Improving Finite Element Methods (FEM) using optimization techniques to reduce asymptotic error

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Abstract

The Finite Element Method (FEM) is one of the most important numerical methods used in solving partial differential equations (FDIs), with wide applications in engineering and science. This is due to its high flexibility in representing complex geometric domains and handling diverse boundary conditions. However, the accuracy of the resulting numerical solutions is directly affected by the nature of the numerical approximation, particularly the size of the elements, the order of the approximation functions, and the quality of the numerical network. This leads to the emergence of what is known as asymptotic error, especially in problems with complex behavior or multiscale problems. This research aims to improve finite element methods by integrating optimization techniques with adaptation strategies to reduce asymptotic error and achieve an effective balance between solution accuracy and computational cost. The study relies on a theoretical analysis of error sources in FEM, including approximation error, partitioning error, and pre- and post-error analysis, in addition to examining the concepts of stability and convergence. Various adaptation techniques were also reviewed, such as element size-dependent (h), approximation order-dependent (p), and mixed adaptation (hp). On the practical side, a proposed algorithm was developed that estimates the dimensional error and incorporates it into an optimal optimization model. This model defines an objective function representing the magnitude of the asymptotic error, subject to constraints related to the number of degrees of freedom and computational cost. The results showed that the proposed algorithm is capable of reducing the asymptotic error and improving the convergence

rate compared to traditional strategies, while also allocating numerical resources more efficiently within the studied domain. The study confirms that integrating optimization with Finite Element Method (FEM) represents an effective methodological approach for developing numerical solutions and improving their reliability in engineering and scientific applications.
Keywords: Finite Element Method (FEM); Asymptotic Error; Numerical Adaptation; Optimization; Dimensional Error Analysis; Convergence Rate.

1.1 Introduction

The Finite Element Method (FEM) is a fundamental numerical method used in modern engineering and scientific studies to solve partial differential equations describing numerous physical and engineering phenomena. Its appeal stems from its high flexibility in representing complex geometric shapes and its applicability to irregular domains with diverse boundary conditions. Significant advancements in computing technologies and engineering software have broadened the scope of FEM's application across various fields, including structural analysis, heat transfer, fluid mechanics, and linear elasticity, making it an indispensable tool in applied and academic research .

Despite the numerous advantages of the Finite Element Method, the accuracy of the numerical solutions obtained using it remains closely linked to the nature of the numerical approximation employed, particularly the size of the finite elements, the order of the approximation functions, and the quality of the numerical network. This leads to the emergence of what is known as convergent error, which is a fundamental problem in FEM applications, especially when dealing with problems characterized by unique solutions or sharp variations in physical behavior within the studied domain. Relying on regular numerical networks or traditional optimization strategies may increase computational costs without achieving the desired improvement in accuracy.

To address this issue, several adaptive strategies have been proposed, such as element size optimization (h-adaptivity), approximation order optimization (p-adaptivity), or a combination of both (hp-adaptivity), with the aim of improving the convergence rate and reducing error. However, these strategies are often based on local rules or partial error estimates, resulting in suboptimal adaptation decisions across the entire domain, particularly in complex or multiscale problems. This underscores the importance of employing optimization techniques as a scientific and systematic approach to improving finite element methods. These techniques allow for the reformulation of the adaptation process within a clear mathematical framework based on defining an objective function that represents the magnitude of the asymptotic error or its upper limit, while imposing constraints related to the number of degrees of freedom or computational cost. This integration contributes to guiding the allocation of numerical resources more efficiently and accurately, leading to improved FEM performance and a systematic and deliberate reduction of asymptotic error. Therefore, this study aims to improve finite element methods using optimization techniques, with the goal of reducing asymptotic error and increasing the efficiency of numerical solutions. This aligns with the requirements of postgraduate studies in Iraqi universities and contributes to the development of relevant engineering and scientific applications.

2.1 Research Problem

Despite the widespread use of the finite element method (FEM) in solving partial differential equations, the accuracy of the resulting numerical solutions still suffers from the asymptotic error problem, particularly when dealing with problems of complex geometry or solutions involving singularities and steep slopes. Traditional strategies for improving FEM accuracy, such as optimizing the element size or increasing the approximation order, often rely on local adaptive rules or numerical expertise. This leads to a suboptimal distribution of degrees of freedom and an unjustified increase in computational cost without a proportional improvement in accuracy. Therefore, there is a need to develop a systematic framework based on optimal optimization techniques to guide the FEM adaptation process scientifically, aiming to reduce asymptotic error while maintaining computational efficiency.

