



Convergence Theorems Via Hybrid Multivalued Mappings

Saddam Muhsin Ghadeer¹  , and Zena Hussein Maibed²  

^{1,2} Department of Mathematics, College of Education for Pure Science (Ibn Al-Haithem),
University of Baghdad, Baghdad, Iraq

*Corresponding Author

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Abstract

It is known that algorithms are of great importance in various fields of mathematics, as they are used in finding fixed points, zeroes of metric projection points, Non-Smooth, Differential Equations, Optimization theory, and Variational Inequality problem. Accordingly, many researchers have focused on investigating and enhancing algorithms in order to utilize their potential amidst the rapid technological developments occurring in our modern world. To ensure success, effectiveness, speed, and superiority of iterative methods over other approximate methods depend on two important factors: The first is the number of iterations, and the second is time. In this paper, we introduce a new iterative method that has been generalized to a number of algorithms, which is considered a generalization of Ishikawa's iteration algorithm. We use a family of hybrid multivalued mappings, nonexpansive single-valued mappings, and $(\varphi, L)^*$ -weak contraction mapping where φ is a comparison function in Hilbert space. The concept of $(\varphi, L)^*$ -weak contraction mapping is a generalization of the concept (φ, L) -weak contraction mapping, and we obtain several convergence theorems under suitable conditions.

Keywords: Hybrid multivalued mapping, $(\varphi, L)^*$ -weak contraction, condition (\tilde{A}) and projection operator

1. Introduction

Let \tilde{E} be a nonempty closed and convex subset of real Hilbert space \tilde{H} with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{BC}(\tilde{E})$ denote the family of all nonempty closed bounded subset of \tilde{E} , while $\tilde{K}(\tilde{E})$ denote the family of all nonempty compact subset of \tilde{E} . An element $p \in \tilde{E}$, is called fixed point (\mathcal{FP}) of mapping $\tilde{T} : \tilde{E} \rightarrow \tilde{E}$ if $p = p\tilde{T}$, in multivalued mapping $\mathcal{T} : \tilde{E} \rightarrow \mathcal{BC}(\tilde{E})$, p is a \mathcal{FP} if $p \in p\mathcal{T}$. A point p is called a common fixed point of \tilde{T} and \mathcal{T} if $p = \tilde{T}p \in p\mathcal{T}$ and denoted by (\mathcal{CFP}) . Many authors have studied extensively the \mathcal{FP} theorems and the existence of \mathcal{FP} of nonexpansive mappings (N-mappings), and they presented many concepts and theorems¹⁻⁴, and other studies have examined the convergence of different iterative methods, as noted in⁵⁻¹², also they studies the equivalence of Some Iterations¹³, and introduced generalization of the Mann's algorithm¹⁴.

Defined a class of nonlinear mapping, which is called hybrid as follows:

$$\|\tilde{u}\mathcal{T} - z\mathcal{T}\|^2 \leq \|\tilde{u} - z\|^2 + \langle \tilde{u} - \tilde{u}\mathcal{T}, z - z\mathcal{T} \rangle, \quad \forall \tilde{u}, z \in \tilde{E}$$

that a mapping $\tilde{T} : \tilde{E} \rightarrow \tilde{H}$ is hybrid if:

$$3\|\tilde{u}\mathcal{T} - z\mathcal{T}\|^2 \leq \|\tilde{u} - z\|^2 + \|\tilde{u} - z\mathcal{T}\|^2 + \|\tilde{u}\mathcal{T} - z\|^2, \quad \forall \tilde{u}, z \in \tilde{E}^{15}$$

Stated and introduced a new concept of mapping $\mathcal{T} : \tilde{E} \rightarrow \mathcal{BC}(\tilde{E})$ in Hilbert space by Hausdorff metric such that (\mathcal{T}) is called hybrid if satisfies the following condition:

$$3H(\tilde{u}\mathcal{T}, z\mathcal{T})^2 \leq \|\tilde{u} - z\|^2 + d(\tilde{u}, z\mathcal{T})^2 + d(z, \tilde{u}\mathcal{T})^2, \quad \forall \tilde{u}, z \in \tilde{E},$$

and if $\mathcal{F}(\mathbb{T})$ nonempty, then \mathbb{T} is a quasi-nonexpansive¹⁶. The approximating \mathcal{FP} of (φ, L) -weak contractions $((\varphi, L)$ -W-contr) it was of attract to some researchers¹⁷ and many scholars and researchers have made generalizations in different directions of contractive mappings; see¹⁸⁻²⁰

The modification of Ishikawa's algorithm for two hybrid multivalued mapping in \check{H} :

$$\begin{cases} \check{u}_1 \in \check{E} \text{ chosen arbitrarily} \\ z_n = \sigma_n \check{u}_n + (1 - \sigma_n) \mathbb{T}_1 \check{u}_n \\ \check{u}_{n+1} = \lambda_n \check{u}_n + (1 - \lambda_n) \mathbb{T}_2 z_n \end{cases}$$

and proved the sequence $\{\check{u}_n\}$ weak converges (W-converges) to a \mathcal{CFP} of $\{\mathbb{T}_1, \mathbb{T}_2\}$, see²¹

The hybrid algorithm was studied through²². After that, the focus was on convergence of the modified Picard-s hybrid iterative scheme, a Picard-S hybrid algorithm and introduced another hybrid scheme see²³⁻²⁶.

