambda}{\alpha}-2}}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2}-\sum_{i=1}^{n}\frac{1}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}(\ln x_i)^2 \quad \textbf{(12)}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \sum_{i=1}^{n}\left[\frac{\beta}{\alpha^2} x_i^{\frac{1}{\alpha}-1} + \frac{\beta}{\alpha^3} x_i^{\frac{1}{\alpha}-1} \ln x_i + \frac{2\beta}{\alpha^3} x_i^{\frac{\lambda}{\alpha}-1} \ln x_i + \frac{\beta}{\alpha^4} x_i^{\frac{1}{\alpha}-1} (\ln x_i)^2 + \frac{\lambda}{\alpha^2} x_i^{\frac{\lambda}{\alpha}-1} + \frac{2\lambda^2}{\alpha^3} x_i^{\frac{\lambda}{\alpha}-1} \ln x_i + \frac{\lambda^3}{\alpha^4} x_i^{\frac{\lambda}{\alpha}-1} (\ln x_i)^2 + \left(\frac{\beta}{\alpha^2} x_i^{\frac{1}{\alpha}-1} \ln x_i\right)\left(-\frac{\beta}{\alpha} x_i^{\frac{\lambda}{\alpha}-1} - \frac{\beta}{\alpha^2} x_i^{\frac{1}{\alpha}-1} \ln x_i - \frac{\lambda}{\alpha} x_i^{\frac{\lambda}{\alpha}-1} - \frac{\lambda^2}{\alpha^2} x_i^{\frac{\lambda}{\alpha}-1} \ln x_i\right)\right]$$

$$/\left(\beta x_i^{\frac{1}{\alpha}-1} + \lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2 + \sum_{i=1}^{n} \frac{-2}{\alpha^3}\left(\beta x_i^{\frac{1}{\alpha}} \ln x_i + \lambda x_i^{\frac{\lambda}{\alpha}-1} \ln x_i\right) + \frac{1}{\alpha^2}\left(\frac{-\beta}{\alpha^2} x_i^{\frac{1}{\alpha}} (\ln x_i)^2 - \frac{\lambda^2}{\alpha^2} x_i^{\frac{\lambda}{\alpha}-1} (\ln x_i)^2\right) \quad \textbf{(13)}$$

$$\frac{\partial^2 \ln L}{\partial \beta\, \partial \alpha} = \sum_{i=1}^{n} \left(\frac{-1}{\alpha^2} x_i^{\frac{1}{\alpha}-1} + \frac{-1}{\alpha 3} x_i^{\frac{1}{\alpha}-1} \ln x_i\right)\left(\frac{\beta}{\alpha} x_i^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x_i^{\frac{1}{\alpha}-1}\right) - \frac{1}{\alpha} x_i^{\frac{1}{\alpha}-1} \left(\frac{-B}{\alpha^2} x_i^{\frac{1}{\alpha}-1} - \frac{\beta}{\alpha^3} x_i^{\frac{1}{\alpha}-1} \ln x_i - \frac{\lambda}{\alpha^2} x_i^{\frac{\lambda}{\alpha}-1} - \frac{\lambda^2}{\alpha^3} x_i^{\frac{\lambda}{\alpha}-1} \ln x_i\right) / \left(\frac{\beta}{\alpha} x_i^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x_i^{\frac{\lambda}{\alpha}-1}\right)^2 - \sum_{i=1}^{n} \frac{-1}{\alpha^2} x_i^{\frac{1}{\alpha}} \ln \quad \textbf{(14)}$$

$$\frac{\partial^2 \ln L}{\partial \beta\, \partial \alpha} = \sum_{i=1}^{n} \frac{1}{\alpha^2} x_i^{\frac{1}{\alpha}} \ln x_i = \frac{\partial^2 \ln L}{\partial \alpha\, \partial \beta} \quad \textbf{(15)}$$

$$\frac{\partial^2 \ln L}{\partial \lambda\, \partial \beta} = \sum_{i=1}^{n} \frac{-x_i^{\frac{1+\lambda}{\alpha}-2}\left(\frac{1}{\alpha^2} + \frac{1}{\alpha^3} \ln x_i\right)}{\left(\frac{\beta}{\alpha} x_i^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x_i^{\frac{\lambda}{\alpha}-1}\right)^2} = \frac{\partial^2 \ln L}{\partial \beta\, \partial \lambda} \quad \textbf{(16)}$$

$$\frac{\partial^2 \ln L}{\partial \lambda\, \partial \alpha} = \sum_{i=1}^{n}\left[-\frac{2\lambda\beta}{\alpha^2} x_i^{\frac{1+\lambda}{\alpha}-2} \ln x_i - \frac{\lambda^2 \beta}{\alpha^3} x_i^{\frac{1+\lambda}{\alpha}-2} (\ln x_i)^2 - \frac{\lambda^2}{\alpha^2} x_i^{\frac{2\lambda}{\alpha}-2} \ln x_i + \frac{\beta}{\alpha^2} x_i^{\frac{1+\lambda}{\alpha}-2} \ln x_i + \frac{\beta\lambda}{\alpha^3} x_i^{\frac{1+\lambda}{\alpha}-2} (\ln x_i)^2\right] / \left(\beta x_i^{\frac{1}{\alpha}-1} + \lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2 - \sum_{i=1}^{n} \frac{1}{\alpha^2} x_i^{\frac{\lambda}{\alpha}} \ln x_i \left[1 + \frac{\lambda}{\alpha} \ln x_i\right] = \frac{\partial^2 \ln L}{\partial \alpha\, \partial \lambda} \quad \textbf{(17)}$$

