Ibn Al-Haitham Journal for Pure and Applied Sciences
Journal homepage: jih.uobaghdad.edu.iq
PISSN: 1609-4042, EISSN: 2521-3407
IHJPAS. 2026; 39(2)

Extension Odd Generalized Rayleigh Inverse Rayleigh Distribution: Structure and Properties

Shihab Ahmed Abdul Reda[1], Ali Talib Mohammed[2*], and Aytekin Enver[3]
[1,2]Department of Mathematics, College of Education for Pure Science (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq.
[3] Department of Mathematics, Gazi University, Teknikokullar, Ankara, Türkiye.
*Corresponding author

Received: 5/Octorber/2025
Accepted: 25/January/2026
Published: 20/April/2026
doi.org/10.30526/39.2.4303

Abstract

The Extension Odd Generalized Rayleigh Inverse Rayleigh (EOGRIR) distribution is a novel statistical model that leverages existing distribution families in a flexible manner. This model enhances the ability of standard distributions to analyze data. It makes it easier and faster to examine and model different types of lifetime and reliability data. The essay thoroughly examines the fundamental mathematical characteristics of the EOGRIR distribution. Some examples include series expansions, moments, the quantile, skewness, Rényi entropy, kurtosis, order statistics, and the moment generating function. These traits help us understand how the proposed distribution works and how its mathematical principles operate. We also examine methods for estimating parameters, with a focus on maximum likelihood estimates (MLE), to ensure that inference and parameter estimates are accurate. The EOGRIR distribution is likely to have a significant impact on statistics, as it offers a more flexible and comprehensive framework that may be superior to the existing ones.

Keywords: T-X family, moment, inverse Rayleigh, ML Estimation, flexible, odd technique, quantile function, entropy.

1. Introduction

Long-term statistical data analysis is a major obstacle in various practical domains, including engineering, healthcare, and finance. It's challenging to guess what will happen in the future. Statistical modeling is a quick and effective way to examine and manage uncertainty in this type of data. Businesses like insurance and banking really need and use lifespan or survival statistics. As a result, these fields demand more advanced statistical models.

Recent studies have produced novel families of probability distributions. The goal of these is to improve current models and offer more flexible statistical approaches. Numerous scholars have expanded this field by developing new parameters that are more suitable for complex datasets. Researchers also examine graphs and analytical characteristics, which are essential for understanding how statistics work. A new family of distributions has formed by raising the probability density function (PDF) and the cumulative distribution function (CDF). This method facilitates the analysis of data in various ways, thereby making statistical modeling more accurate and efficient. The model also describes a broader range of occurrences better than traditional distributions. The researchers addressed the following research. Cubic rank transmuted inverse Rayleigh distribution: Properties and applications[1], a study on properties of generalized Rayleigh distributions[2] , discussed the Development of On Rayleigh-exponentiated odd generalized-Pareto distribution with its applications[3], Truncated Inverse Generalized Rayleigh Distribution and Some Properties[4], New extensions of Rayleigh distribution based on inverted-Weibull and Weibull distributions[5], The Odd Inverse Rayleigh Family of Distributions: Simulation & Application to Real Data[6], The Gamma Kumaraswamy-G family of distributions: theory, inference and applications[7] and developed the Bayesian estimation under different loss functions for the case of inverse Rayleigh distribution[8] , Extended Odd Fréchet-G Family of Distributions[9]. On the Weibull-X family of distributions[10], Parameter estimation of inverse exponential Rayleigh distribution based on classical methods[11], Odd Generalized Rayleigh- Exponential distribution[12], Bayesian Estimation for Two Parameters of Weibull Distribution under Generalized Weighted Loss Function[13], And a researcher addressed the The odd generalized exponential family of distributions with applications[14], A new three-parameter weibull inverse rayleigh distribution: theoretical development and applications[15], proposed another he extended odd Weibull-G family: properties and applications[16],and The researcher focused on Extended Rayleigh Probability Distribution to Higher Dimensions[17], Elgarhy introduced the inverse Fréchet–Rayleigh distribution[18] ,A new three-parameter weibull inverse rayleigh distribution: theoretical development and applications[19], Generalized Rayleigh distribution: different methods of estimations[20], Compared the Shanker–Weibull lifetime distribution with several other lifetime models[21], Exponential transformed inverse Rayleigh distribution: Statistical properties and different methods of estimation[22], A New Topp–Leone Odd Weibull Flexible-G Family of Distributions with Applications[23], finally, the researcher presented the Exponentiated TX family of distributions with some applications[24]. The CDF and PDF will be awarded to determine GR as follows:

$$F_{GR}(x) = \left(1 - e^{-b x^2}\right)^{c} \quad \textbf{(1)}$$

$$f_{GR}(x) = 2cb x e^{-b x^2}\left(1 - e^{-b x^2}\right)^{c-1} \quad \textbf{(2)}$$

A novel approach for constructing families of continuous probability distributions was introduced by[25]

$$G(x) = \int_0^{F(x,\mathfrak{h})} \mathcal{Q}(\mathfrak{s})\,d\mathfrak{s}$$

Here, $\mathcal{Q}(\mathfrak{s})$ denotes the pdf and $F(x,\xi)$ represents the cdf of x on range $[0,1]$. The Gamma-G type-3 distribution was presented, for which the cdf is defined as follows;

$$G(x) = \int_0^{\frac{F(x,\mathfrak{h})}{1-F(x,\mathfrak{h})}} \mathcal{Q}(\mathfrak{s})\,d\mathfrak{s}$$

This study proposes the development of a new family of generated distributions, referred to as the Extension Odd Generalized Rayleigh (EOGR-G) family. The proposed family is formulated by modifying the upper bound of the Gamma-G Type-3 distribution, as detailed below:

$$\mathfrak{H}(F(x;\mathfrak{h})) = \frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1 - F^{\mathfrak{a}}(x;\mathfrak{h})}$$