In this work, we construct a new iterative scheme, that modifies the above iterative algorithm by using two hybrid multivalued mappings, two N-mappings and two $(\varphi, L)^*$ -weak contraction $((\varphi, L)^*$ -W-contr) mappings in real Hilbert space.

2. Preliminaries:

We recall the following:

2.1. Definition: A mapping $\mathbb{T} : \check{E} \rightarrow \mathcal{BC}(\check{E})$ is said to be a hybrid multivalued mapping (HM-mapping) if satisfies the following condition:

$$3H(\check{u}\mathbb{T}, z\mathbb{T})^2 \leq ||\check{u} - z||^2 + d(\check{u}, z\mathbb{T})^2 + d(z, \check{u}\mathbb{T})^2, \quad \forall \check{u}, z \in \check{E}$$

and if $\mathcal{F}(\mathbb{T}) \neq \emptyset$, then \mathbb{T} is a quasi-nonexpansive¹⁶.

2.2. Definition: Let (\check{E}, d) be a metric space and $\mathbb{T} : \check{E} \rightarrow \check{E}$ any operator. Then, \mathbb{T} is called (φ, L) -weak contraction if there exists a some $L \geq 0$ and a \mathcal{CF} φ such that:

$$d(\check{u}\mathbb{T}, z\mathbb{T}) \leq \varphi d(\check{u}, z) + L d(z, \check{u}\mathbb{T}) \quad \forall \check{u}, z \in \check{E} \quad 17$$

2.3. Definition: Let $\mathbb{T} : \check{E} \rightarrow \check{E}$ self-mapping then \mathbb{T} is called N- mapping if:

$$||\check{u}\mathbb{T} - z\mathbb{T}|| \leq ||\check{u} - z|| \quad \text{for all } \check{u}, z \in \check{E} \quad 27.$$

2.4. Definition: A map $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a comparison function (\mathcal{CF}) if it satisfies:

(i) $r_1 < r_2 \Rightarrow \varphi r_1 \leq \varphi r_2$ for all $r_1, r_2 \in \mathbb{R}^+$

(ii) the sequence $\{\varphi^n r\}_{n=0}^\infty$ converges to zero, $\forall r \in \mathbb{R}^+$ such that φ^n is a stand for the n^{th} iterate of φ see²⁸.

2.5. Definition: Let (\check{E}, d) be a metric space and $\mathbb{T} : \check{E} \rightarrow \check{E}$ any operator. Then, \mathbb{T} is called $(\varphi, L)^*$ -weak contraction if there exists any a \mathcal{CF} φ such that:

$$d(\check{u}\mathbb{T}, z\mathbb{T}) \leq \varphi d(\check{u}, z) + L \min\{d(z, \check{u}\mathbb{T}), d(z, z\mathbb{T})\} \quad \forall \check{u}, z \in \check{E}$$

And if $\min\{d(z, \check{u}\mathbb{T}), d(z, z\mathbb{T})\} = d(z, \check{u}\mathbb{T})$ then $L \geq 0$

if $\min\{d(z, \check{u}\mathbb{T}), d(z, z\mathbb{T})\} = d(z, z\mathbb{T})$ then $0 \leq L < 1$

2.6. Remark: Clearly, the $(\varphi, L)^*$ -weak contraction to be (φ, L) -weak contraction if:

$$\min\{d(z, \check{u}\mathbb{T}), d(z, z\mathbb{T})\} = d(z, \check{u}\mathbb{T})$$

2.7. Definition: A mapping $\mathbb{T} : \check{E} \rightarrow \mathcal{CB}(\check{E})$ is said to satisfy Condition (\check{A}) if:

$$||\check{u} - p|| = d(\check{u}, p\mathbb{T}) \quad \forall \check{u} \in \check{H}, p \in \mathcal{F}(\mathbb{T}) \text{ see}^{29}$$

2.8. Lemma: Let $\mathbb{T} : \check{E} \rightarrow \check{K}(\check{E})$ be an HM-mapping and $\{\check{u}_n\}$ be a sequence in \check{E} such that $\check{u}_n \rightarrow u$ and $\lim_{n \rightarrow \infty} ||\check{u}_n - z_n|| = 0$ for some $z_n \in \mathbb{T}\check{u}_n$. Then $u \in \mathbb{T}u$ ¹⁶

2.9. Lemma: Let $\mathbb{T} : \check{E} \rightarrow \check{K}(\check{E})$ be an HM-mapping such that $\mathcal{F}(\mathbb{T})$ is nonempty, then $\mathcal{F}(\mathbb{T})$ is closed¹⁶.