3. The Maximum Entropy Estimation Method (MEE)

The Maximum Entropy Estimation Method (MEE) is a statistical inference tool that addresses statistical entropy in relation to uncertainty, particularly when there is a defect associated with the probability density function. To determine the probability density function that maximizes entropy information while adhering to the specified information constraints, we can employ the method of Lagrange multipliers for calculation. Ultimately, we derive the probability density function that satisfies the established constraints[24-26]. The maximum entropy estimation method consists of four steps to estimate the three parameters of the IDAL distribution. We start with the probability density function for distribution, which is **Equation 1.**

$$\mathrm{f}(x;\ \alpha, \beta, \lambda) = \left(\frac{\beta}{\alpha} x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x^{\frac{\lambda}{\alpha}-1}\right) e^{-\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)}$$

$$\delta = -\int_0^\infty \ln \mathrm{f}(x) . \mathrm{f}(x) dx \quad \textbf{(18)}$$

$$\ln \mathrm{f}(x) = \ln\left[\left(\frac{\beta}{\alpha} x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x^{\frac{\lambda}{\alpha}-1}\right) e^{-\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)}\right] \quad \textbf{(19)}$$

By substitution **Equation 18**

$$\delta = -\int_0^\infty \left[\ln \left(\frac{\beta}{\alpha} x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha} x^{\frac{\lambda}{\alpha}-1}\right) - \left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)\right] \mathrm{f}(x) dx \quad \textbf{(20)}$$

E (gi(x))=$\int_0^\infty$ gi$(x) f(x) dx =$ Ci ,=1,2,..,n

$\int_0^\infty f(x)dx = 1$

C2=$\int_0^\infty \left(\ln\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right) f(x)dx = \text{ E}\left[\ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)\right]$

$$\int_0^\infty \left(\ln\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right) f(x)dx = \text{E}\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right) = \text{C3}$$

$\int_0^\infty f(x)dx = 1 = \text{C1}$, Where Ci=;i=1,2,3 **(21)**

Step: Structure of the LaGrange multipliers based on **Equation 18**

$f(x) = exp(-\lambda_0 - \sum_{i=1}^{n} \lambda_i g_i(x))$and

$$f(x) = exp\left(-\lambda_0 - \left\{\lambda_i \ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right) - \lambda_2\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)\right\}\right) \quad \textbf{(22)}$$

Where λ_0, λ_1 and λ_2 are LaGrange multipliers subsiding **Equation 21** in **Equation 22** the following are obtained:

$$\int_0^\infty f(x)dx = \int_0^\infty \exp\left(-\lambda_0 - \left\{\lambda_i \ln\left(-\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right) - \lambda_2\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)\right\}\right) dx = 1$$

Such that

$$\int_0^\infty \exp(-\lambda_0).\ \exp\left[-\lambda_1 \ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right) + \lambda_2\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)\right] dx = 1$$

$$\int_0^\infty \frac{e^{\lambda 2\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)}}{\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)^{\lambda 1}} \text{dx} = e^{\lambda_0} \quad \textbf{(23)}$$

Recall that

$$\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1} + \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)^{\lambda 1} = \sum_{j=0}^{\lambda 1} \text{C}_j^{\lambda 1}\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}\right)^j \left(\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)^{\lambda 1 - j}$$

$$= \left(\frac{\beta}{\alpha} + \frac{\lambda}{\alpha}\right)^{\lambda_1} (x)^{\frac{j(1-\lambda)+\lambda_1(\lambda-\alpha)}{\alpha}}$$

By substituting this formula in **Equation 23,** we have

$$\frac{1}{\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1}}^{\lambda_1} \int_0^\infty (x)^{\left(\frac{j(1-\lambda)}{\alpha}\right)+\left(\frac{\lambda_1(\lambda-\alpha)}{\alpha}\right)} = e^{\lambda_2\left(\beta x^{\frac{1}{\alpha}} + x^{\frac{\lambda}{\alpha}}\right)} \text{d}x = e^{\lambda_0} \quad \text{, then}$$

$$e^{\lambda_2 \beta x^{\frac{1}{\alpha}}} = \sum_{r=0}^{\infty}\left(\frac{\lambda_2 \beta x^{\frac{1}{\alpha}}}{r!}\right)^r = \sum_{r=0}^{\infty}\frac{(\beta\lambda_2)^r x^{\frac{r}{\alpha}}}{r!}$$

$$\frac{\sum_{r=0}^{\infty}\frac{(\beta\lambda_2)^r}{r!}}{\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1}} \int_0^\infty (x)^{x^{\frac{r}{\alpha}} + \frac{j(1-\lambda)}{\alpha} + \frac{\lambda_1(\lambda-\alpha)}{\alpha}}.\ e^{\lambda_2 x^{\frac{\lambda}{\alpha}}} \text{d}x = e^{\lambda_0} \quad \textbf{(24)}$$

Let Z=$x^{\frac{\lambda}{\alpha}}$ $\longrightarrow$ x=$z^{\frac{\alpha}{\lambda}}$ $\longrightarrow$ d$x = \frac{\alpha}{\lambda}\ z^{\frac{\alpha}{\lambda}-1}$ dz

$$\frac{\sum_{r=0}^{\infty}\frac{(\beta\lambda_2)^r}{r!}}{\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1}} \int_0^\infty Z^{\frac{\alpha}{\lambda}}\ ^{(\frac{r}{\alpha} + \frac{j(1-\lambda)}{\alpha} + \frac{\lambda_1(\lambda-\alpha)}{\alpha}}.\ e^{\lambda_2 z}\ \frac{\alpha}{\lambda} z^{\frac{\alpha}{\lambda}-1}\text{d}z \quad \textbf{(25)}$$

$$\frac{\alpha\sum_{r=0}^{\infty}\frac{(\beta\lambda_2)^r}{r!}}{\lambda\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1}}\int_0^{\infty} Z^{\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}-1} .\ e^{-(-\lambda_2 z)}\,\mathrm{d}z = e^{-\lambda} \qquad \textbf{(26)}$$