Now, according to the method of Alzaatreh, the cdf of the EOGR-G distribution as detailed below:

$$G(x) = \int_0^{\mathfrak{H}(F(x;\mathfrak{h}))} 2cb t e^{-b t^2}\left(1 - e^{-b t^2}\right)^{c-1} dt$$

The CDF and PDF of the EOGR-G distribution are defined as follows to describe its main mathematical structure:

$$G_{EOGR-G}(x) = \left(1 - e^{-b\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}\right)^{c} \quad \textbf{(3)}$$

$$g_{EOGR-G}(x) = 2cb\left(\frac{\mathfrak{a} f(x;\mathfrak{h})\, F^{2\mathfrak{a}-1}(x;\mathfrak{h})}{(1 - F^{\mathfrak{a}}(x;\mathfrak{h}))^3}\right)\left(e^{-b\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}\right)\left(1 - e^{-b\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}\right)^{c-1} \quad \textbf{(4)}$$

2. The Extension Odd Generalized Rayleigh Inverse Rayleigh distribution (EOGRIR)

The EOGRIR distribution is discussed, where its definition is based on the inverse Rayleigh distribution with specific parameters. The corresponding CDF and PDF are subsequently derived to characterize its statistical properties.

$$F_{IR}(x) = e^{-\frac{\lambda}{x^2}} \tag{5}$$

$$f_{IR}(x) = \frac{2\lambda}{x^3} e^{-\frac{\beta}{x^2}} \quad , \lambda, x > 0 \tag{6}$$

Using substitution of (4) into (2), we introduce the new EOGRIR distribution:

$$G_{EOGRIR}(x) = \left(1 - e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right)^c \tag{7}$$

The corresponding PDF is defined as shown in **Equation (3)**

$$g_{EOGRIR}(x) = 2acb \left(\frac{\frac{2\lambda}{x^3} e^{-\frac{\lambda}{x^2}} \left(e^{-\frac{\lambda}{x^2}}\right)^{2a-1}}{\left(1 - e^{-\frac{a\lambda}{x^2}}\right)^3} \right) \left(e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right) \left(1 - e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right)^{c-1} \tag{8}$$

For the EOGRIR distribution, the survival and hazard functions are derived and presented as follows.

$$S_{EOGRIR}(x) = 1 - \left(1 - e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right)^c \tag{9}$$

$$h_{EOGRIR}(x) = \frac{2acb \left(\frac{\frac{2\lambda}{x^3} e^{-\frac{\lambda}{x^2}} \left(e^{-\frac{\lambda}{x^2}}\right)^{2a-1}}{\left(1 - e^{-\frac{a\lambda}{x^2}}\right)^3} \right) \left(e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right) \left(1 - e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right)^{c-1}}{1 - \left(1 - e^{-b\left(\frac{e^{-\frac{a\lambda}{x^2}}}{1-e^{-\frac{a\lambda}{x^2}}}\right)^2} \right)^c} \tag{10}$$

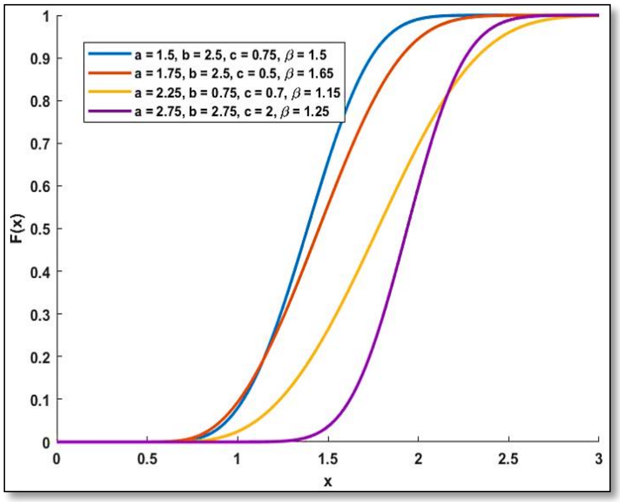

Figure 1. plots of cdf

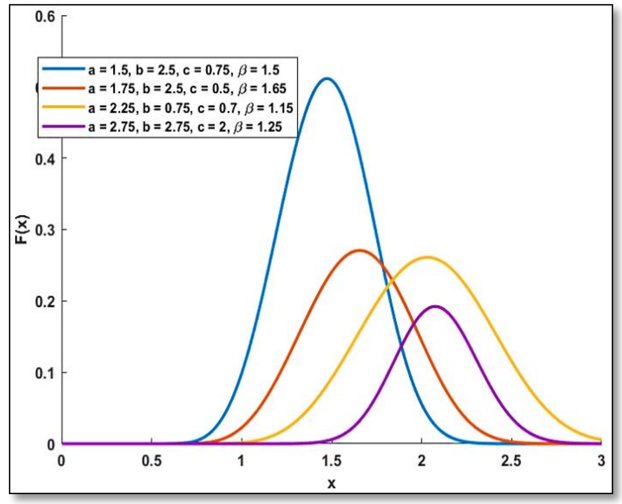

Figure 2. plots of pdf

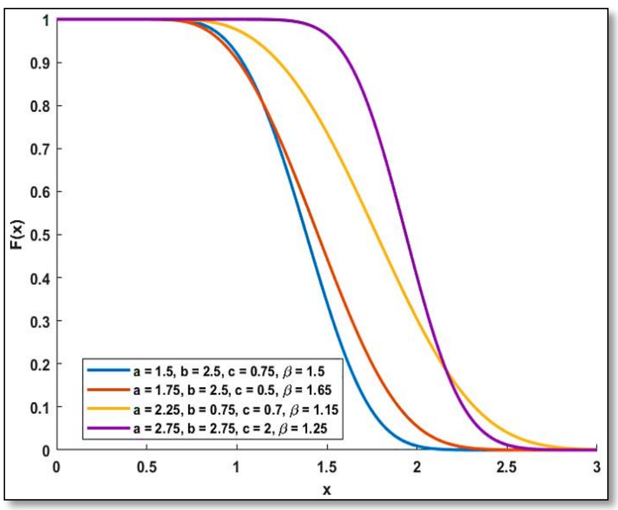

Figure 3. plots of S(x)