2.10. Lemma: Let $\mathbb{T} : \check{E} \rightarrow \check{K}(\check{E})$ be an HM-mapping such that $\mathcal{F}(\mathbb{T})$ is nonempty. If \mathbb{T} satisfies Condition (\check{A}) , then $\mathcal{F}(\mathbb{T})$ is convex¹⁶.

2.11. Lemma: For all \check{u} and z in \check{H} and $\alpha \in [0, 1]$ the following is hold:

(i) $||\check{u} - z||^2 = ||\check{u}||^2 + ||z||^2 - 2\langle \check{u}, z \rangle$

(ii) $|\alpha \check{u} - (1 - \alpha)z|^2 = \alpha|\check{u}|^2 + (1 - \alpha)|z|^2 - \alpha(1 - \alpha)|\check{u} - z|^2$

(iii) If $\{\check{u}_n\}$ is a sequence in \check{H} such that $\check{u}_n \rightarrow \check{u}$, then

$$\limsup_{n \rightarrow \infty} |\check{u}_n - z|^2 = \limsup_{n \rightarrow \infty} (|\check{u}_n - \check{u}|^2 + |\check{u} - z|^2). \quad 30$$

2.12. Lemma: Let $P_{\check{E}}: \check{H} \rightarrow \check{E}$ be the metric projection from \check{H} onto \check{E} then:

$$|z - P_{\check{E}} \check{u}|^2 + |\check{u} - P_{\check{E}} \check{u}|^2 \leq |\check{u} - z|^2, \quad \forall \check{u} \in \check{H} \text{ and } z \in \check{E} \quad 31$$

2.13. Lemma : Let \check{E} be a nonempty closed and convex subset of \check{H} , then the set $\check{K} = \{\check{s} \in \check{E} : |\check{z} - \check{s}|^2 \leq |\check{u} - \check{s}|^2 + \langle \check{a}, \check{s} \rangle + \check{r}\}$ is closed and convex for each $\check{u}, \check{z} \in \check{H}$ and $\check{r} \in \mathbb{R}$ see³²

2.14. Lemma: Let $\mathcal{T}: \check{H} \rightarrow \check{H}$ an N-mapping, then $\mathcal{F}(\mathcal{T})$ is either empty or closed and convex³³. Also from fact³⁴, if an N-mapping $\mathcal{T}: \check{H} \rightarrow \check{H}$ has at least one \mathcal{FP} , $\mathcal{F}(\mathcal{T}) \subset \check{H}$ is closed and is closed and convex and expressed as:

$$\mathcal{F}(\mathcal{T}) = \bigcap_{\check{u} \in \check{H}} \{z \in \check{H}: \langle \check{u} - \check{u}\mathcal{T}, z \rangle \leq |\check{u}|^2 - |\check{u}\mathcal{T}|^2 \}$$

3. Results and Discussion

Studied Approximating \mathcal{FP} of (φ, L) -weak contractions it has attracted the interest of some researchers¹⁸, while introduced²⁴ the modification for two hybrid multivalued mapping in \check{H} . Also¹¹ Common fixed points for hybrid pair of generalized non-expensive mappings by a three-step iterative scheme. In the other hand³⁴ studied Strong convergence theorems for nonexpansive mappings. In this study, a convergence theorems-W to \mathcal{CFP} by multivalued maps are proved by using new algorithms.

3.1. Lemma: Let $\mathcal{T}: \check{H} \rightarrow \check{H}$ be $(\varphi, L)^*$ -W-contr- mapping where φ is a \mathcal{CF} then $\mathcal{F}(\mathcal{T})$ is nonempty.

Proof: To prove that $\mathcal{F}(\mathcal{T}) \neq \emptyset$, let $\check{u}_0 \in \check{H}$, a sequence $\{\check{u}_n\}_{n=0}^\infty$ defined by $\check{u}_{n+1} = \mathcal{T}\check{u}_n$. Since \mathcal{T} is a $(\varphi, L)^*$ -W-contr, there exists a \mathcal{CF} φ and some $L \geq 0$ where $\min\{d(z, \check{u}\mathcal{T}), d(z, z\mathcal{T})\} = d(z, \check{u}\mathcal{T})$, and $0 \leq L < 1$ where $\min\{d(z, \check{u}\mathcal{T}), d(z, z\mathcal{T})\} = d(z, z\mathcal{T})$ such that:

$$d(\check{u}\mathcal{T}, z\mathcal{T}) \leq \varphi d(\check{u}, z) + L \min\{d(z, \check{u}\mathcal{T}), d(z, z\mathcal{T})\}, \quad \forall \check{u}, z \in \check{H} \quad (1)$$

Take $\check{u} := \check{u}_{n-1}$, $y := \check{u}_n$ in (1). We obtain

$$d(\check{u}_n, \check{u}_{n+1}) \leq \varphi d(\check{u}_{n-1}, \check{u}_n), \quad \forall n = 1, 2, \dots \quad (2)$$