The border integral is a negative gamma distribution, then we must multiply and divide by $\frac{\beta^{\alpha}}{\mu\alpha}$ we have $\alpha = \frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}$, $\beta = +(-\lambda_2)$

$$\frac{\sum_{r=0}^{\infty}\frac{(\beta\lambda_2)^r}{r!}}{\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1}} \cdot\frac{\alpha}{\lambda}\cdot\frac{\mu\left(\frac{r}{\lambda}+\frac{j(1-\lambda)}{\lambda}+\frac{\lambda_1(\lambda-\alpha)}{\lambda}+\frac{\alpha}{\lambda}\right)}{(-\lambda_2)^{\frac{r}{\lambda}+\frac{j(1-\lambda)}{\lambda}+\frac{\lambda_1(\lambda-\alpha)}{\lambda}+\frac{\alpha}{\lambda}}} = e^{\lambda_0}$$

$$e^{\beta\lambda_2}\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1} \cdot\frac{\alpha}{\lambda}\cdot\frac{\mu\left(\frac{r}{\lambda}+\frac{j(1-\lambda)}{\lambda}+\frac{\lambda_1(\lambda-\alpha)}{\lambda}+\frac{\alpha}{\lambda}\right)}{(-\lambda_2)^{\frac{r}{\lambda}+\frac{j(1-\lambda)}{\lambda}+\frac{\lambda_1(\lambda-\alpha)}{\lambda}+\frac{\alpha}{\lambda}}} = e^{\lambda_0} \qquad \textbf{(27)}$$

Equation 27 expresses the Zero$^{\text{th}}$ Lagrange λ_0 multiplier as a function of Lagrange multiplier $\lambda_1 and\ \lambda_2$.

Step 3: Derivation of the entropy function of the distribution. Substituting **Equation 27** into **Equation 21**

$$\mathrm{f}(x)=\frac{\lambda\left(\frac{\beta+\lambda}{\alpha}\right)^{\lambda_1}\ (-\lambda_2)^{\left(\frac{r}{\lambda}+\frac{j(1-\lambda)}{\lambda}+\frac{\lambda_1(\lambda-\alpha)}{\lambda}+\frac{\alpha}{\lambda}\right)}.\ e^{-\lambda_2\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)}}{\alpha\ e^{\beta\lambda_2}\ \mu\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)^{\lambda_1}} \qquad \textbf{(28)}$$

By taking the normal logarithms, then we get

$$\ln\ \mathrm{f}(x)=\ln\ \lambda+\lambda_1\ln\ \left(\frac{\beta+\lambda}{\alpha}\right)+\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)^{\ln(-\lambda_2)} + -\lambda_2\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)-\ln x-\beta\lambda_2-\ln\mu \left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)-\lambda_1\ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)$$

Hence, from the definition of entropy, **Equation 13**

$$\ln\ \mathrm{f}(x)=\ln\left[\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)-\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)\right]$$

$$S=\ln\lambda+\lambda_1\ln\left(\frac{\beta+\lambda}{\alpha}\right)+\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)^{\ln(-\lambda_2)}+\lambda_2\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)-\ln\alpha-\beta\lambda_2- \ln\mu\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)-\lambda_1\ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right) \qquad \textbf{(29)}$$

Step 4: Derivation of the relation between the Lagrange multipliers and constraints

Let $a = \frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}, a = \frac{r}{\lambda}+\frac{j}{\lambda}-1+\lambda_1-\frac{\alpha\lambda_1}{\lambda}+\frac{\alpha}{\lambda}$

$$\frac{\partial a}{\partial\alpha}=\frac{\lambda_1(-1)}{\lambda}+\frac{1}{\lambda}=+\frac{1}{\lambda}-\frac{\lambda_1}{\lambda}$$

We know that $\frac{\mu(a)}{\mu(a)}=\ \psi(a)$

$$\frac{\partial a}{\partial\alpha}\ln\mu\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)=\left(\frac{1}{\lambda}-\frac{\lambda_1}{\lambda}\right)\psi(a),\ \text{Since}\frac{\partial}{\partial\alpha}\ln\mu(x)=\psi(\alpha)$$

The **Equation 29** contains five parameters: α, β, λ, and λ_1. To optimize **Equation 29,** we must establish the following partial derivative accordingly:

$$\frac{\partial S}{\partial\alpha}=\frac{\lambda_1}{\alpha}-\frac{(1-\lambda_1)}{\lambda}\ln\lambda_2 E\ \left(\frac{-\beta}{\alpha^2}x^{\frac{1}{\alpha}}\ln x+\frac{-\lambda}{\alpha^2}x^{\frac{\lambda}{\alpha}}\ln x\right)+\frac{1}{\alpha}+\frac{(1-\lambda_1)}{\lambda}\ \psi(\alpha)-\lambda_1 E \left[-\frac{1}{\alpha}+\frac{\frac{\ln x}{\alpha^2}-\left(\beta x^{\frac{1}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}\right)}{\beta x^{\frac{1}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}}\right] \qquad \textbf{(30)}$$

$$\frac{\partial S}{\partial \lambda}=-\frac{1}{\lambda}-\frac{\lambda_1}{\beta+\lambda}+\frac{r+j+\alpha\lambda_1+\alpha}{\lambda^2}\ln(-\lambda_2)-\lambda_2 E\left(\frac{\ln x}{\alpha}x^{\frac{\lambda}{\alpha}}\right)-\frac{\partial}{\partial\lambda}\ln\mu\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right)+$$
$$\lambda_1 E\left(\frac{x^{\frac{\lambda}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\ln x}{\beta x^{\frac{\lambda}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}}\right)\frac{-r-j+\lambda_1\alpha+\alpha}{\lambda^2} \quad \textbf{(31)}$$

$$\frac{\partial S}{\partial \beta}=-\frac{\lambda_1}{\beta+\lambda}-\lambda_2 E\left(x^{\frac{1}{\alpha}}\right)+\lambda_2+\lambda_1 E\left(\frac{x^{\frac{1}{\alpha}-1}}{\beta x^{\frac{\lambda}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}}\right) \quad \textbf{(32)}$$

$$b=\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}$$

$\frac{\partial b}{\partial\lambda_1}=\frac{\lambda-\alpha}{\lambda}=1-\frac{\alpha}{\lambda}$, *Since* $\frac{\partial}{\partial t}\ln\mu(t)=\psi(\mathrm{t})$ where $\psi(\mathrm{t})=\frac{\mu(\acute{b})}{\mu(b)}$ is the digamma, it is followed:

$$\frac{\partial b}{\partial\lambda_1}=\left(1-\frac{\alpha}{\lambda}\right)\psi(\mathrm{b})$$

$$\frac{\partial S}{\partial\lambda_1}=\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right).\frac{-1}{\lambda^2}-E\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)+\beta \quad \textbf{(33)}$$

$$\frac{\partial S}{\partial\lambda_2}=\beta-E\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)-\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right).\frac{1}{\lambda^2} \quad \textbf{(34)}$$