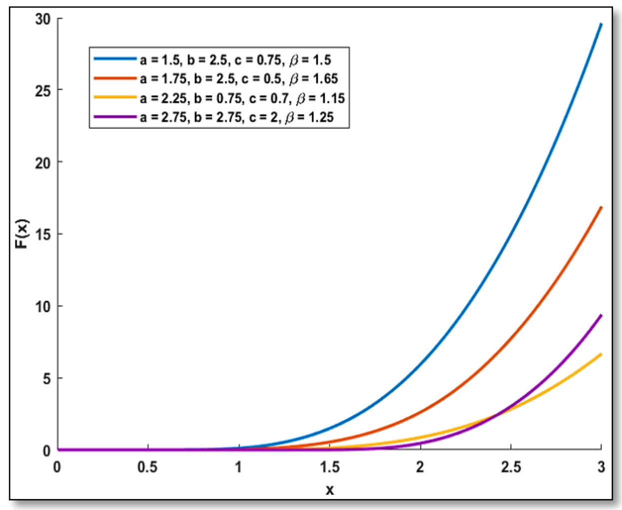

Figure 4. plots of h(x)

3. Mathematical Properties

3.1. Useful Expansion

The scaling process is essential for both (CDF) and (PDF) to facilitate and simplify mathematical properties. Simplifying **Equations (2) and (3)** relies on applying the algebraic formulation to expand them $e^{-\mathcal{Z}} = \sum_{\mathfrak{h}=0}^{\infty}\frac{(-\mathcal{Z})^{\mathfrak{h}}}{\mathfrak{h}!}$,, $[1-\mathfrak{U}]^{\mathcal{Z}} = \sum_{\mathfrak{h}=0}^{\infty}\binom{\mathcal{Z}}{\mathfrak{h}}(-\mathfrak{U})^{\mathfrak{h}}$,,$[1-\mathfrak{U}]^{-\mathcal{Z}} = \sum_{j=0}^{\infty}\frac{\Gamma(\mathcal{Z}+j)}{j!\,\Gamma(\mathcal{Z})}u^{j}$ and $|\mathfrak{U}|<1\,,\mathcal{Z}>0$.

$$g_{EOGR-G}(x) = 2\mathfrak{c}\mathfrak{b}\left(\frac{\mathfrak{a}f(x;\mathfrak{h})\,F^{\mathfrak{a}-1}(x;\mathfrak{h})}{\left(1-F^{\mathfrak{a}}(x;\mathfrak{h})\right)^{2}}\right)\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)\left(e^{-\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2}}\right)\left(1-e^{-\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2}}\right)^{\mathfrak{c}-1}$$

where

$$\left(1-e^{-\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2}}\right)^{\mathfrak{c}-1} = \sum_{i=0}^{\infty}\binom{\mathfrak{c}-1}{i}(-1)^{i}\,e^{-i\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2}}$$

Then

$$g_{EOGR-G}(x) = \sum_{ị=0}^{\infty}\binom{c-1}{ị}(-1)^{ị}\, 2\mathfrak{a}c\mathfrak{b}\left(\frac{f(x;\mathfrak{h})\,F^{\mathfrak{a}-1}(x;\mathfrak{h})}{\left(1-F^{\mathfrak{a}}(x;\mathfrak{h})\right)^2}\right)\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)e^{-(ị+1)\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}$$

Where

$$e^{-(ị+1)\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\xi)}{1-F^{\mathfrak{a}}(x;\xi)}\right)^2} = \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(ị+1)^{\ell}\mathfrak{b}^{\ell}}{\ell!}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2\ell}$$

$$g(x)_{EOGR-G} = \sum_{ị=\ell=0}^{\infty}\binom{c-1}{ị}\frac{(-1)^{ị+\ell}(ị+1)^{\ell}\mathfrak{b}^{\ell}}{\ell!}\,2\mathfrak{a}c\mathfrak{b}\left(\frac{f(x;\mathfrak{h})\,F^{\mathfrak{a}-1}(x;\mathfrak{h})}{\left(1-F^{\mathfrak{a}}(x;\mathfrak{h})\right)^2}\right)\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2j}$$

$$= \sum_{ị=\ell=0}^{\infty}\binom{c-1}{ị}\frac{(-1)^{ị+\ell}(ị+1)^{\ell}\mathfrak{b}^{\ell}}{\ell!}\,2\mathfrak{a}c\mathfrak{b}f(x;\mathfrak{h})\left(1-F^{\mathfrak{a}}(x;\mathfrak{h})\right)^{-(3+2\ell)}\,F^{2\mathfrak{a}(\ell+1)-1}(x;\mathfrak{h})$$

Where

$$\left(1-F^{\mathfrak{a}}(x)\right)^{-(3+2\ell)} = \sum_{\hbar=0}^{\infty}\frac{\Gamma(3+2\ell+\hbar)}{\hbar!\,\Gamma(3+2\ell)}\,F^{\mathfrak{a}\hbar}(x)$$

$$g_{EOGR-G}(x) = \sum_{ị=\ell=\hbar=0}^{\infty}\binom{c-1}{ị}\frac{(-1)^{ị+\ell}(ị+1)^{\ell}\mathfrak{b}^{\ell}\Gamma(3+2\ell+\hbar)}{\ell!\,\hbar!\,\Gamma(3+2\ell)}\,2\mathfrak{a}c\mathfrak{b}f(x;\mathfrak{h})F^{2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar-1}(x;\mathfrak{h})$$