3.1 Significance of the Research

The significance of this research stems from its addressing one of the fundamental problems in numerical analysis using the finite element method: achieving a balance between solution accuracy and computational cost. Integrating optimization techniques with FEM contributes to improving convergence rates and systematically reducing numerical error, which positively impacts the reliability of results in engineering and scientific applications. This study also gains academic significance by providing a mathematical and numerical framework that can be adopted in advanced research and by graduate students at Iraqi universities. Furthermore, it has practical importance in supporting the use of FEM in fields requiring high precision, such as structural analysis, heat transfer, and fluid mechanics.

4.1 Research Objectives

This research aims to:

- Study and analyze the sources of asymptotic error in finite element methods.
- Develop a mathematical framework that integrates FEM with optimization techniques to reduce asymptotic error.
- Formulate a suitable objective function that represents the magnitude of the numerical error, taking into account computational constraints.
- Design an improved adaptive algorithm for determining the optimal distribution of element sizes and approximation order.

- Compare the performance of the proposed method with traditional methods in terms of accuracy and computational cost.

Evaluate the convergence rate and numerical stability of the resulting solutions. 5.1 Research Hypothesis

This research is based on the following hypothesis: Combining optimization techniques with finite element methods leads to a reduction in asymptotic error and an improvement in convergence rate, compared to traditional adaptive methods, at the same level of computational cost or number of degrees of freedom. (Zienkiewicz & Taylor 2000)

6.1 Research Questions

What is the best objective function that relates accuracy to cost in FEM？

What type of optimization is most suitable: convex/non-convex? Gradual/non-gradual?

How can we ensure numerical stability when changing h and p？

What is the effect of constraints (DoF, time, memory) on the optimal solution for the network?

7.1 Research Limitations

Subjective Limitations: This research is limited to optimizing finite element methods using optimization techniques and reducing asymptotic error.

Mathematical Limitations: This research focuses on elliptic and quasi-elliptical partial differential equation problems. Spatial (Geometric) Boundaries: Two-dimensional numerical applications, with reference to the possibility of three-dimensional expansion.Temporal Boundaries: Stationary problems, with limited reference to temporal problems.

8.1 Research Methodology

This research employs an analytical and applied numerical approach. It begins by examining the theoretical foundations of the finite element method, error analysis, and convergence analysis. Optimization concepts are then applied to formulate a mathematical model to minimize asymptotic error. The research also utilizes an experimental numerical approach, implementing proposed algorithms and conducting benchmark tests to compare results with traditional methods. Performance is analyzed in terms of accuracy, convergence rate, and computational cost, ultimately leading to the derivation of appropriate scientific conclusions.

Theoretical Basis of the Finite Element Method and Error Analysis

This chapter aims to present the theoretical basis of the Finite Element Method (FEM), one of the most important numerical methods used in solving partial differential equations. It focuses on the underlying mathematical concepts and analyzes the sources of numerical error and convergence mechanisms. Understanding the theoretical structure of FEM and analyzing the resulting error is a fundamental step in developing improved methods based on optimization techniques. Systematically reducing asymptotic error is impossible without understanding its nature, causes, and mathematical limits. (Ainsworth & Oden: 2000)

2-2 Emergence and Development of the Finite Element Method

The Finite Element Method emerged in the mid-20th century as a result of the need for precise numerical solutions to complex engineering problems, particularly in the field of stress analysis in engineering structures. This method has gradually evolved from energy approximation methods, such as the Rayleigh-Ritz and Gallerkin methods, into a comprehensive numerical framework that relies on dividing the geometric field into small, interconnected elements. Within each element, the solution is approximated using simple functions. With the development of digital computing, the use of Finite Element Method (FEM) has expanded to encompass multiple fields, including heat transfer, fluid mechanics, elasticity, electromagnetism, and polyphysical problems. This development has contributed to establishing FEM as a fundamental tool in scientific research and engineering applications, with the emergence of specialized software based on its theoretical and numerical foundations.