But φ is not decreasing and from **Equation 2**, we obtain,

$$d(\check{u}_n, \check{u}_{n+1}) \leq \varphi^n d(\check{u}_0, \check{u}_1)$$

implies that $\{\check{u}_n\}_{n=0}^\infty$ is a Cauchy sequence and by completely of \check{H} , we have $\check{u}_n \rightarrow p$, we shall prove that $p \in \mathcal{F}(\mathcal{T})$. Indeed

$$\begin{aligned} d(p, p\mathcal{T}) &\leq d(p, \check{u}_{n+1}) + d(\check{u}_{n+1}, p\mathcal{T}) \\ &= d(\check{u}_{n+1}, p) + d(\mathcal{T}\check{u}_n, p\mathcal{T}) \end{aligned}$$

By **Equation 1**, we obtain

$$d(\mathcal{T}\check{u}_n, p\mathcal{T}) \leq \varphi d(\check{u}_n, p) + L \min\{d(p, \mathcal{T}\check{u}_n), d(p, p\mathcal{T})\}$$

Therefore,

$$d(p, p\mathcal{T}) \leq d(\check{u}_{n+1}, p) + \varphi d(\check{u}_n, p) + L \min\{d(\check{u}_{n+1}, p), d(p, p\mathcal{T})\} \quad (3)$$

Case 1: $\min\{d(\check{u}_{n+1}, p), d(p, p\mathcal{T})\} = d(\check{u}_{n+1}, p)$, we have

$$d(p, p\mathcal{T}) \leq (1 + L)d(\check{u}_{n+1}, p) + \varphi d(\check{u}_n, p) \quad (4)$$

Now suppose that $n \rightarrow \infty$ in (4), we obtain

$$d(p, p\mathcal{T}) = 0. \text{ Therefore } p \in \mathcal{F}(\mathcal{T})$$

Case 2: $\min\{d(\check{u}_{n+1}, p), d(p, p\mathcal{T})\} = d(p, p\mathcal{T})$, we have

$$(1 - L)d(p, p\mathcal{T}) \leq d(\check{u}_{n+1}, p) + \varphi d(\check{u}_n, p) \quad (5)$$

As $n \rightarrow \infty$ in **Equation 5**, we obtain $d(p, p\mathcal{T}) = 0$. Therefore, $p \in \mathcal{F}(\mathcal{T})$

3.2.Theorem: Let $\mathcal{T}_1, \mathcal{T}_2: \check{E} \rightarrow \check{K}(\check{E})$ be HM-mapping $\mathfrak{f}_1, \mathfrak{f}_2: \check{H} \rightarrow \check{H}$ are an N-mapping, and $G_1, G_2: \check{H} \rightarrow \check{H}$ are $(\varphi, L)^*$ -W-contr mapping where φ is a \mathcal{CF} define by $\varphi(x) = x\delta, 0 < \delta \leq 1$ such that $\Gamma := (\cap_{i=1}^{\infty} \mathcal{F}(\mathcal{T}_i)) \cap (\cap_{i=1}^{\infty} \mathcal{F}(\mathfrak{f}_i)) \cap (\cap_{i=1}^{\infty} \mathcal{F}(G_i)) \neq \emptyset$. Let $\{\check{u}_n\}$ be a generated by:

$$\begin{cases} \check{u}_1 \in \check{E} \text{ chosen arbitrary.} \\ z_n \in \mathfrak{t}_n[\mathfrak{b}_n G_1 \check{u}_n + (1 - \mathfrak{b}_n)\mathcal{T}_1 \check{u}_n] + (1 - \mathfrak{t}_n)\mathfrak{f}_1 \check{u}_n \\ \check{u}_{n+1} \in \mathfrak{d}_n[\mathfrak{j}_n \mathfrak{f}_2 \check{u}_n + (1 - \mathfrak{j}_n)G_2 \check{u}_n] + (1 - \mathfrak{d}_n)\mathcal{T}_2 z_n \end{cases} \tag{6}$$

For all $n \geq 1$ where $\{\mathfrak{t}_n\}, \{\mathfrak{b}_n\}, \{\mathfrak{d}_n\}$ and $\{\mathfrak{j}_n\} \subset (0, 1]$. Assume that:

- (i) $\sum_{n=0}^{\infty} \|G_1 \check{u}_n - \check{u}_n\| < \infty, \sum_{n=0}^{\infty} \|G_2 \check{u}_n - \check{u}_n\| < \infty$
- (ii) $\lim_{n \rightarrow \infty} \mathfrak{t}_n < 1, \lim_{n \rightarrow \infty} \mathfrak{d}_n < 1$

If \mathcal{T}_1 and \mathcal{T}_2 satisfy Condition (A), then $\{\check{u}_n\}$ converges-W to \mathcal{CFP} of \mathcal{T}_1 and \mathcal{T}_2 .