By applying PDF **Equation (5)** where $f_{IR}(x) = \frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}$

$$F^{2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar-1}(x;\mathfrak{h}) = \left(e^{-\frac{\lambda}{x^2}}\right)^{2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar-1} = e^{-\frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar-1)} = e^{-\frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar)}e^{\frac{\lambda}{x^2}}$$

$$g_{EOGRIR}(x) = \sum_{ị=\ell=\hbar=0}^{\infty}\binom{c-1}{ị}\frac{(-1)^{\ell}(i+1)^{\ell}\mathfrak{b}^{\ell}\Gamma(3+2\ell+\hbar)}{\ell!\,\hbar!\,\Gamma(3+2\ell)}\,4\mathfrak{a}c\mathfrak{b}\frac{\lambda}{x^3}e^{-\frac{\lambda}{x^2}}e^{-\frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar)}e^{\frac{\lambda}{x^2}}$$

$$g_{EOGRIR}(x) = \sum_{ị=\ell=\hbar=0}^{\infty}\binom{c-1}{ị}\frac{(-1)^{\ell}(ị+1)^{\ell}\mathfrak{b}^{\ell}\Gamma(3+2\ell+\hbar)}{\ell!\,\hbar!\,\Gamma(3+2\ell)}\,4\mathfrak{a}c\mathfrak{b}\lambda\frac{1}{x^3}e^{-\frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar)}$$

$$let\ Ꝁ_{ị,\ell,\hbar} = \sum_{ị=\ell=\hbar=0}^{\infty}\binom{c-1}{ị}\frac{(-1)^{\ell}(ị+1)^{\ell}\mathfrak{b}^{\ell}\Gamma(3+2\ell+\hbar)}{\ell!\hbar!\Gamma(3+2\ell)}$$

$$g_{EOGRIR}(x) = 4Ꝁ_{ị,l,\hbar}\mathfrak{a}c\mathfrak{b}\lambda\frac{1}{x^3}e^{-\frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}\hbar)} \qquad \textbf{(11)}$$

And Expansion of cdf **Equation (2)**

$$G_{EOGR-G}(x) = \left(1-e^{-\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}\right)^{c}$$

$$\left(1-e^{-\mathfrak{b}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}\right)^{c} = \sum_{m=0}^{\infty}\binom{c}{m}(-1)^{m}\,e^{-\mathfrak{b}m\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2}$$

$$e^{-\mathfrak{b}m\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^2} = \sum_{q=0}^{\infty}\frac{(-1)^{q}\mathfrak{b}^{q}m^{q}}{q!}\left(\frac{F^{\mathfrak{a}}(x;\mathfrak{h})}{1-F^{\mathfrak{a}}(x;\mathfrak{h})}\right)^{2q}$$

$$G_{EOGR-G}(x) = \sum_{m=q=0}^{\infty}\binom{c}{m}\frac{(-1)^{m+q}\mathfrak{b}^{q}m^{q}}{q!}F^{2\mathfrak{a}q}(x;\mathfrak{h})\left(1-F^{\mathfrak{a}}(x;\mathfrak{h})\right)^{-2q}$$

$$\left(1-\mathrm{F}^{\alpha}(x;\mathfrak{h})\right)^{-2q}=\sum_{\mathcal{Z}=0}^{\infty}\frac{\Gamma(2q+\mathcal{Z})}{\mathcal{Z}!\,\Gamma(2q)}\mathrm{F}^{\alpha\mathcal{Z}}(x;\mathfrak{h})$$

$$\mathrm{G}(x)_{EOGR-G}=\sum_{m=q=\mathcal{Z}=0}^{\infty}\binom{c}{m}\frac{(-1)^{m+q}\mathfrak{b}^{q}m^{q}\Gamma(2q+\mathcal{Z})}{q!\,\mathcal{Z}!\,\Gamma(2q)}\mathrm{F}^{\alpha(2q+\mathcal{Z})}(x;\mathfrak{h})$$

Using **Equation (4),** where

$$\mathrm{F}_{IR}(x)=e^{-\frac{\lambda}{x^2}}$$

$$\mathrm{G}_{EOGRIR}(x)=\sum_{m=q=\mathcal{Z}=0}^{\infty}\binom{c}{m}\frac{(-1)^{m+q}\mathfrak{b}^{q}m^{q}\Gamma(2q+\mathcal{Z})}{q!\,\mathcal{Z}!\,\Gamma(2q)}\left(e^{-\frac{\lambda}{x^2}}\right)^{\alpha(2q+\mathcal{Z})}$$

$$\mathrm{G}_{EOGRIR}(x)=\sum_{m=q=\mathcal{Z}=0}^{\infty}\binom{c}{m}\frac{(-1)^{m+q}\mathfrak{b}^{q}m^{q}\Gamma(2q+\mathcal{Z})}{q!\,\mathcal{Z}!\,\Gamma(2q)}e^{-\frac{\lambda\alpha(2q+\mathcal{Z})}{x^2}}$$

$$\mathrm{G}_{EOGRIR}(x)=\sum_{m=q=\mathcal{Z}=0}^{\infty}\binom{c}{m}\frac{(-1)^{m+q}\mathfrak{b}^{q}m^{q}\Gamma(2q+\mathcal{Z})}{q!\,\mathcal{Z}!\,\Gamma(2q)}e^{-\lambda\alpha(2q+\mathcal{Z})x^{-2}}$$