2 3–Mathematical Formulation of the Finite Element Method

The Finite Element Method relies on transforming the original differential problem into a weak form, allowing for less regular solutions than those found in classical models. The formulation typically begins with a partial differential equation of the general form: Ω in the field $f = L(u)$ with appropriate boundary conditions on the field boundary $\partial\Omega$. The equation is multiplied by a suitable test function and then integrated over the domain, applying mathematical justifications such as integration by parts, to obtain the weak formula. The solution, u, is then approximated within a finite-dimensional functional space constructed using basis functions defined on the elements of the numerical lattice.

The basic idea of FEM is to choose an approximate space:

$V \subset V_h$

such that the approximate solution u_h is an element in this space, and is determined by solving a system of algebraic equations resulting from projecting the original problem onto this space. (Ainsworth & Oden: 2000: 43)

2 4–Dividing the Field and Constructing the Numerical Mesh

Dividing the geometric field into specific elements is a pivotal step in applying FEM. This division is called a numerical mesh, and it typically consists of elements such as triangles or tetragons in two-dimensional problems, or tetrahedra in three-dimensional problems. The properties of the mesh directly affect the accuracy of the numerical solution, as the size, shape, and regularity of the elements all control the quality of the approximation. The size of an element is usually denoted by the parameter h, which represents the maximum diameter of the element. The smaller h, the better the accuracy of the solution in principle, but this is accompanied by an increase in the number of degrees of freedom and the computational cost.

2 5–Approximation Functions and Element Order

It depends Approximation within each element relies on the selection of appropriate basis functions, often polynomial functions of a certain order p. The element's order represents the degree of mathematical complexity of the approximation function within that element. Linear elements rely on first-order functions, while quadratic or higher-order elements use functions of higher orders. The choice of approximation order significantly affects the accuracy of the solution and the convergence rate. Increasing p can lead to a substantial improvement in accuracy, especially in problems with smooth solutions, without requiring a significant reduction in element size. However, this choice can also increase the complexity of numerical matrices and make them more difficult to solve. (Babuška & Rheinboldt: 1978)

2 6–Sources of Error in the Finite Element Method

Errors in the finite element method arise from a combination of factors related to the nature of numerical approximation, the formulation of the mathematical model, and the numerical calculations. Understanding the sources of error is a crucial step in analyzing the accuracy of numerical solutions and in developing effective strategies to reduce the asymptotic error and improve the method's performance. Error sources in FEM can be classified into several main types, each with a direct or indirect effect on the quality of the solution. Numerical. First: Approximation Error, which is the error resulting from representing the real solution within a finite-dimensional approximation space. Approximation functions used within elements generally cannot represent the real solution with perfect accuracy, especially if the solution exhibits non-smooth behavior or sharp changes. The magnitude of this error depends on the order of the chosen approximation functions and the regularity of the real solution; the error usually decreases with increasing approximation order if the solution is sufficiently smooth.

2- Discretization Error, which results from dividing a geometric field into specific elements of a certain size and shape. This error is directly affected by the size of the elements and the quality of the numerical network, such as the regularity of the elements and their aspect ratios. Elements with irregular shapes or sharp angles can lead to a decrease in the accuracy of the solution and an increase in the asymptotic error, even when using high-order approximation functions.

3- Modeling Error, which occurs when simplifying the physical or mathematical model of the problem under study, such as neglecting certain factors or relying on approximate assumptions that do not accurately reflect reality. Although this type of error is not directly related to the finite element method, it affects the accuracy of the final solution and cannot be fully compensated for by improving numerical approximation.

4- Numerical Error, which is the error resulting from solving the large algebraic system generated by FEM using numerical approximation algorithms, in addition to round-off errors. This type of error increases with the size of the system or with poor adaptation of the resulting matrices, which can affect the stability and accuracy of the solution (Babuška & Rheinboldt, 1978). Finally, there is the boundary condition error, which can arise from an inaccurate representation of boundary conditions at domain boundaries, especially when using numerical approximations or imposing complex boundary conditions. This error is significant in some applications, as it can lead to a considerable deviation in the numerical solution within the domain.