The values of parameters are $\alpha,\beta,\lambda>0$, $\lambda_2<0, \infty<\lambda_1<\infty$. By the definition of expectation of random variable
$(y)=\sum_{all\ y} yp(Y=y)$. Assume that p(Y=y) =1, then the **Equations** become as follows:

$$E\left(t^{\frac{1}{\lambda}}\right)=\sum_{i=1}^{n} t_i^{\frac{1}{\lambda}}$$

$$E\left(\ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)=\sum_{i=1}^{n}\ln\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)\right)$$

$$E\left(x^{\frac{1}{\alpha}}\right)=\sum_{i=1}^{n} x_i^{\frac{1}{\alpha}}$$

$$E\left(\frac{x^{\frac{1}{\alpha}-1}}{\left(\beta x^{\frac{1}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}\right)}\right)=\sum_{i=1}^{n}\frac{x^{\frac{1}{\alpha}-1}}{\beta x^{\frac{1}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}} \quad \textbf{(35)}$$

$$f(\lambda)=\frac{\partial S}{\partial\lambda}=\frac{1}{\lambda}-\frac{\lambda}{\beta+\lambda}+\frac{r+j+\alpha+\alpha\lambda_1}{\lambda^2}\ln(-\lambda_2)-\lambda_2\sum_{i=1}^{n}\left(\frac{\ln x}{\alpha}x_i^{\frac{\lambda}{\alpha}}\right) \quad \textbf{(36)}$$

Then the **Equation** becomes as follows:

$$E\left(t^{\frac{1}{\lambda}}\right)=\sum_{i=1}^{n} t_i^{\frac{1}{\lambda}}$$

$$E\left(\ln\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)=\sum_{i=1}^{n}\ln\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)\right)$$

$$E\left(x^{\frac{1}{\alpha}}\right)=\sum_{i=1}^{n} x_i^{\frac{1}{\alpha}}$$

$$E\left(\frac{x^{\frac{1}{\alpha}-1}}{\left(\beta x^{\frac{1}{\alpha}-1}+\lambda x^{\frac{\lambda}{\alpha}-1}\right)}\right)=\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\alpha}-1}}{\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}}$$

$$\frac{\partial S}{\partial\beta}=f(\beta)=\frac{-\lambda}{\beta+\lambda}+\lambda_2\sum_{i=1}^{n} x_i^{\frac{1}{\alpha}}+\lambda_2+\lambda_1\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\alpha}-1}}{\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}} \quad \textbf{(37)}$$

$$f(\lambda)=\frac{\partial S}{\partial \lambda}=\frac{1}{\lambda}-\frac{\lambda}{\beta+\lambda}+\frac{r+j+\alpha+\alpha\lambda_1}{\lambda^2}\ln(-\lambda_2)-\lambda_2\sum_{i=1}^{n}\left(\frac{\ln x}{\alpha}x_i^{\frac{\lambda}{\alpha}}\right)+\frac{-r+j(\lambda_1-1)\alpha}{\lambda^2}\psi(\mathrm{c})+$$

$$\lambda_1\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x}{\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}}$$

$$c=\left(\frac{r+j(1-\lambda)+\lambda_1(\lambda-\alpha)+\alpha}{\lambda}\right),\ Where\ \frac{\partial \mathrm{c}}{\partial \lambda}=\frac{-r-j+\lambda_1\,\alpha-\alpha}{\lambda^2}$$

$$\frac{\partial \mathrm{c}}{\partial \lambda}ln\,\mu(c)\frac{-r-j+\lambda_1\,\alpha-\alpha}{\lambda^2}\ \psi(\mathrm{c})$$

$$\frac{\partial S}{\partial \alpha}=\frac{\lambda_1}{\alpha}\left(\frac{\lambda_1}{\lambda}+\frac{1}{\lambda}\right)\ln(-\lambda_2)+\lambda_2\sum_{i=1}^{n}\left(\beta x_i^{\frac{1}{\alpha}-1}\ln x_i+\lambda x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)$$

$$+\frac{1}{\alpha}+\left(\frac{1}{\lambda}+\frac{\lambda_1}{\lambda}\right)\psi(\mathrm{a})+\lambda_1\sum_{i=1}^{n}\beta x_i^{\frac{1}{\alpha}-1}\ln x_i+\lambda x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\frac{\left(\beta x_i^{\frac{1}{\alpha}-1}+\frac{\beta}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i+\frac{\lambda^2}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)}{\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}} \quad \textbf{(38)}$$

In order to determine the estimate of the three parameters, the aforementioned Equations form a system of nonlinear Equations that can be solved using the Newton-Raphson technique. This method relies on the Jacobean matrix, which can be expressed as follows:

$$\frac{\partial f(\alpha)}{\partial \alpha}=\frac{-\lambda_1}{\alpha^2}+\lambda_2\sum_{i=1}^{n}\frac{-2}{\alpha^3}\left(\beta x_i^{\frac{1}{\alpha}-1}\ln x_i+\lambda x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)-\left(\frac{\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i+\frac{\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\right)\frac{1}{\alpha^2}$$

$$-\frac{1}{\alpha^2}-\lambda_1\sum_{i=1}^{n}$$

$$\left[\left[\frac{-1}{\alpha^2}\left(\beta x_i^{\frac{1}{\alpha}-1}+\beta x_i^{\frac{1}{\alpha}-1}\ln x_i+\frac{\lambda^2}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)+\left(\frac{-\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i+\frac{-\beta}{\alpha^3}x_i^{\frac{\lambda}{\alpha}-1}.(\ln x_i)^2-\right.\right.\right.$$

$$\left.\left.\frac{\lambda^3}{\alpha^3}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i-\frac{\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}+\frac{-\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i+\frac{-\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i\right)\frac{1}{\alpha}\right]\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)-$$