Let $\mathbb{G}_{m,q,\mathcal{Z},p}=\sum_{m=q=\mathcal{Z}=0}^{\infty}\binom{c}{m}\frac{(-1)^{m+q}\mathfrak{b}^{q}m^{q}\Gamma(2q+\mathcal{Z})}{q!\mathcal{Z}!\Gamma(2q)}$

$$\mathrm{G}_{EOGRIR}(x)=\mathbb{G}_{m,q,\mathcal{Z},p}\,e^{-\lambda\alpha(2q+\mathcal{Z})x^{-2}} \quad \textbf{(12)}$$

3.2. Moment, Skewness and Kurtosis

This is considered one of the most important distinguishing features, as it enables the determination of the mean, kurtosis ($\mathcal{KU}$), variance, and skewness ($\mathcal{SK}$).

$$\mu_r'=E(x^r)=\int_0^{\infty}x^r g(x;\mathfrak{h})dx=4\mathrm{K}_{i,\ell,\hbar}\alpha c\mathfrak{b}\lambda\int_0^{\infty}x^{r-3}e^{-\frac{\lambda}{x^2}(2\alpha(\ell+1)+\alpha\hbar)}dx$$

Let $y=\frac{\lambda}{x^2}(2\alpha(\ell+1)+\alpha\hbar)$, *then* $x=\left(\frac{\lambda(2\alpha(\ell+1)+\alpha\hbar)}{y}\right)^{\frac{1}{2}}$

$dy=-2x^{-3}\lambda(2\alpha(\ell+1)+\alpha\hbar)dx$, $dx=\frac{-x^3}{2\lambda(2\alpha(\ell+1)+\alpha\hbar)}dy$

$$\mu_r'=4\mathrm{K}_{i,\ell,\hbar}\alpha c\mathfrak{b}\lambda\int_0^{\infty}\left(\frac{\lambda(2\alpha(\ell+1)+\alpha\hbar)}{y}\right)^{\frac{r-3}{2}}e^{-y}\frac{x^3}{2\lambda(2\alpha(\ell+1)+\alpha\hbar)}dy$$

$$\mu_r'=2\mathrm{K}_{i,l,\hbar}\alpha c\mathfrak{b}\lambda\int_0^{\infty}[\lambda(2\alpha(\ell+1)+\alpha\hbar)]^{\frac{r-3}{2}-1}y^{\frac{-r+3}{2}}e^{-y}\left(\frac{\lambda(2\alpha(\ell+1)+\alpha\hbar)}{y}\right)^{\frac{3}{2}}dy$$

$$\mu_r'=2\mathrm{K}_{i,j,\hbar}\alpha c\mathfrak{b}\lambda[\lambda(2\alpha(\ell+1)+\alpha\hbar)]^{\frac{r}{2}-1}\int_0^{\infty}y^{-\frac{r}{2}}e^{-y}dy$$

$$\mu_r'=2\mathrm{K}_{i,l,\hbar}\alpha c\mathfrak{b}\lambda[\lambda(2\alpha(\ell+1)+\alpha\hbar)]^{\frac{r}{2}-1}\Gamma\left(1-\frac{r}{2}\right) \quad \textbf{(13)}$$

When we use **Equation (13)** we get

$$\mu_1'=2\mathrm{K}_{i,l,\hbar}\alpha c\mathfrak{b}\lambda[\lambda(2\alpha(\ell+1)+\alpha\hbar)]^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)$$

$$\mu_2'=\Gamma(0)$$

$$\mu_3'=2\mathrm{K}_{i,l,\hbar}\alpha c\mathfrak{b}\lambda[\lambda(2\alpha(\ell+1)+\alpha\hbar)]^{\frac{1}{2}}\Gamma\left(-\frac{1}{2}\right)$$

$\mu_4' = 2\mathrm{K}_{i,j,k}\mathfrak{acb}\lambda[\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)]^1\Gamma(-1)$

$var(x) = \mu_2' - (\mu_1')^2$

$\mathcal{SK} = \frac{\mu_3'}{\sqrt[2]{(\mu_2')^3}}$

$\mathcal{KU} = \frac{\mu_4'}{(\mu_2')^2}$

Table 1. 1st - 4th moments, skewness, kurtosis, variance

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	β	μ_1'	μ_2'	μ_3'	μ_4'	var	$\mathcal{SK}$	$\mathcal{KU}$
0.75	3.25	2.5	0.5	0.2799	0.1804	0.1177	0.0778	0.1020	1.5368	-0.6088
	1.25	1.5	2.5	0.6742	1.0838	1.7837	3.0006	0.6293	1.5809	-0.4455
1.5	2.25	2.5	0.1	0.1876	0.0829	0.0372	0.0169	0.0478	1.5554	-0.5504
	0.25	4.5	0.7	1.1348	2.1480	4.1199	8.0037	0.8603	1.3086	-1.2654
2.5	3.5	3.75	1.5	0.3390	0.7357	1.6098	3.5512	0.6208	2.5509	3.5604
	3	2.75	0.25	0.1386	0.1238	0.1118	0.1019	0.1046	2.5648	3.6426

Table 1 shows that the proposed distribution outperforms the current distribution. A statistical comparison was conducted based on the distributional characteristics. This indicates that it has significant flexibility in modeling asymmetric data with both light and heavy tails, thus resulting in a clear statistical advantage over current distributions.