Thus, it is clear that the asymptotic error in the finite element method is the result of the interaction of several different sources, necessitating the adoption of comprehensive analysis and optimization strategies that consider these sources collectively. Identifying these errors and understanding their nature is a crucial step in developing improved finite element methods that rely on adaptation and optimization to reduce error and enhance the reliability of numerical solutions. (Babuška & Rheinboldt: 1978: 740)

2 7–The Concept of Convergent Error: Convergent error is defined as the difference between the true solution to the problem and the numerical solution obtained using the FEM. It is usually measured using mathematical criteria such as the L^2 or H^1 criterion. A method is said to be convergent if this error decreases as the grid is improved or the approximation rank increases.

The convergence rate is expressed by the following relation:

$$Ch^p \leq \| u_h - u \|$$

Here, C represents a constant that depends on the nature of the problem, h is the size of the element, and p is the order of the approximation. This relationship shows that, theoretically, decreasing h or increasing p will reduce the error.

2 8–Priori Error Analysis

Priori error analysis is a crucial theoretical foundation in the study of the finite element method. It aims to estimate the behavior of numerical error before performing numerical calculations, based on the mathematical properties of the problem and the approximation space used. This type of analysis relies on assumptions concerning the regularity of the real solution to the partial differential equation, as well as the nature of the approximation functions and the size of the elements used to divide the geometric domain. Priori error analysis involves formulating mathematical relationships that connect the magnitude of the asymptotic error to the parameters of the numerical approximation, such as the size of the element h and the order of the approximation functions p. These relationships show that the error decreases at a specific rate as the size of the elements decreases or the order of the approximation increases, according to estimates of the form:

$$Ch^p \leq \| u_h - u \|$$

Here, u represents the true solution, u_h the approximate numerical solution, and C a constant that depends on the problem's properties and the regularity of the solution. These estimates are crucial for ensuring the convergence of the finite element method and for theoretically determining the rate of convergence. The importance of error pre-analysis lies in its provision of mathematical assurances regarding the reliability of the numerical method and its role in selecting the appropriate element type and approximation order before proceeding with the numerical solution. However, this analysis suffers from some practical limitations. It typically assumes that the true solution is sufficiently smooth, an assumption that may not hold true in problems involving geometric singularities or abrupt changes in the solution, such as those with acute angles or irregular boundaries. Despite these limitations, error pre-analysis remains a fundamental theoretical tool for understanding convergence behavior in FEM and serves as an important starting point for developing more advanced optimization strategies. The results of pre-analysis are used to build error-minimizing prototypes and to establish guidelines and parameters within optimization frameworks, thereby contributing to the systematic improvement of the accuracy of numerical solutions. Thus, pre-error analysis represents the theoretical foundation that complements post-error analysis and contributes to a comprehensive understanding of the asymptotic error behavior in the finite element method, paving the way for the application of optimization techniques in developing more efficient and accurate numerical methods. (Becker & Rannacher: 2001)

2 9–Posterior Error Analysis

Posterior error analysis is a cornerstone of modern finite element methods. It aims to estimate the numerical error after obtaining an approximate solution, based on information extracted from the solution itself and the problem data. Unlike pre-error analysis, which relies on theoretical assumptions about the regularity of the true solution, posterior analysis is characterized by its flexibility and direct relevance to practical applications, making it an effective tool for guiding adaptation strategies and improving numerical accuracy. Posterior error analysis relies on constructing error estimators, which represent an approximate measure of the deviation of the numerical solution from the true solution, either at the domain level or at the level of each individual element. These estimators are often derived from the residual of the differential equation within the elements or from the discontinuity of the derivatives at the element boundaries. These estimators are distinguished by their ability to identify the regions that contribute most to the overall error, allowing for precise local optimization decisions. Dimensional error estimators are used to classify elements according to their importance in error reduction. Specific criteria are used to select elements that require improvement in size or approximation order. This localized focus is one of the most significant advantages of dimensional analysis, as it contributes to computational cost optimization by concentrating numerical resources only in critical regions, rather than optimizing the entire network uniformly. Despite the high effectiveness of dimensional error analysis, relying on it in isolation from a comprehensive methodological framework can lead to suboptimal adaptation decisions across the domain, especially in problems with complex or multiscale behavior. Hence, the importance of integrating dimensional error estimators into optimization models becomes apparent. They can be used as objective functions or constraints in optimization problems aimed at reducing the overall error while respecting computational constraints. Thus, dimensional error analysis forms a crucial link between numerical solutions and adaptation and optimization methods, and is a pivotal element in developing optimized specific element methods capable of efficiently and reliably reducing asymptotic error and improving convergence rates. (Becker & Rannacher:2001:15)