$$\left.\left(\frac{-\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i+\frac{-\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)\frac{1}{\alpha}\left(\beta x_i^{\frac{1}{\alpha}-1}+\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}\ln x_i+\frac{\lambda^2}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)\right]\Big/$$

$$\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2$$

$$\frac{\partial f(\alpha)}{\partial \lambda}=\left(\frac{1-\lambda_1}{\lambda^2}\right)\ln(-\lambda_2)+\lambda_2\sum_{i=1}^{n}\frac{1}{\alpha^2}\left(\lambda x_i^{\frac{\lambda}{\alpha}-1}(\ln x_i)^2+x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)+\left(\frac{\lambda_1-1}{\lambda^2}\right)\psi(\mathrm{a})$$

$$\frac{\partial f(\alpha)}{\partial \beta}=\lambda_2\sum_{i=1}^{n}\frac{1}{\alpha^2}\left(x_i^{\frac{1}{\alpha}}\ln x_i\right)-\frac{\lambda_1}{\alpha}\sum_{i=1}^{n}\left(\frac{\frac{\lambda}{\alpha}\ln x_i\,x_i^{\frac{1+\lambda}{\alpha}-2}(1-\lambda)}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2}\right) \quad \textbf{(39)}$$

$$\frac{\partial f(\beta)}{\partial \beta}=\frac{\lambda_1}{(\beta+\lambda)^2}+\lambda_1\sum_{i=1}^{n}\frac{-\left(x_i^{\frac{\lambda}{\alpha}-1}\right)\left(x_i^{\frac{\lambda}{\alpha}-1}\right)}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2} \quad \textbf{(40)}$$

$$\frac{\partial f(\beta)}{\partial \lambda}=\frac{+\lambda_1}{(\beta+\lambda)^2}+\lambda_1\sum_{i=1}^{n}\frac{\left(x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)\left(x_i^{\frac{\lambda}{\alpha}-1}\right)}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2} \quad \textbf{(41)}$$

$$\frac{\partial f(\beta)}{\partial \alpha}=\frac{\lambda^2}{\alpha^2}\sum_{i=1}^{n}x_i^{\frac{1}{\alpha}}\ln x_i-\frac{\lambda_1}{\alpha^2}\sum_{i=1}^{n}\frac{x_i^{\frac{\lambda}{\alpha}-1}\ln x_i}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)}-\frac{\lambda_1}{\alpha^2}\sum_{i=1}^{n}\frac{\left(\beta x_i^{\frac{1}{\alpha}-1}\ln x_i+\lambda^2 x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)x_i^{\frac{1}{\alpha}}}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2}$$

$$\frac{\partial f(\lambda)}{\partial \beta}=+\frac{\lambda_1}{(\beta+\lambda)^2}+\lambda_1\sum_{i=1}^{n}\frac{\left(x_i^{\frac{\lambda}{\alpha}-1}\right)\left(x_i^{\frac{\lambda}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2} \tag{42}$$

$$\frac{\partial f(\lambda)}{\partial \alpha}=+\left(\frac{1+\lambda_1}{\lambda^2}\right)\ln(-\lambda_2)-\lambda_2\sum_{i=1}^{n}\left(\frac{-\ln x_i}{\alpha^2}x_i^{\frac{\lambda}{\alpha}}-\frac{\lambda}{\alpha^3}x_i^{\frac{\lambda}{\alpha}}(\ln x_i)^2\right)\frac{(\lambda_1+1)}{\lambda^2}\ \psi(\mathrm{c})$$
$$+\lambda_1\sum_{i=1}^{n}\frac{\left(\frac{-\lambda}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i+\frac{-\lambda}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i+\frac{-\lambda^2}{\alpha^3}x_i^{\frac{\lambda}{\alpha}-1}(\ln x_i)^2\right)\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2}-\left(\frac{-\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i-\right.$$
$$\left.\frac{-\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\right)\left(x_i^{\frac{\lambda}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)$$

$$\frac{\partial f(\lambda)}{\partial \lambda}=\frac{1}{\lambda^2}+\frac{\lambda_1}{(\beta+\lambda)^2}-\frac{-2(r+j+\alpha+\alpha\lambda_1)}{\lambda^3}\ln(-\lambda_2)-\lambda_2\sum_{i=1}^{n}\left(\frac{(\ln x_i)^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}}\right)$$
$$+\frac{2(r+j-1+\lambda_1)}{\lambda^3}\ \psi(\mathrm{c})-\lambda_1\sum_{i=1}^{n}\frac{\lambda}{\alpha^2}\frac{x_i^{\frac{\lambda}{\alpha}}\ln x_i\left[2\frac{\lambda}{\alpha}\ln x_i\right]}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)}--\lambda_1\sum_{i=1}^{n}\frac{x_i^{2\frac{\lambda}{\alpha}-2}\left(1+\frac{\lambda}{\alpha}\right)^2}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}}\ln x_i\right)^2} \tag{43}$$

4. Results and Discussion

The simulation procedure results for the estimators of the reliability model of the new IDAL distribution have good results, but for the studied methods, we made different sample sizes (n=30, 50, 100) to represent small, moderate, and large sample sizes. several values of parameter α = 0.25, 0.75, λ = 0.1, 0.2, and β = 1; five values of lifetime t = (0.1, 0.2, 0.3, 0.4, 0.5). The simulation program was written by using the MATLAB program and built on 1000 replications depending on mean square error (MSE) to the comparison indicators for the reliability function between the two studied methods of estimation [27-28].

The simulation process and the outcome results of simulation will be listed as shown in **Tables 2-5** and **Figures 1** and **2**.

Note that, from the previous **Tables 4** and **5** the best estimation method for estimated of parameter (α) was (MLE) according to (mse).

