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RESEARCH ARTICLE

Generalization of Prolla's Theorem in Approximation

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ABSTRACT

The principal factor in the proofs of many so-called intersection results and associated best approximation theorems—such as the famed Brouwer fixed-point theorem) is Knaster-Kuratowski-Mazurkiewicz theorem (KKM), which was extended from R^l to Hausdorff vector spaces by Ky Fan. This kind of theory necessitates introducing some form of abstract convex structure on a topological space. In both pure and applied mathematics, the theory of approximation plays a central role. The connection between best approximation and fixed-point theory provides a powerful tool for analyzing the nonlinear maps, the convergence of iterative processes, and the solving of variational problems.

This paper is dedicated to presenting a new extension of the well-known Prolla's theorem in the field of invariant approximation from a normed space to a kind of metric space. First, we will introduce the basic concepts of vector metric spaces, define the best approximation, and describe its relationship to fixed points. Second, we will substantiate the best approximation theorem using the state of KKM-map (where the convex combination of any finite elements in $\Theta \subset \Pi$ entirely belonged to the union of their images under the KKM-map, N, $N : \Theta \to 2^{\Pi}$). This theoretical construction has been applied to specific cases that were previously discussed as the best approximations in earlier papers.

Keywords: Best approximation, Fixed point, KKM-map, Metric spaces, Prolla's theorem

Introduction

The familiar facts regarding best approximation are of essential importance in various branches of analysis, such as variational inequalities, approximation theory, minimum and maximum theorems, game theory, and fixed-point theory. The usual problem in approximation theory can be defined as follows: Suppose that Θ is a subspace of a normed vector space Π and $s \in \Pi$, s the best approximation to s from the elements of Θ is a vector $\theta \in \Theta$ such that $\| s - \theta \| = d(s, \Theta)$. [1]

For example, $\Pi = R^2$, the best approximation to $s = (s_1, s_2)$, along a straight line passing through the origin, can be found by dropping a perpendicular from $s = (s_1, s_2)$, to the line. Important questions related to K include:

How can θ be found? Can θ be described? Is θ unique? Is $|| \theta - s|| \to 0$, ($is\ \theta \in \underset{\sim}{K}$)?

A principal problem in approximation theory, and indeed in functional analysis, is finding the best approximation element under a non-self map $h : \Theta \to \Pi$, where $\Pi$is a normed space and $\Theta \subset \Pi$, meaning: [1,2] does $\exists \theta \in \Theta$ is such that $\| \theta - h(\theta) \| = D(h(\theta), \Theta) := inf\{\| h(\theta), s \| : s \in \Theta\}$? In 1969, Ky Fan [3,4] offered a solution to this question for the case in which h is continuous and Θ is convex compact. [5] Insufficient availability of compact sets in infinite-dimensional normed spaces complicates this problem.

Nakatsukasa et al. [6] computed minimax approximations by rational functions on real or complex sets. In a modular metric space, Asadi et al. [7] proved the uniqueness of the best proximity for each ω-quasi

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contraction map (carrying the ω-proximity condition). The existence of a common invariant best approximation is presented for maps that are a Banach operator pair under demi-closeness or Opial's condition.[8] Regarding the best proximity point for p-proximal contractive maps, several researchers produced many results, like,[9] also seen in the next sentence.[10,11] We mention the work by Jabbar et al.[12] in which two results were proven about the relationship between approximation and semi-compactness in b-modular space and then demonstrated a result that includes existence. Recently, for cyclic (and almost cyclic) contraction maps, S. Basha[13] and Magadevan et al.[14] established the existence of a best approximation in the context of uniformly convex Banach spaces. Research on multivariate approximation has concluded with geometrical considerations of necessary and sufficient optimality conditions.[15,16] Based on finding in,[17] different results have appeared to derive the best proximity points for the F-proximal contractive map in a general formula by Beg.[18] For more results in this line, see.[19–21] Building on original results in,[22] Tatar et al.[23] presented various forms of integral-contractive maps and used them to characterize fixed points. Their work also introduced the definition of the best proximity pair and applied it to specific examples, depending on the best approximation method. Recently, in b-metric spaces, Youns et al.[24] calculated results on best proximity and coincidence points for multi-valued proximity contractions (using an alternating distance function). Furthermore, they employed this framework to describe the equation of motion by solving a second-order boundary value problem. Additional discussions on these topics can be found in.[25,26] In this article, we present a study on best approximation in vector metric spaces, aligning with previous research. We will now restore the definition of a vector metric space.

Materials and methods

Recall the following definition:

Definition 1:[24] A metric space (M, L) forms a vector metric space (VMS) if it contains the following structures:

M is a vector space with addition $+$ and scalar multiplication,

It holds that addition and scalar multiplication are continuous.

$L(s' + a, n + a) = L(s', n)$, for all s', n, $a \in M$. (Translation invariant of L).