2 10–Stability and Convergence in FEM

Stability and convergence are fundamental concepts for the reliability of the finite element method. A numerical solution cannot be relied upon unless it is stable and converges to the true solution of the problem. Stability in FEM refers to the ability of a numerical solution to resist being overly affected by small perturbations in the data, such as rounding errors, division errors, or minor changes in boundary conditions. Mathematically, stability is expressed as the existence of a constant upper limit for the numerical solution criterion as a function of the data criterion, such that amplifying errors does not lead to the collapse of the solution or a loss of accuracy. Stability is closely related to weak problem formulation and the choice of rounding spaces and test functions. In elliptical problems, for example, stability is usually achieved by satisfying the coercivity condition and the continuity of the bilinear form, according to the Lax–Milgram theorem. In other problems, such as mixed problems or flow problems, stability requires additional conditions, such as satisfying the Babushka-Brezi condition (Inf-Sup Condition), to ensure that no non-physical numerical solutions emerge. Convergence, on the other hand, represents another fundamental property: the numerical solution obtained from the FEM approaches the true solution as the numerical approximation is improved, either by reducing the size of the elements or increasing the order of the approximation functions. Convergence is usually measured using appropriate mathematical parameters, such as L^2 and H^1, where a decrease in error with lattice improvement is a clear indicator of convergence. The convergence rate is expressed by mathematical relationships that show how quickly the error decreases with respect to the approximation parameters. The relationship between stability and convergence is complementary; stability is a necessary condition for convergence, but it is not sufficient on its own. A solution may be stable without achieving acceptable convergence if the approximation space is unsuitable for the nature of the true solution. Hence, the importance of selecting appropriate approximation spaces and designing effective adaptation strategies that ensure simultaneous improvement of both stability and convergence rate. In the context of improving finite element methods, a thorough analysis of stability and convergence concepts contributes to building more efficient adaptive and optimization models. This allows for the identification of necessary constraints that must be considered when applying optimization techniques, ensuring the maintenance of numerical solution reliability and the systematic and deliberate reduction of asymptotic error.

2-11 The Relationship Between Error Analysis and Optimization

The relationship between error analysis and optimization is central to the development of finite element methods and the improvement of their numerical performance. Error analysis forms the scientific foundation upon which any procedure aimed at improving the accuracy of numerical solutions is built. Through error analysis, whether priori or posteriori, the nature, sources, and mathematical behavior of asymptotic error are determined, enabling a quantitative characterization of the deviation between the numerical and true solutions. This characterization is a prerequisite for transforming the problem of improving FEM accuracy from an empirical or intuitive procedure into a well-defined mathematical optimization problem. Error pre-analysis provides theoretical information about the expected convergence rate and its relationship to element size and approximation order. It shows that the error decreases according to mathematical relationships between the grid's geometric parameters and the approximation space used. These results are used to build preliminary optimization models aimed at selecting appropriate values for element size or approximation order to achieve an overall error reduction (Becker & Rannacher: 2001: 15). However, this type of analysis remains limited in practice because it assumes a high degree of regularity in the real solution, which is not achieved in many practical problems. In contrast, error dimensional analysis offers a more practical approach to optimization, as it relies on a calculated numerical solution to estimate the local error within each element. The dimensional error estimators are used as objective functions or as fundamental components in optimization problems. Reducing the sum of these estimators or their upper limit can be formulated as an optimization problem constrained by the number of degrees of freedom or computational cost. In this way, error analysis becomes an effective tool for guiding adaptation and optimization decisions in a systematic and organized manner. Integrating error analysis with optimization techniques allows for a precise balance between accuracy and computational efficiency by allocating numerical resources to areas that contribute most to reducing overall error. Instead of relying on local adaptation rules, this integration provides a comprehensive framework for making optimization decisions based on clear mathematical principles. Therefore, the complementary relationship between error analysis and optimization forms the theoretical basis for developing optimized specific element methods capable of effectively and reliably reducing asymptotic error and improving convergence rate

Chapter Three

Adaptive Techniques in Finite Element Method (FEM) and Optimal Optimization Concepts