3.3. Moment Generating Function (MGF)

MGF is one of the important properties that we can find the μ and it can be given by

$$M_x(t)_{EOGIR} = \mu(e^{tx}) = \int_0^\infty e^{tx} g(x;\mathfrak{h})dx$$

$$M_x(t)_{EOGRIR} = 4\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda \int_0^\infty e^{tx}\,\frac{1}{x^3} e^{-\frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)} dx$$

$Let\ y = \frac{\lambda}{x^2}(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)\ , then\ x = \left(\frac{\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)}{y}\right)^{\frac{1}{2}}, dy = -2x^{-3}\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)dx$

$dx = \frac{-x^3}{2\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)}dy\ ,\ M_x(t)_{EOGRIR} = 4\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda\sum_{i=0}^{\infty}\frac{t^{h}x^{h}}{h!}\int_0^\infty \frac{1}{x^3}e^{-y}dx$

$$M_x(t)_{EOGRIR} = -4\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda \sum_{h=0}^{\infty}\frac{t^{h}}{h!}\int_0^\infty x^{-3+h}e^{-y}\frac{x^3}{2\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)}dy$$

$$M_x(t)_{EOGRIR} = -4\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda \frac{1}{2\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)}\sum_{h=0}^{\infty}\frac{t^{h}}{h!}\int_0^\infty x^{h}e^{-y}dy$$

$$M_x(t)_{EOGRIR} = -4\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda \frac{1}{2\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)}\sum_{h=0}^{\infty}\frac{t^{h}}{h!}\int_0^\infty \left(\left(\frac{\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)}{y}\right)^{\frac{1}{2}}\right)^{h} e^{-y}dy$$

$$M_x(t)_{EOGRIR} = -2\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda\left(\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)\right)^{\frac{h}{2}-1}\sum_{h=0}^{\infty}\frac{t^{h}}{h!}\int_0^\infty y^{-\frac{h}{2}}e^{-y}dy$$

$$M_x(t)_{EOGRIR} = -2\mathrm{K}_{i,j,k}\mathfrak{acb}\lambda\left(\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)\right)^{\frac{h}{2}-1}\sum_{h=0}^{\infty}\frac{t^{h}}{h!}\Gamma\left(1-\frac{h}{2}\right)$$

$$M_x(t)_{EOGRIR} = -2\mathrm{K}_{i,\ell,k}\mathfrak{acb}\lambda\left(\lambda(2\mathfrak{a}(\ell+1)+\mathfrak{a}k)\right)^{\frac{h}{2}-1}\sum_{h=0}^{\infty}\frac{t^{h}}{h!}\Gamma\left(1-\frac{h}{2}\right) \quad (14)$$

3.4. Quantile function

It is one of the most important mathematical properties through which random numbers are generated in the simulation using **Equation (6)**

$$\mathcal{U} = \left(1 - e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}} \quad \textbf{(15)}$$

$$Q(\mathcal{U}) = \left[\mathfrak{a}\lambda\left(-ln\left[\left(\frac{-1}{\mathfrak{b}}ln\left(1-\mathcal{U}^{\frac{1}{\mathfrak{c}}}\right)^{\frac{-1}{2}}+1\right)^{-1}\right]\right)^{-1}\right]^{\frac{1}{2}} \quad \textbf{(16)}$$

3.5. Order Statistic

Let$x_{1:\acute{k}} \le x_{2:\acute{k}} \le x_{3:\acute{k}} \le \cdots \le x_{\mathcal{K}:\mathcal{K}}$ The order statistics are expressed $x_1, x_2, \ldots, x_{\acute{k}}$ of size k from EOGRIR.

$$g_{\mathcal{P}:\acute{k}}(x;\mathfrak{h}) = \frac{\acute{k}!}{(\mathcal{P}-1)!\,(\acute{k}-\mathcal{P})!}\left(G_{EOGRIR}(x;\mathfrak{h})\right)^{\mathcal{P}-1}\left(1-G_{EOGRIR}(x;\mathfrak{h})\right)^{\acute{k}-\mathcal{P}} g_{EOGRIR}(x;\mathfrak{h})$$

From **Equations (6) and (7)** into$g_{p,k}(x;\mathfrak{h})$, The result is as follows:

$$g_{\mathcal{P}:\acute{k}}(x;\acute{k}) = \frac{\acute{k}!}{(\mathcal{P}-1)!\,(\acute{k}-\mathcal{P})!}\left[\left(1 - e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}}\right]^{\mathcal{P}-1}\left[1 - \left(1 - e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\acute{k}}{x^2}}}\right)^2}\right)^{\mathfrak{c}}\right]^{\acute{k}-\mathcal{P}} 2\mathfrak{a}\mathfrak{c}\mathfrak{b}\left(\frac{\frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}\left(e^{-\frac{\lambda}{x^2}}\right)^{2\mathfrak{a}-1}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}\right)^3}\right)\left(e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)\left(1 - e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}-1} \quad \textbf{(17)}$$

When $\mathcal{P} = 1$, and $\mathcal{P} = \acute{k}$that will be given:

$$g_{1:\mathcal{K}}(x;\mathfrak{h}) = 2\mathfrak{a}\mathfrak{c}\mathfrak{b}\frac{\acute{k}!}{(\acute{k}-1)!}\left[1-\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}}\right]^{\acute{k}-1}\left(\frac{\frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}\left(e^{-\frac{\lambda}{x^2}}\right)^{2\mathfrak{a}-1}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}\right)^3}\right)\left(e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}-1} \tag{18}$$

$$g_{\acute{k}:\acute{k}}(x;\mathfrak{h}) = 2\mathfrak{a}\mathfrak{c}\mathfrak{b}\frac{\acute{k}!}{(\acute{k}-1)!}\left[\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}}\right]^{\acute{k}-1}\left(\frac{\frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}\left(e^{-\frac{\lambda}{x^2}}\right)^{2\mathfrak{a}-1}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}\right)^3}\right)\left(e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}-1} \tag{19}$$