Proof: Let $p \in \Gamma$, we have

$$\begin{aligned} \|\check{u}_{n+1} - p\|^2 &= \|\mathfrak{d}_n[\mathfrak{j}_n \mathfrak{f}_2 \check{u}_n + (1 - \mathfrak{j}_n)G_2 \check{u}_n] + (1 - \mathfrak{d}_n)\check{r}_n - p\|^2 \\ &\leq \mathfrak{d}_n \|\mathfrak{j}_n \mathfrak{f}_2 \check{u}_n + (1 - \mathfrak{j}_n)G_2 \check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) \|\check{r}_n - p\|^2 \\ &\leq \mathfrak{d}_n \mathfrak{j}_n \|\mathfrak{f}_2 \check{u}_n - p\|^2 + (1 - \mathfrak{j}_n) \|G_2 \check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) \|\check{r}_n - p\|^2 \\ &= \mathfrak{d}_n \mathfrak{j}_n \|\mathfrak{f}_2 \check{u}_n - \mathfrak{f}_2 p\|^2 + (1 - \mathfrak{j}_n) \|G_2 \check{u}_n - G_2 p\|^2 + (1 - \mathfrak{d}_n) d(\check{r}_n, \mathcal{T}_2 p)^2 \\ &\leq \mathfrak{d}_n \mathfrak{j}_n \|\check{u}_n - p\|^2 + (1 - \mathfrak{j}_n) (\varphi \|\check{u}_n - p\|^2 + L \min\{\|\check{u}_n - G_2 p\|^2, \|p - G_2 p\|^2\}) \\ &\quad + (1 - \mathfrak{d}_n) H(\mathcal{T}_2 z_n, \mathcal{T}_2 p)^2 \\ &\leq \mathfrak{d}_n \mathfrak{j}_n \|\check{u}_n - p\|^2 + (1 - \mathfrak{j}_n) \delta \|\check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) \|z_n - p\|^2 \\ &= \mathfrak{d}_n \|\check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) \|z_n - p\|^2 \end{aligned} \tag{7}$$

And

$$\begin{aligned} \|z_n - p\|^2 &= \|\mathfrak{t}_n[\mathfrak{b}_n G_1 \check{u}_n + (1 - \mathfrak{b}_n)\check{s}_n] + (1 - \mathfrak{t}_n)\mathfrak{f}_1 \check{u}_n - p\|^2 \\ &= \mathfrak{t}_n \|\mathfrak{b}_n G_1 \check{u}_n + (1 - \mathfrak{b}_n)\check{s}_n - p\|^2 + (1 - \mathfrak{t}_n) \|\mathfrak{f}_1 \check{u}_n - p\|^2 \\ &\quad - \mathfrak{t}_n(1 - \mathfrak{t}_n) \|\mathfrak{b}_n G_1 \check{u}_n + (1 - \mathfrak{b}_n)\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2 \\ &\leq \mathfrak{t}_n [\mathfrak{b}_n \|G_1 \check{u}_n - G_1 p\|^2 + (1 - \mathfrak{b}_n) \|\check{s}_n - p\|^2] + (1 - \mathfrak{t}_n) \|\mathfrak{f}_1 \check{u}_n - \mathfrak{f}_1 p\|^2 \\ &\quad - \mathfrak{t}_n(1 - \mathfrak{t}_n) [\mathfrak{b}_n \|G_1 \check{u}_n - \check{s}_n\|^2 + \|\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2] \\ &\leq \mathfrak{t}_n [\mathfrak{b}_n (\varphi \|\check{u}_n - p\|^2 + L \min\{\|\check{u}_n - G_1 p\|^2, \|p - G_1 p\|^2\}) + (1 - \mathfrak{b}_n) d(\check{s}_n, \mathcal{T}_1 p)^2] \\ &\quad + (1 - \mathfrak{t}_n) \|\check{u}_n - p\|^2 - \mathfrak{t}_n(1 - \mathfrak{t}_n) [\mathfrak{b}_n \|G_1 \check{u}_n - \check{s}_n\|^2 + \|\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2] \\ &\leq \mathfrak{t}_n [\mathfrak{b}_n (\delta \|\check{u}_n - p\|^2 + L \|p - G_1 p\|^2) + (1 - \mathfrak{b}_n) H(\mathcal{T}_1 \check{u}_n, \mathcal{T}_1 p)^2] \\ &\quad + (1 - \mathfrak{t}_n) \|\check{u}_n - p\|^2 - \mathfrak{t}_n(1 - \mathfrak{t}_n) [\mathfrak{b}_n \|G_1 \check{u}_n - \check{s}_n\|^2 + \|\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2] \\ &= \|\check{u}_n - p\|^2 - \mathfrak{t}_n(1 - \mathfrak{t}_n) [\mathfrak{b}_n \|G_1 \check{u}_n - \check{s}_n\|^2 + \|\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2] \end{aligned} \tag{8}$$