Table 2. Shown estimators and mean square estimators for reliability with $(\alpha = 0.25)$&β=1

β	λ	n	t	R	$\hat{R}_1$	$MSE_{\hat{R}_1}$	$\hat{R}_2$	$MSE_{\hat{R}_2}$	*Best*
1	0.1	30	0.1	0.671523	0.707702	5.07E-04	0.662579	2.14E-04	2
1	0.1	30	0.2	0.590429	0.620328	8.50E-04	0.583337	3.57E-04	2
1	0.1	30	0.3	0.534779	0.558797	1.07E-03	0.529514	4.47E-04	2
1	0.1	30	0.4	0.487364	0.506087	1.21E-03	0.483966	5.01E-04	2
1	0.1	30	0.5	0.440274	0.454195	1.28E-03	0.438727	5.30E-04	2
1	0.1	50	0.1	0.671523	0.693764	1.56E-03	0.700354	1.01E-03	2
1	0.1	50	0.2	0.590429	0.608765	1.75E-03	0.614282	1.34E-03	2
1	0.1	50	0.3	0.534779	0.549725	1.87E-03	0.554257	1.55E-03	2
1	0.1	50	0.4	0.487364	0.49948	1.95E-03	0.503183	1.67E-03	2
1	0.1	50	0.5	0.440274	0.449949	1.99E-03	0.452957	1.74E-03	2
1	0.1	100	0.1	0.671523	0.692377	2.33E-03	0.691879	2.09E-03	2
1	0.1	100	0.2	0.590429	0.607566	2.56E-03	0.607098	2.32E-03	2
1	0.1	100	0.3	0.534779	0.54863	2.71E-03	0.548085	2.47E-03	2
1	0.1	100	0.4	0.487364	0.498408	2.81E-03	0.497654	2.56E-03	2
1	0.1	100	0.5	0.440274	0.44889	2.87E-03	0.447822	2.62E-03	2
1	0.2	30	0.1	0.853347	0.859805	2.90E-03	0.85594	2.69E-03	2
1	0.2	30	0.2	0.757641	0.764652	2.93E-03	0.76044	2.78E-03	2
1	0.2	30	0.3	0.677207	0.683654	2.95E-03	0.679767	2.85E-03	2

β	λ	n	t	R	$\hat{R}_1$	$MSE_{\hat{R}_1}$	$\hat{R}_2$	$MSE_{\hat{R}_2}$	$Best$
1	0.2	30	0.4	0.602872	0.608265	2.97E-03	0.605002	2.91E-03	2
1	0.2	30	0.5	0.528956	0.533073	2.98E-03	0.530574	2.94E-03	2
1	0.2	50	0.1	0.853347	0.86212	3.05E-03	0.855623	2.97E-03	2
1	0.2	50	0.2	0.757641	0.767239	3.12E-03	0.760123	3.01E-03	2
1	0.2	50	0.3	0.677207	0.686172	3.19E-03	0.679549	3.05E-03	2
1	0.2	50	0.4	0.602872	0.610607	3.23E-03	0.604947	3.07E-03	2
1	0.2	50	0.5	0.528956	0.535192	3.27E-03	0.530711	3.09E-03	2
1	0.2	100	0.1	0.853347	0.855594	3.30E-03	0.865007	3.15E-03	2
1	0.2	100	0.2	0.757641	0.760089	0.003342	0.770426	3.22E-03	2
1	0.2	100	0.3	0.677207	0.679513	3.38E-03	0.689121	3.29E-03	2
1	0.2	100	0.4	0.602872	0.604906	3.40E-03	0.613074	3.33E-03	2
1	0.2	100	0.5	0.528956	0.530659	3.42E-03	0.537064	3.36E-03	2

Table 3. Shown estimators and mean square estimators for reliability with ($\alpha = 0.25$)&β=2

β	λ	n	t	R	$\hat{R}_1$	$MSE_{\hat{R}_1}$	$\hat{R}_2$	$MSE_{\hat{R}_2}$	$Best$
2	0.1	30	0.1	0.671456	0.684414	3.77E-03	0.668896	3.52E-03	2
2	0.1	30	0.2	0.589485	0.599938	4.00E-03	0.587224	3.63E-03	2
2	0.1	30	0.3	0.530465	0.538453	4.15E-03	0.528134	3.70E-03	2
2	0.1	30	0.4	0.475045	0.480592	4.24E-03	0.472282	3.74E-03	2
2	0.1	30	0.5	0.413599	0.416798	4.30E-03	0.410248	3.77E-03	2
2	0.1	50	0.1	0.671456	0.681646	4.40E-03	0.655272	4.29E-03	2
2	0.1	50	0.2	0.589485	0.597519	4.47E-03	0.57636	4.65E-03	1
2	0.1	50	0.3	0.530465	0.536114	4.51E-03	0.519759	4.89E-03	1
2	0.1	50	0.4	0.475045	0.478068	4.53E-03	0.466234	5.05E-03	1
2	0.1	50	0.5	0.413599	0.413996	4.55E-03	0.406377	5.15E-03	1
2	0.1	100	0.1	0.671456	0.673477	4.66E-03	0.685366	5.60E-03	1
2	0.1	100	0.2	0.589485	0.590965	4.74E-03	0.600985	5.91E-03	1
2	0.1	100	0.3	0.530465	0.531193	4.80E-03	0.540047	6.10E-03	1
2	0.1	100	0.4	0.475045	0.474812	4.84E-03	0.483193	6.21E-03	1
2	0.1	100	0.5	0.413599	0.412361	0.004863	0.420575	6.28E-03	1
2	0.2	30	0.1	0.853261	0.855243	4.91E-03	0.865059	6.32E-03	1
2	0.2	30	0.2	0.75643	0.758567	4.98E-03	0.76936	6.38E-03	1
2	0.2	30	0.3	0.671744	0.673693	5.03E-03	0.683747	6.42E-03	1
2	0.2	30	0.4	0.587635	0.589237	5.06E-03	0.597792	0.006455	1
2	0.2	30	0.5	0.496909	0.498087	5.09E-03	0.504766	6.47E-03	1
2	0.2	50	0.1	0.853261	0.856688	5.12E-03	0.863736	6.52E-03	1
2	0.2	50	0.2	0.75643	0.760098	5.16E-03	0.767907	6.57E-03	1
2	0.2	50	0.3	0.671744	0.675	5.19E-03	0.682437	6.62E-03	1
2	0.2	50	0.4	0.587635	0.590156	5.21E-03	0.596756	6.65E-03	1
2	0.2	50	0.5	0.496909	0.498562	5.22E-03	0.504062	6.67E-03	1
2	0.2	100	0.1	0.853261	0.854339	5.26E-03	0.859977	6.75E-03	1
2	0.2	100	0.2	0.75643	0.757595	5.29E-03	0.763728	6.85E-03	1
2	0.2	100	0.3	0.671744	0.672822	5.33E-03	0.678459	6.93E-03	1
2	0.2	100	0.4	0.587635	0.588547	5.35E-03	0.593242	6.99E-03	1
2	0.2	100	0.5	0.496909	0.497613	5.36E-03	0.50116	7.03E-03	1

Note that, from the previous table, the best estimation method for the estimated parameter (α) was (MEE) when β=1& (MLHE) and when β=2, according to (mse).