3.6. Rėnyi Entropy

One of the most important properties that plays a fundamental role in extracting information is the randomness measure defined as follows:

$$Ŧ_R(Ŋ) = \frac{1}{1-Ŋ}log\int_0^{\infty} g^{Ŋ}{}_{EOGIR}(x;\mathfrak{h})dx$$

By using **Equation (7),** we get:

$$Ŧ_R(Ŋ) = \frac{1}{1-Ŋ}log\int_0^{\infty}\left[2\mathfrak{a}\mathfrak{c}\mathfrak{b}\left(\frac{\frac{2\lambda}{x^3}e^{-\frac{\lambda}{x^2}}\left(e^{-\frac{\lambda}{x^2}}\right)^{2\mathfrak{a}-1}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}\right)^3}\right)\left(e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x^2}}}\right)^2}\right)^{\mathfrak{c}-1}\right]^{Ŋ}dx \tag{20}$$

4. Maximum Likelihood Estimation method

Let. $x_1, x_2, \ldots\ldots, x_{ḱ}$ be a random of EOGRIR. The likelihood is:

$$L(x;\mathfrak{h}) = \prod_{\mathfrak{s}=1}^{ḱ} g(x;\mathfrak{h})$$

$$L(x;\mathfrak{h}) = \prod_{\mathfrak{s}=1}^{ḱ} 4\mathfrak{acb}\lambda \left(\frac{x^{-3}e^{-\frac{\lambda}{x_\mathfrak{s}^2}}\left(e^{-\frac{\lambda}{x_\mathfrak{s}^2}}\right)^{2\mathfrak{a}-1}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}\right)^3}\right)\left(e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right)\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right)^{\mathfrak{c}-1}$$

$$L(x;\mathfrak{h}) = \prod_{\mathfrak{s}=1}^{ḱ} 4\mathfrak{acb}\beta \left(\frac{x^{-3}\left(e^{-\frac{\lambda}{x_\mathfrak{s}^2}}\right)^{2\mathfrak{a}}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}\right)^3}\right)\left(e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right)\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right)^{\mathfrak{c}-1}$$

Now the log-Likelihood of EOGRIR is:

$$\ell = ḱ\log 4 + ḱ\log\mathfrak{a} + ḱ\log\mathfrak{c} + ḱ\log\mathfrak{b} + ḱ\log\beta - 3\sum_{\mathfrak{s}=0}^{ḱ}\log x_\mathfrak{s} - 2\mathfrak{a}\lambda(1+\mathfrak{b})\sum_{\mathfrak{s}=1}^{ḱ}\frac{1}{x_\mathfrak{s}^2}$$
$$-(3+2\mathfrak{b})\sum_{\mathfrak{s}=1}^{ḱ}\log\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}\right) + (\mathfrak{c}-1)\sum_{\mathfrak{s}=1}^{ḱ}\log\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right)$$

$$\frac{\partial(\ell)}{\partial\mathfrak{c}} = \frac{ḱ}{\mathfrak{c}} + \sum_{\mathfrak{s}=1}^{ḱ}\log\left(1-e^{-\mathfrak{b}\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right) \quad \textbf{(21)}$$

$$\frac{\partial(\ell)}{\partial\mathfrak{b}} = \frac{ḱ}{\mathfrak{b}} - 2\mathfrak{a}\lambda\sum_{\mathfrak{s}=1}^{ḱ}\frac{1}{x_\mathfrak{s}^2} - 2\sum_{\mathfrak{s}=1}^{ḱ}\ln\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}\right) \quad \textbf{(22)}$$

$$\frac{\partial(\ell)}{\partial\lambda} = \frac{ḱ}{\lambda} - 2\mathfrak{a}(1+\mathfrak{b})\sum_{\mathfrak{s}=1}^{ḱ}\frac{1}{x_\mathfrak{s}^2} - (3+2\mathfrak{b})\sum_{\mathfrak{s}=1}^{ḱ}\frac{\mathfrak{a}}{x_\mathfrak{s}^2}\cdot\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}$$
$$+(\mathfrak{c}-1)\sum_{\mathfrak{s}=1}^{ḱ}\frac{2e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}\, e^{-\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}}{\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}\right)\left(1-e^{-\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}\right)^2}\right)^2}\cdot\frac{-\mathfrak{a}}{x_\mathfrak{s}^2}\cdot\frac{e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_\mathfrak{s}^2}}} \quad \textbf{(23)}$$

$$\frac{\partial(\ell)}{\partial \mathfrak{a}} = \frac{\acute{k}}{\mathfrak{a}} - 2\lambda(1+\mathfrak{b})\sum_{\mathfrak{s}=1}^{\acute{k}} \frac{1}{x_{\mathfrak{s}}^{2}} - (3+2\mathfrak{b})\lambda \sum_{\mathfrak{s}=1}^{\acute{k}} \frac{e^{-\frac{\mathfrak{a}\lambda}{x_{\mathfrak{s}}^{2}}}}{x_{\mathfrak{s}}^{2}\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_{\mathfrak{s}}^{2}}}\right)} - 2(\mathfrak{c}-1)\lambda \sum_{\mathfrak{s}=1}^{\acute{k}} \frac{\left(e^{-\frac{\mathfrak{a}\lambda}{x_{\mathfrak{s}}^{2}}}\right)^{2}}{x_{\mathfrak{s}}^{2}\left(1-e^{-\frac{\mathfrak{a}\lambda}{x_{\mathfrak{s}}^{2}}}\right)^{3}\left(1-e^{-\left(\frac{e^{-\frac{\mathfrak{a}\lambda}{x_{\mathfrak{s}}^{2}}}}{1-e^{-\frac{\mathfrak{a}\lambda}{x_{\mathfrak{s}}^{2}}}}\right)^{2}}\right)} \qquad \textbf{(24)}$$

The above **Equations**, after setting them equal to 0, are solved using R to find the estimated values of the parameters.