From **Equations 7 and 8**, we obtain

$$\begin{aligned} \|\check{u}_{n+1} - p\|^2 &\leq \mathfrak{d}_n \|\check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) \|\check{u}_n - p\|^2 \\ &\quad - (1 - \mathfrak{d}_n) \mathfrak{t}_n(1 - \mathfrak{t}_n) [\mathfrak{b}_n \|G_1 \check{u}_n - \check{s}_n\|^2 + \|\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2] \\ &= \|\check{u}_n - p\|^2 - (1 - \mathfrak{d}_n) \mathfrak{t}_n(1 - \mathfrak{t}_n) [\mathfrak{b}_n \|G_1 \check{u}_n - \check{s}_n\|^2 + \|\check{s}_n - \mathfrak{f}_1 \check{u}_n\|^2] \end{aligned} \tag{9}$$

We have, $\|\check{u}_{n+1} - p\|^2 \leq \|\check{u}_n - p\|^2$

Then, $\{\check{u}_n\}$ is bounded and decreasing, also $\{z_n\}$. Therefore, $\lim_{n \rightarrow \infty} \|\check{u}_n - p\|$ exists, thus there

exists $\{\check{s}_{n_k}\}$ of $\{\check{s}_n\}$ such that $\check{s}_{n_k} \rightarrow u$

To prove u is \mathcal{CFP} of \mathcal{T}_1 and \mathcal{T}_2 , by Lemma 1.10(ii), we get

$$\begin{aligned} \|\check{u}_{n+1} - p\|^2 &= \|\mathfrak{d}_n[\mathfrak{j}_n \mathfrak{f}_2 \check{u}_n + (1 - \mathfrak{j}_n)G_2 \check{u}_n] + (1 - \mathfrak{d}_n)\check{r}_n - p\|^2 \\ &= \mathfrak{d}_n \|\mathfrak{j}_n \mathfrak{f}_2 \check{u}_n + (1 - \mathfrak{j}_n)G_2 \check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) \|\check{r}_n - p\|^2 \\ &\quad - \mathfrak{d}_n(1 - \mathfrak{d}_n) \|\mathfrak{j}_n \mathfrak{f}_2 \check{u}_n + (1 - \mathfrak{j}_n)G_2 \check{u}_n - \check{r}_n\|^2 \\ &\leq \mathfrak{d}_n \mathfrak{j}_n \|\mathfrak{f}_2 \check{u}_n - p\|^2 + (1 - \mathfrak{j}_n) \|G_2 \check{u}_n - p\|^2 + (1 - \mathfrak{d}_n) d(\check{r}_n, \mathcal{T}_2 p)^2 \\ &\quad - \mathfrak{d}_n(1 - \mathfrak{d}_n) [\mathfrak{j}_n \|\mathfrak{f}_2 \check{u}_n - G_2 \check{u}_n\|^2 + \|G_2 \check{u}_n - \check{r}_n\|^2] \\ &\leq \mathfrak{d}_n \mathfrak{j}_n \|\mathfrak{f}_2 \check{u}_n - \mathfrak{f}_2 p\|^2 + (1 - \mathfrak{j}_n) \|G_2 \check{u}_n - G_2 p\|^2 + (1 - \mathfrak{d}_n) H(\mathcal{T}_2 z_n, \mathcal{T}_2 p)^2 \end{aligned}$$

$$\begin{aligned}
 & -\mathfrak{d}_n(1 - \mathfrak{d}_n)[j_n || \mathfrak{f}_2 \hat{u}_n - G_2 \hat{u}_n ||^2 + ||G_2 \hat{u}_n - \check{r}_n ||^2] \\
 = & \mathfrak{d}_n ||\hat{u}_n - p ||^2 + (1 - \mathfrak{d}_n) ||z_n - p ||^2 - \mathfrak{d}_n(1 - \mathfrak{d}_n)[j_n || \mathfrak{f}_2 \hat{u}_n - G_2 \hat{u}_n ||^2 + ||G_2 \hat{u}_n - \check{r}_n ||^2]
 \end{aligned} \tag{10}$$

From **Equations 8 and 10** we obtain

$$\begin{aligned}
 ||\hat{u}_{n+1} - p ||^2 \leq & ||\hat{u}_n - p ||^2 - (1 - \mathfrak{d}_n)\mathfrak{t}_n(1 - \mathfrak{t}_n)[\mathfrak{b}_n ||G_1 \hat{u}_n - \check{s}_n ||^2 + ||\check{s}_n - \mathfrak{f}_1 \hat{u}_n ||^2] \\
 & - \mathfrak{d}_n(1 - \mathfrak{d}_n)[j_n || \mathfrak{f}_2 \hat{u}_n - G_2 \hat{u}_n ||^2 + ||G_2 \hat{u}_n - \check{r}_n ||^2]
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & (1 - \mathfrak{d}_n)\mathfrak{t}_n(1 - \mathfrak{t}_n)[\mathfrak{b}_n ||G_1 \hat{u}_n - \check{s}_n ||^2 + ||\check{s}_n - \mathfrak{f}_1 \hat{u}_n ||^2] \\
 & + \mathfrak{d}_n(1 - \mathfrak{d}_n)[j_n || \mathfrak{f}_2 \hat{u}_n - G_2 \hat{u}_n ||^2 + ||G_2 \hat{u}_n - \check{r}_n ||^2] \leq ||\hat{u}_n - p ||^2 - ||\hat{u}_{n+1} - p ||^2
 \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} ||G_1 \hat{u}_n - \check{s}_n || = \lim_{n \rightarrow \infty} ||\check{s}_n - \mathfrak{f}_1 \hat{u}_n ||$