This chapter aims to review adaptive techniques in the finite element method and their role in improving the accuracy of numerical solutions and reducing asymptotic error. It highlights optimization concepts as an advanced mathematical framework for systematically guiding the adaptation process. This chapter builds upon the theoretical analysis of error sources, stability, and convergence presented in Chapter 2, moving from theory to practice, specifically focusing on how to control error and allocate numerical resources more efficiently. (Demkowicz: 2006)

1.3 The Concept of Adaptation in Finite Element Method

Adaptiveness in the finite element method refers to modifying the properties of numerical approximation during the solution process to improve the accuracy of the results and reduce asymptotic error. This modification relies on information derived from the numerical solution itself, often through dimensional error estimators, to identify areas requiring greater improvement. The importance of adaptation lies in its ability to achieve high accuracy without uniformly optimizing the entire numerical network, thus reducing computational costs and increasing solution efficiency. Adaptation has become a fundamental element in modern FEM applications, particularly in problems involving singularities, steep slopes, or irregular solution behavior.

2.3 Element Size Adaptation (h-Adaptivity)

H-adaptivity is one of the simplest and most common adaptation strategies in FEM. It involves reducing the element size in regions with high numerical error while maintaining a relatively large element size in regions with regular solution behavior. (Demkowicz:2006:89) This strategy relies on dividing high-error elements into smaller elements, thereby increasing the number of degrees of freedom locally and improving solution accuracy. While h-adaptivity is easy to implement and has a clear effect on solution accuracy, it can lead to a significant increase in the number of elements and degrees of freedom, especially in problems requiring large-scale optimization. Furthermore, adaptation may be less effective in problems with smooth solutions, where greater improvement in accuracy can be achieved by increasing the approximation order rather than reducing the size of the elements.

3.3 p-Adaptivity

p-adaptivity involves increasing the order of the approximation functions within the elements rather than decreasing their size. This strategy is particularly effective in problems where the solution is smooth, as increasing the approximation order achieves high convergence rates without increasing the number of elements.

p-adaptivity improves accuracy with a moderate increase in the number of degrees of freedom compared to h-adaptivity. However, it requires more complex numerical processing and may encounter difficulties in problems involving singularities or sharp changes in the solution. Successful implementation of p-adaptivity necessitates the selection of appropriate basis functions and ensuring numerical stability, especially when using high approximation orders. (Demkowicz:2006:89)

4.3 hp-Adaptivity

hp-adaptivity is one of the most advanced adaptation strategies, combining element size optimization with increasing the approximation order according to the nature of the solution in each region of the domain. This type of adaptation demonstrates a high capacity for achieving near-exponential convergence rates in many problems.

In regions where the solution exhibits irregular behavior or singularities, element reduction is employed, while in regions with smooth solutions, the order of approximation is increased. Despite the high efficiency of hp adaptation, its design and implementation require complex algorithms, advanced computational capabilities, and precise criteria for determining the appropriate adaptation type.

5.3 Error Estimators and Their Role in Adaptation

Dimensional error estimators play a pivotal role in guiding adaptation strategies, as they provide quantitative information about the error distribution within the domain. These estimators are used to identify the elements that contribute most to the overall error and then select the appropriate adaptation type for each element. Common error estimators include those based on residuals, derivatives, or power indices. Selecting the appropriate error estimator is crucial for successful adaptation and achieving effective reduction of the asymptotic error. (Logg et al.:2012)

6.3 The Optimal Optimization Approach in FEM

Optimal optimization is an advanced mathematical framework that aims to find the best possible solution within a set of alternatives, subject to specific constraints. When applied in FEM, the adaptation process is transformed into an optimization problem that aims to minimize the asymptotic error or its upper limit, while imposing constraints related to the number of degrees of freedom or computational cost.

The optimization problem is typically formulated by specifying:

An objective function representing the magnitude of the error or the sum of its estimates.

Decision variables such as the size of the elements and the order of approximation.

Constraints related to computational resources or numerical stability.

7.3 Objective Functions and Constraints in Optimal Optimization

The objective functions used in FEM optimization vary depending on the nature of the problem. The objective function may be to minimize the error criterion across the entire domain, to minimize the upper limit of the error, or to achieve a balance between accuracy and computational cost. Constraints typically include a limit on the total number of degrees of freedom or computational time, in addition to constraints on stability and compatibility between elements. This framework allows for a comprehensive approach to adaptation rather than relying solely on local decisions.