5. Conclusion

This study focused on presenting a new family for generating distributions (EOGR-G), where the mathematical structure of the basic statistical functions of the family was presented. In terms of application, a sub-model called (EOGRIR) was presented, and the functions and statistical properties were studied, in addition to estimating parameters using the Maximum Likelihood Estimation method. The study also explores how new submodels and broader applications can be developed from this model.

Acknowledgment

The researchers sincerely thanks the editor-in-chief and the preparatory committee of the Ibn Al-Haitham Journal for Pure and Applied Sciences.

Conflict of Interest

The authors declare no conflict of interest.

Funding

There is no funding for the article

References

1. Tanış C, Saraçoğlu B. Cubic rank transmuted inverse Rayleigh distribution: Properties and applications. Sigma J Eng Nat Sci. 2022;40(2):421-432. https://doi.org/10.14744/sigma.2022.00042
2. Blumenson L, Miller K. Properties of generalized Rayleigh distributions. Ann Math Stat. 1963;34(3):903-910. https://doi.org/10.1214/aoms/1177704013
3. Yahaya A, Doguwa S. On Rayleigh-exponentiated odd generalized-Pareto distribution with its applications. Benin J Stat. 2022;5:89-107.
4. Smadi M, Alrefaei M. New extensions of Rayleigh distribution based on inverted-Weibull and Weibull distributions. Int J Electr Comput Eng. 2021;11(6): 5107-5118. https://doi.org/10.11591/ijece.v11i6
5. Hemeda S, Ul Haq M. The odd inverse Rayleigh family of distributions: simulation and application to real data. Appl Appl Math. 2020;15(2):7.
6. Arshad R, Tahir M, Chesneau C, Jamal F. The Gamma Kumaraswamy-G family of distributions: theory, inference and applications. Stat Transit New Ser. 2020;21(5):17-40. https://doi.org/10.21307/stattrans-2020-053%0A
7. Nasiru S. Extended odd Fréchet-G family of distributions. J Probab Stat. 2018;2018(1):2931326. https://doi.org/10.1155/2018/2931326
8. Alzaatreh A, Ghosh I. On the Weibull-X family of distributions. J Stat Theory Appl. 2015;14(2):169-183. https://doi.org/10.2991/jsta.2015.14.2.5
9. Mohammed A, Hamdani H, Zakari Y, Abdullahi J, Sadiq I, Ouertani M, Nasiru S, Elgarhy M. On the

Rayleigh exponentiated odd generalized-inverse exponential distribution with properties and applications. Eng Rep. 2025;7(11):e70457. https://doi.org/10.1002/eng2.70457

10. Monef A . Odd generalized Rayleigh-exponential distribution statistical properties with real data application. Kirkuk J Sci. 2025;20(1):11-22. https://doi.org/10.32894/kujss.2024.152614.1172
11. Kadhim A , Rasheed H . Bayesian estimation for two parameters of weibull distribution under generalized weighted loss function. Ibn Al-Haitham J Pure Appl Sci. 2021;34(4):116-129. https://doi.org/10.30526/34.4.2709
12. Tahir M , Cordeiro G , Alizadeh M, Mansoor M, Zubair M, Hamedani G G. The odd generalized exponential family of distributions with applications. J Stat Distrib Appl. 2015;2(1):1. https://doi.org/10.1186/s40488-014-0024-2
13. Ogunsanya A, Yahya W, Mobolaji A, Christiana, Aderele O, Ekum M. A new three-parameter weibull inverse rayleigh distribution: theoretical development and applications. Math Stat. 2021;9(3):249-272. https://doi.org/10.13189/ms.2021.090306
14. Alizadeh M, Altun E, Afify A, Ozel G. The extended odd Weibull-G family: properties and applications. Fac Sci Univ Ankara Ser A1 Math Stat. 2018;68(1):161-186.
15. Yirsaw A, Goshu A. Extended Rayleigh probability distribution to higher dimensions. J Probab Stat. 2024;2024(1):7677855. https://doi.org/10.1155/2024/7677855
16. Elgarhy M, Alrajhi S. The odd Fréchet inverse Rayleigh distribution: statistical properties and applications. J Nonlinear Sci Appl. 2019;12:291-299. https://doi.org/10.22436/jnsa.012.05.03
17. Mahdi G, Kalaf B, Khaleel M. Enhanced supervised principal component analysis for cancer classification. Iraqi J Sci. 2021:1321-1333. https://doi.org/10.24996/ijs.2021.62.4.28
18. Kundu D, Raqab M. Generalized Rayleigh distribution: different methods of estimations. Comput Stat Data Anal. 2005;49(1):187-200.
19. Rezk H. Extended reciprocal Rayleigh distribution: copula, properties and real data modeling. Pak J Stat Oper Res. 2020;16(1):35-52.
20. Banerjee P, Bhunia S. Exponential transformed inverse rayleigh distribution: statistical properties and different methods of estimation. Austrian J Stat. 2022;51(4):60-75. https://doi.org/10.17713/ajs.v51i4.1338
21. Alzaghal A, Famoye F, Lee C. Exponentiated TX family of distributions with some applications. Int J Stat Probab. 2013;2(3):31. https://doi.org/10.5539/ijsp.v2n3p31
22. Alexander C, Cordeiro G, Ortega E, Sarabia J. Generalized beta-generated distributions. Comput Stat Data Anal. 2012;56(6):1880-1897. https://doi.org/10.1016/j.csda.2011.11.015
23. Suksaengrakcharoen S, Bodhisuwan W. A new family of generalized gamma distribution and its application. J Math Stat. 2014;10(2):211. https://doi.org/10.3844/jmssp.2014.211.220
24. Torabi H, Montazeri N. The gamma-uniform distribution and its applications. kybernetika. 2012;48(1):16-30.
25. Merovci F, Elbatal I. Weibull Rayleigh distribution: Theory and applications. Appl Math Inf Sci. 2015;9(4):2127.