$$= \lim_{n \rightarrow \infty} ||\mathfrak{f}_2 \hat{u}_n - G_2 \hat{u}_n || = \lim_{n \rightarrow \infty} ||G_2 \hat{u}_n - \check{r}_n || = 0 \tag{11}$$

Therefore, $||\hat{u}_n - \check{s}_n || \leq ||\hat{u}_n - G_1 \hat{u}_n || + ||G_1 \hat{u}_n - \check{s}_n ||$. Then $||\hat{u}_n - \check{s}_n || \rightarrow 0$

Also, $||z_n - \hat{u}_n || = ||\mathfrak{t}_n[\mathfrak{b}_n G_1 \hat{u}_n + (1 - \mathfrak{b}_n)\check{s}_n] + (1 - \mathfrak{t}_n)\mathfrak{f}_1 \hat{u}_n - \hat{u}_n ||$

$\leq \mathfrak{t}_n[\mathfrak{b}_n ||G_1 \hat{u}_n - \hat{u}_n || + (1 - \mathfrak{b}_n) ||\check{s}_n - \hat{u}_n ||] + (1 - \mathfrak{t}_n)[||\mathfrak{f}_1 \hat{u}_n - \check{s}_n || + ||\check{s}_n - \hat{u}_n ||]$

Then $||z_n - \hat{u}_n || \rightarrow 0$

Also,

$$||z_n - \check{r}_n || \leq ||z_n - \hat{u}_n || + ||\hat{u}_n - G_2 \hat{u}_n || + ||G_2 \hat{u}_n - \check{r}_n ||$$

Then $||z_n - \check{r}_n || \rightarrow 0$

Now, because the sequence $\{\hat{u}_n\}$ is a bounded, there exists subsequence $\{\hat{u}_{n_k}\}$ of $\{\hat{u}_n\}$ such that $\hat{u}_{n_k} \rightarrow u$ for some $u \in \check{E}$, by Lemma 1.7, we have $u \in \mathcal{T}_1 u$. But $||\hat{u}_n - \check{s}_n || \rightarrow 0$ then $||\hat{u}_{n_k} - \check{s}_{n_k} || \rightarrow 0$ (i.e $\check{s}_{n_k} \rightarrow u$) hence $u \in \mathcal{T}_1 \hat{u}_n$. Again, by Lemma 1.7, we can show that $u \in \mathcal{T}_2 u$ but $||z_n - \hat{u}_n || \rightarrow 0$ then $||z_{n_k} - \hat{u}_{n_k} || \rightarrow 0$ (i.e $z_{n_k} \rightarrow u$) hence $u \in \mathcal{T}_2 z_n$. Therefore, $u \in \mathcal{F}(\mathcal{T}_1) \cap \mathcal{F}(\mathcal{T}_2)$

4. Converges Strongly to Common Fixed Point

4.1.Theorem: Let $\mathcal{T}_1, \mathcal{T}_2: \check{E} \rightarrow \check{K}(\check{E})$ be an HM-mapping, $\mathfrak{f}_1, \mathfrak{f}_2: \check{H} \rightarrow \check{H}$ are a N-mapping, and $G_1, G_2: \check{H} \rightarrow \check{H}$ are $(\varphi, L)^*$ -W- contr mapping where φ is a \mathcal{CF} defined by $\varphi(x) = x\delta, 0 < \delta \leq 1$ such that $\Gamma := (\cap_{i=1}^2 \mathcal{F}(\mathcal{T}_i)) \cap (\cap_{i=1}^2 \mathcal{F}(\mathfrak{f}_i)) \cap (\cap_{i=1}^2 \mathcal{F}(G_i)) \neq \emptyset$. Let $\{\hat{u}_n\}$ be generated by:

$$\begin{cases}
 \hat{u}_1 \in \check{E}, \check{E}_1 = \check{E} \\
 z_n \in \mathfrak{t}_n[\mathfrak{b}_n G_1 \hat{u}_n + (1 - \mathfrak{b}_n)\mathcal{T}_1 \hat{u}_n] + (1 - \mathfrak{t}_n)\mathfrak{f}_1 \hat{u}_n \\
 y_n \in \mathfrak{d}_n[j_n \mathfrak{f}_2 \hat{u}_n + (1 - j_n)G_2 \hat{u}_n] + (1 - \mathfrak{d}_n)\mathcal{T}_2 z_n \\
 \check{E}_{n+1} = \{z \in \check{E}_n: ||y_n - z || \leq ||\hat{u}_n - z ||\} \\
 \hat{u}_{n+1} = P_{\check{E}_{n+1}} \hat{u}_0, \forall n \geq 1
 \end{cases} \tag{12}$$