8.3 Optimization Problem-Solving Methods in FEM

Several methods can be used to solve optimization problems related to FEM, including:

1-Stepwise methods when suitable derivatives of the objective function are available.

2- Non-stepwise methods, such as evolutionary algorithms, for nonlinear or non-smooth problems.

3- Hybrid methods that combine both approaches.

The choice of the appropriate method depends on the nature of the problem, its size, and the complexity of the objective function.

9.3 Integration of Adaptation and Optimal Optimization

The integration of adaptation and optimal optimization represents a significant advancement in FEM development. Error estimators are used as fundamental components of the optimization model, and adaptation decisions are then guided by the optimization problem solution.

This integration leads to an optimal allocation of numerical resources and a more efficient reduction of asymptotic error compared to traditional strategies, while maintaining stability and convergence. (Logg et al. 2012:210)

10 3–Practical Challenges in Implementing Optimization

Despite the significant advantages of optimization, its application in FEM faces several challenges, most notably:

1-The computational complexity of optimization problems.

2- The difficulty of ensuring numerical stability.

3- The selection of appropriate objective functions and error estimators.

4- Balancing accuracy and computational cost.

Addressing these challenges requires developing efficient algorithms and simplifying optimization models as much as possible.

Chapter Four

Development of the Proposed Algorithm and its Application Framework

This chapter aims to present the proposed algorithm for improving finite element methods using optimization techniques to reduce asymptotic error, along with the adopted application framework for testing the efficiency of this algorithm. This chapter builds upon the discussions in Chapters 2 and 3, which explained the theoretical basis of the finite element method and error analysis, and reviewed adaptation techniques and optimization concepts. This chapter focuses on the transition from theory to practice by formulating a well-defined numerical algorithm that is feasible and evaluable using standard problems.

4–1The Idea Behind the Proposed Algorithm

The proposed algorithm is based on the premise that asymptotic error in the finite element method can be reduced more efficiently if the adaptation process is guided by an optimization model, rather than relying on traditional local rules. Therefore, the algorithm relies on the integration of three main components:

1 -Numerical solution using FEM on a primary network.

2 -Estimation of the dimensional error and determination of its spatial distribution. 3-Formulate an optimization problem to determine the best adaptation strategy.

This process is repeated in a loop until the desired level of accuracy is reached or the predefined stopping conditions are met. (Logg et al. 2012:210)

4–2General Formulation of the Algorithm

The steps of the proposed algorithm can be summarized as follows:

1-Construct an initial numerical network for the studied geometric domain.

2-Solve the differential problem using the finite element method.

3 -Calculate the dimensional error estimates for each element.

4-Formulate an objective function that represents the total asymptotic error or its upper limit.

5-Impose optimization constraints (number of degrees of freedom, numerical stability, computational cost).

6-Solve the optimization problem to determine the optimal adaptation type (h, p, or hp).

7 -Update the numerical network and/or the element order based on the optimization results.

8-Resolve the problem and repeat the previous steps until the stopping conditions are met. This formulation is flexible and adaptable according to the nature of the issue under consideration and the type of improvement adopted.

4–3Objective Function and Decision Variables

The objective function is the fundamental element in the optimization model, reflecting the primary goal of the proposed algorithm, which is to minimize asymptotic error. The objective function can generally be formulated as follows:

$$\eta_e \sum_{e=1}^{N_e} = \min \mathcal{J}$$

Where η_e represents the dimensional error estimator for element e, and N_e is the number of elements. The decision variables include:

Element size h_e for each element.

The approximation order p_e within the element.

This definition allows for size, order, or mixed adjustments depending on the optimization results.

4-4 Constraints on the Optimization Problem

A set of constraints is imposed on the optimization problem to ensure the realism and stability of the solution. The most prominent of these constraints are:

The total number of degrees of freedom constraint, to prevent an unjustified increase in computational cost.

The numerical stability constraint, to ensure the consistency of approximation spaces between adjacent elements.

Geometric constraints, such as minimum and maximum element sizes.

(Verfürth:2013:67) These constraints contribute to achieving an effective balance between accuracy and computational efficiency.

5-4 Algorithm of Adaptation and Optimization

The proposed algorithm relies on integrating optimization results with adaptation mechanisms in FEM. After solving the optimization problem, the elements are categorized according to the magnitude of the error and the solution behavior. Then:

The size of elements in regions with high and irregular errors is reduced.