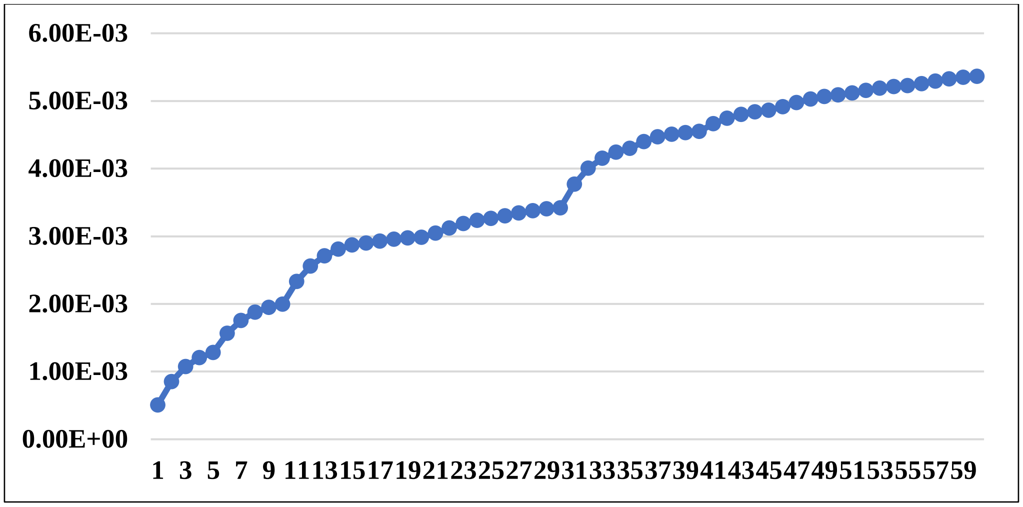

Figure 1. Best MSE for (Reliability) with ($\alpha = 0.25$)

Table 4. Shown estimators and mean square estimators for reliability with ($\alpha = 0.75$)&β=1

β	λ	n	t	R	$\hat{R}_1$	$MSE_{\hat{R}_1}$	$\hat{R}_2$	$MSE_{\hat{R}_2}$	**Best**
1	0.1	30	0.1	0.457464	0.461324	5.43E-03	0.456356	7.09E-03	1
1	0.1	30	0.2	0.396995	0.399702	5.46E-03	0.396762	7.12E-03	1
1	0.1	30	0.3	0.349056	0.351012	5.47E-03	0.349335	7.14E-03	1
1	0.1	30	0.4	0.307365	0.308753	5.48E-03	0.307942	7.15E-03	1
1	0.1	30	0.5	0.270205	0.27114	5.48E-03	0.270937	7.15E-03	1
1	0.1	50	0.1	0.457464	0.462837	5.51E-03	0.462791	7.23E-03	1
1	0.1	50	0.2	0.396995	0.400277	5.52E-03	0.400723	7.27E-03	1
1	0.1	50	0.3	0.349056	0.351183	5.53E-03	0.351851	7.29E-03	1
1	0.1	50	0.4	0.307365	0.308811	5.53E-03	0.309534	7.30E-03	1
1	0.1	50	0.5	0.270205	0.271257	5.54E-03	0.271929	7.30E-03	1
1	0.1	100	0.1	0.457464	0.462391	5.61E-03	0.46509	7.35E-03	1
1	0.1	100	0.2	0.396995	0.400404	5.64E-03	0.402366	7.38E-03	1
1	0.1	100	0.3	0.349056	0.351546	5.65E-03	0.353068	7.39E-03	1
1	0.1	100	0.4	0.307365	0.309215	5.66E-03	0.310429	7.39E-03	1
1	0.1	100	0.5	0.270205	0.271584	5.66E-03	0.272565	7.40E-03	1
1	0.2	30	0.1	0.555667	0.559341	5.70E-03	0.557647	7.42E-03	1
1	0.2	30	0.2	0.463939	0.466523	5.71E-03	0.465267	7.44E-03	1
1	0.2	30	0.3	0.396052	0.397893	5.72E-03	0.396945	7.45E-03	1
1	0.2	30	0.4	0.340296	0.341599	5.73E-03	0.340885	7.45E-03	1
1	0.2	30	0.5	0.292852	0.293755	5.73E-03	0.293224	7.45E-03	1
1	0.2	50	0.1	0.555667	0.560915	5.77E-03	0.563268	7.47E-03	1
1	0.2	50	0.2	0.463939	0.46757	5.78E-03	0.469318	7.48E-03	1
1	0.2	50	0.3	0.396052	0.398592	5.79E-03	0.399917	7.49E-03	1
1	0.2	50	0.4	0.340296	0.342056	5.80E-03	0.343063	7.49E-03	1
1	0.2	50	0.5	0.292852	0.294043	5.80E-03	0.294804	7.49E-03	1
1	0.2	100	0.1	0.555667	0.560414	5.82E-03	0.558229	7.53E-03	1
1	0.2	100	0.2	0.463939	0.467357	5.83E-03	0.465733	7.54E-03	1
1	0.2	100	0.3	0.396052	0.398546	5.83E-03	0.397335	7.55E-03	1
1	0.2	100	0.4	0.340296	0.342104	5.83E-03	0.341217	7.56E-03	1
1	0.2	100	0.5	0.292852	0.294138	5.84E-03	0.293511	7.56E-03	1