For all $n \geq 1$ where $\{\mathfrak{t}_n\}, \{\mathfrak{b}_n\}, \{\mathfrak{d}_n\}$ and $\{j_n\} \subset (0,1]$. Assume that

(i) $\sum_{n=0}^{\infty} ||G_1 \hat{u}_n - \hat{u}_n || < \infty, \sum_{n=0}^{\infty} ||G_2 \hat{u}_n - \hat{u}_n || < \infty$

(ii) $\lim_{n \rightarrow \infty} \mathfrak{t}_n < 1, \lim_{n \rightarrow \infty} \mathfrak{d}_n < 1$

If \mathcal{T}_1 and \mathcal{T}_2 satisfy Condition (A), then

- \check{E}_n is a nonempty closed convex.
- $\Gamma \subseteq \check{E}_n$ for each $n \geq 1$.
- The sequence converges-S to \mathcal{CFP} of \mathcal{T}_1 and \mathcal{T}_2
- $\lim_{n \rightarrow \infty} ||\hat{u}_n - \check{s}_n || = \lim_{n \rightarrow \infty} ||z_n - \check{r}_n || = 0$
- $u = P_{\Gamma} \hat{u}_0$

Proof: Following the same proof method above, we get what is required

4.2. Theorem: Let $\mathcal{T}_1, \mathcal{T}_2: \check{E} \rightarrow \check{K}(\check{E})$ be an HM-mapping and $\mathfrak{f}_1, \mathfrak{f}_2: \check{H} \rightarrow \check{H}$ are an N-mapping, $G_1, G_2: \check{H} \rightarrow \check{H}$ are $(\varphi, L)^*$ -W- contr mapping where φ is a \mathcal{CF} defined by $\varphi(x) =$

$\chi\delta, 0 < \delta \leq 1$ such that $\Gamma := (\cap_{i=1}^2 \mathcal{F}(\mathbb{T}_i)) \cap (\cap_{i=1}^2 \mathcal{F}(\mathbb{F}_i)) \cap (\cap_{i=1}^2 \mathcal{F}(\mathbb{G}_i)) \neq \emptyset$. Let $\{\hat{u}_n\}$ be generated by:

$$\begin{cases} \hat{u}_1 \in \check{E} \text{ chosen arbitrarily} \\ z_n \in t_n[b_n G_1 \hat{u}_n + (1 - b_n)\mathbb{T}_1 \hat{u}_n] + (1 - t_n)f_1 \hat{u}_n \\ y_n \in d_n[j_n f_2 \hat{u}_n + (1 - j_n)G_2 \hat{u}_n] + (1 - d_n)\mathbb{T}_2 z_n \\ \check{E}_n = \{z \in \check{E}: \|y_n - z\| \leq \|\hat{u}_n - z\|\} \\ \check{N}_n = \{z \in \check{E}: \langle \hat{u}_0 - \hat{u}_n, \hat{u}_n - z \rangle \geq 0\} \\ \hat{u}_{n+1} = P_{\check{E}_n \cap \check{N}_n} \hat{u}_0, \forall n \geq 1 \end{cases} \quad (13)$$

For all $n \geq 1$ where $\{t_n\}, \{b_n\}, \{d_n\}$ and $\{j_n\} \subset (0,1]$. Assume that

- (i) $\sum_{n=0}^{\infty} \|G_1 \hat{u}_n - \hat{u}_n\| < \infty, \sum_{n=0}^{\infty} \|G_2 \hat{u}_n - \hat{u}_n\| < \infty$
- (ii) $\lim_{n \rightarrow \infty} t_n < 1, \lim_{n \rightarrow \infty} d_n < 1$

If \mathbb{T}_1 and \mathbb{T}_2 satisfy Condition (A) then

- \check{E}_n and \check{N}_n are nonempty closed and convex
- $\Gamma \subseteq \check{E}_n \cap \check{N}_n$ for each $n \geq 1$.
- The sequence converges-S to \mathcal{CFP} of \mathbb{T}_1 and \mathbb{T}_2
- $\lim_{n \rightarrow \infty} \|\hat{u}_n - \check{s}_n\| = \lim_{n \rightarrow \infty} \|z_n - \check{r}_n\| = 0$
- $u = P_{\Gamma} \hat{u}_0$

Proof: Following the same proof method above, we get what is required

5. Conclusion

In this study, we introduced the concept of $(\varphi, L)^*$ -weak contraction mapping which is a generalization of the concept (φ, L) -weak contraction mapping. New iterative techniques in \check{H} are introduced, convergence-S and convergence-W theorems to \mathcal{CFP} via HM mapping, nonexpansive single-valued mappings, and $(\varphi, L)^*$ -weak contr mappings are proved.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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