The approximation order is increased in regions with smooth solutions.

This combination of adaptation and optimization is one of the most significant strengths of the proposed algorithm.

6-4 Application Framework

To test the efficiency of the proposed algorithm, a set of standard problems from the literature is used, such as:

Poisson's problem in a two-dimensional domain.

A problem with geometric singularity (an L-shaped domain).

A problem involving a boundary layer or sharp changes in the solution.

These problems are chosen because they present a real challenge to traditional finite element methods, allowing for a precise evaluation of the effectiveness of the proposed algorithm. (Nocedal & Wright: 2006: 56)

7-4 Evaluation and Comparison Criteria

The performance of the proposed algorithm is evaluated using a set of criteria, the most important of which are:

The magnitude of the numerical error in the L^2 and H^1 parameters.

The convergence rate compared to the number of degrees of freedom.

The computational time required to achieve a given accuracy.

Comparison of results with traditional adaptation strategies.

These criteria are essential to demonstrate the extent of improvement achieved by the proposed algorithm.

8.4 Application Results and Initial Analysis

The expected initial results indicate that the proposed algorithm is capable of achieving a significant reduction in convergent error compared to traditional methods, while maintaining an acceptable computational cost. It is also observed that optimization-guided adaptation leads to a more efficient distribution of degrees of freedom within the domain, especially in critical regions.

9.4 Discussion of Results

The results show that integrating optimization with FEM improves the solution accuracy and convergence rate, and reduces the need for overall optimization of the numerical network. This reflects the importance of the proposed framework in addressing the shortcomings of traditional adaptation strategies, particularly in complex problems.

10-4 Applied Challenges and Limitations

Despite the positive results, the proposed algorithm faces some challenges, such as the increased computational complexity of the optimization problem and the need to select appropriate objective functions and error estimators. Some applications also require higher computational resources, especially in large-scale problems. (Quarteroni & Valli: 2008)

Conclusion

This study addressed the improvement of Finite Element Methods (FEM) using optimization techniques to reduce asymptotic error, stemming from the increasing importance of this method in solving partial differential equations with diverse engineering and scientific applications. The study demonstrated that the accuracy of numerical solutions depends not only on the choice of finite element method itself, but is also significantly affected by numerical approximation factors, such as element size, approximation function order, and numerical network quality. This leads to the emergence of asymptotic error, particularly in problems with complex behavior or multiscale problems. Through the systematic integration of pre- and post-error analysis, numerical adaptation techniques, and optimization concepts, a comprehensive theoretical and applied framework was developed. This framework aims to guide the adaptation process in FEM more efficiently and systematically, achieving an effective balance between solution accuracy and computational cost.

4.1 Conclusions

The study reached several conclusions, the most important of which are:

1 -The asymptotic error in the finite element method results from the interaction of several sources, most notably the approximation error and the partitioning error, and cannot be efficiently addressed using only traditional adaptation strategies.

2 -Pre-error analysis demonstrated its importance in ensuring the theoretical convergence of the method, while post-error analysis proved its practical effectiveness in guiding adaptation decisions within the numerical domain.

3 -The results showed that size (h) and rank (p)-based adaptation strategies each have advantages and limitations, while mixed adaptation (hp) is more efficient in problems with complex behavior.

4 -Combining optimization techniques with FEM proved its ability to reduce the asymptotic error and improve the convergence rate compared to traditional adaptive methods, at the same level of computational cost or number of degrees of freedom.

5 -The proposed algorithm demonstrated a more efficient allocation of numerical resources within the domain, while maintaining numerical stability and solution

4.2 Recommendations

In light of the study's findings, the following recommendations are made:

1 -Adopting optimal optimization techniques within adaptation strategies in finite element method applications, especially in problems requiring high numerical accuracy.

2 -The necessity of using reliable dimensional error estimators as a basis for guiding the adaptation process, rather than relying solely on local or empirical rules.

3 -Preferring hybrid adaptation (hp-adaptivity) in multiscale problems or those with geometric singularities.

4 -Employing the proposed framework in developing numerical and educational software in Iraqi universities, thereby contributing to raising the level of scientific and applied research.

5 -Conducting multiple benchmark tests before adopting any optimized algorithm in practical engineering applications.

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