Table 5. Shown estimators and mean square estimators for reliability with ($\alpha = 0.75$)&β=2

β	λ	n	t	R	$\hat{R}_1$	$MSE_{\hat{R}_1}$	$\hat{R}_2$	$MSE_{\hat{R}_2}$	*Best*
2	0.1	30	0.1	0.436715	0.440091	5.89E-03	0.441332	7.63E-03	1
2	0.1	30	0.2	0.353175	0.35583	5.92E-03	0.356805	7.66E-03	1
2	0.1	30	0.3	0.285546	0.28757	5.93E-03	0.288456	7.67E-03	1
2	0.1	30	0.4	0.228907	0.230345	5.93E-03	0.2312	7.68E-03	1
2	0.1	30	0.5	0.181696	0.182619	5.94E-03	0.183452	7.68E-03	1
2	0.1	50	0.1	0.436715	0.43656	5.99E-03	0.4355	7.74E-03	1
2	0.1	50	0.2	0.353175	0.352894	6.01E-03	0.352217	7.76E-03	1
2	0.1	50	0.3	0.285546	0.285216	6.02E-03	0.284732	7.77E-03	1
2	0.1	50	0.4	0.228907	0.228576	6.02E-03	0.228199	7.78E-03	1
2	0.1	50	0.5	0.181696	0.181392	6.03E-03	0.181079	7.78E-03	1
2	0.1	100	0.1	0.436715	0.447839	6.09E-03	0.452059	7.84E-03	1
2	0.1	100	0.2	0.353175	0.360598	0.006111	0.363905	7.87E-03	1
2	0.1	100	0.3	0.285546	0.290686	6.12E-03	0.293277	7.88E-03	1
2	0.1	100	0.4	0.228907	0.232467	6.13E-03	0.234431	7.89E-03	1
2	0.1	100	0.5	0.181696	0.184127	6.13E-03	0.185546	7.89E-03	1
2	0.2	30	0.1	0.530465	0.535752	6.16E-03	0.538951	7.92E-03	1
2	0.2	30	0.2	0.41273	0.416264	6.18E-03	0.418382	7.93E-03	1
2	0.2	30	0.3	0.323991	0.32635	6.18E-03	0.327749	7.93E-03	1
2	0.2	30	0.4	0.253431	0.254976	6.19E-03	0.255885	7.94E-03	1
2	0.2	30	0.5	0.196924	0.197904	6.19E-03	0.198477	7.94E-03	1
2	0.2	50	0.1	0.530465	0.538768	6.22E-03	0.53681	7.97E-03	1
2	0.2	50	0.2	0.41273	0.418205	6.23E-03	0.416946	7.98E-03	1
2	0.2	50	0.3	0.323991	0.327587	6.24E-03	0.32679	7.99E-03	1
2	0.2	50	0.4	0.253431	0.255744	6.24E-03	0.255256	7.99E-03	1
2	0.2	50	0.5	0.196924	0.198359	6.24E-03	0.198078	7.99E-03	1
2	0.2	100	0.1	0.530465	0.537264	6.26E-03	0.534774	8.03E-03	1
2	0.2	100	0.2	0.41273	0.417186	6.27E-03	0.415525	8.04E-03	1
2	0.2	100	0.3	0.323991	0.326917	6.27E-03	0.325807	8.05E-03	1
2	0.2	100	0.4	0.253431	0.255327	6.27E-03	0.254593	8.05E-03	1
2	0.2	100	0.5	0.196924	0.198124	6.27E-03	0.197649	8.05E-03	1

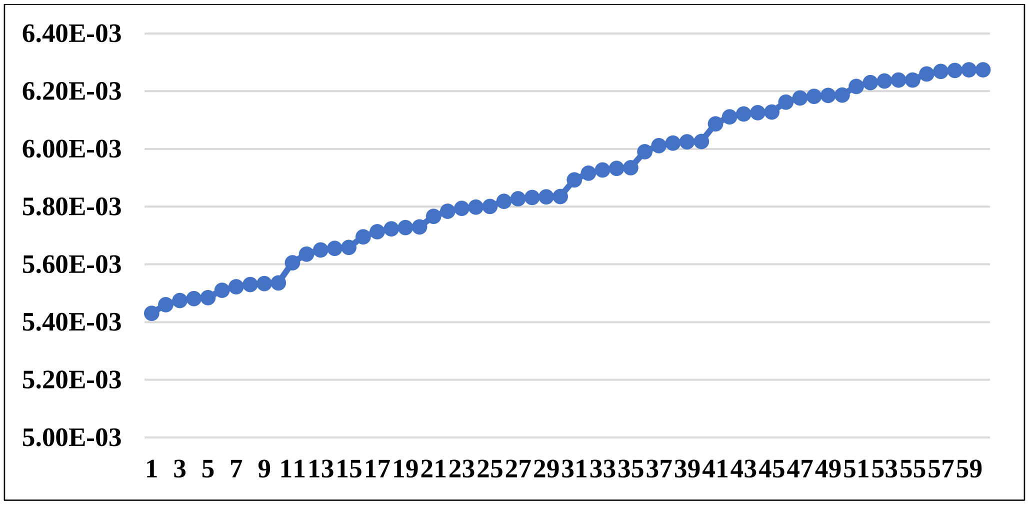

Figure 2. Best MSE for Reliability with ($\alpha = 0.75$)

5. Conclusion

In Tables 2-5, when we compared reliability estimation between the two studied methods of estimation, the first one was the Maximum Likelihood Estimation (MLE) and the second method was the Maximum Entropy Estimation (MEE) for a novel distribution. Simulation study work was used to generate different sizes of samples. It is clear that the Maximum Likelihood Estimation (MLE) is the best in performance when α=0.25 & β=2, and the Maximum Entropy

Estimation Method (MEE) when α=0.75 in all cases of the β, and this result is true for all values of parameters and sample sizes used in the study.

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Conflict of Interest

The author declares no conflict of interest.

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