Definition 2:[23] A VMS is said to satisfy condition $(\mathscr{C})$ if

$$L(\vartheta s' + (1-\vartheta)n, a) \leq \vartheta L(s', a) + (1-\vartheta) L(n, a),$$
$$for\ all\ s', n, a \in X\ and\ 0 \leq \vartheta \leq 1.$$

Clearly, every normed vector space satisfies condition $(\mathscr{C})$. Below is an example of a VMS that is not normed but satisfies condition $(\mathscr{C})$.

Example 1: *The vector space of sequences $M = l\infty = \{s' = (s'_i)_{i=1}^{\infty}, : sup|s'_i| < \infty\}$, where $(s'_i)_{i=1}^{\infty}$ be a sequence of real (or complex) numbers. Here, Mis a metric space w.r.t. $L(s', n) = sup_i \frac{|s'_i - n_i|}{1+|s'_i - n_i|}$, for $s' = (s'_i)$, $n = (n_i)$and satisfying condition $(\mathscr{C})$, so, Mis VMS. However, this metric is not induced by a norm as the function $|\frac{|s'_i - n_i|}{1+|s'_i - n_i|}$, i.e., the homogeneity property of a norm $(\| \vartheta s' \| = | \vartheta | \| s' \|$ fails.*

Below, Fig. 1 shows the relationships among spaces: metric, normed, and vector metric. Normed spaces are a special class of metric spaces, and vector (or linear) metric spaces are also included within metric

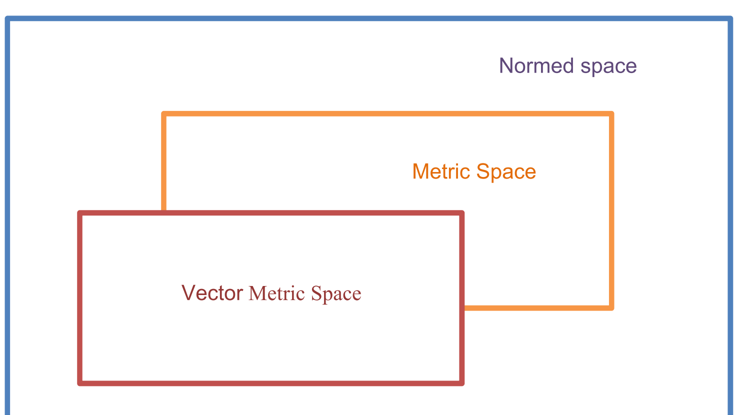

Fig. 1. The relationship between spaces.

spaces, but they are not always a subclass of normed spaces.

As is known, Prolla's theorem has been established to deal with approximating functions in normed spaces, providing hypotheses under which the best approximation of a function by a point in a given subspace exists and is unique. Prolla introduced an extension of Ky Fan's theorem in approximation in which he used Himmelberg's fixed-point result for set-valued maps.[8] The importance of Prolla's theorem is evident in several aspects: it develops Chebyshev's alternation theorem (a classical result in polynomial approximation), it provides a deductive method to find the best approximation with guaranteed uniqueness, and, in polynomial or spline approximation (in finite-dimensional approximation problems), it is appropriate. By using the ($Knaster - Kuratowski - Mazurkiewicz$-map) principle, here, a proof of the general version of Prolla's result will be presented in the context of VMS.

In this paper, we will extend Prolla's theorem to operate in spaces whose algebraic background is a vector space and has a type of distance, but is not a normed space.

For this purpose, we begin with some definitions.

Definition 3: Suppose(M, L) be VMS, $\emptyset \neq K \subset M$ and $s' \in M$, an element $m' \in K$ is called a best approximation (or a nearest point) to s' in K if $L(s', m') = infinf\ \{l(s', m') : m' \in K\} \equiv l(s', K)$.

As in the usual case, the set of all such $m' \in K$, denoted by $PK(s')$ and the set-valued map $PK : M \to K$ is called a best approximation map (or a metric projection). Now, for a subset K of a vector space M, 2^K =: class of all nonempty subsets of K.

Definition 4: A set-valued map $N : K\ 2^K$ is named (KKM-map) if $co\{s'_j\}_{j=1}^{i} \cup_{j=0}^{i} N(s'_j)$, where $co\{s'_j\}_{j=0}^{i}$is the convex hull of any finite subset $\{s'_j\}_{j=0}^{i}$ of K.

For example, if M is a vector space endowed with a semi-norm ρ and $\gamma : K \to K$ is a map, the set-valued map $N : K \to K$ defined by $N(s') = \{a'K : (\gamma(a') - a')\ (\gamma(a') - s')\}$, for $s'K$ is (KKM-map) 1.

Definition 5: Let (M, L) be VMS and $\emptyset \neq K \subset M$, if K is closed and convex. The map $N : K\ M$ is called:

an almost affine map if for s'_1, $s'_2 \in K$, $a' \in M$and $0 < \vartheta < 1$,

$$L\ (N(\vartheta\ s'_1 + (1 - \vartheta)s'_2), a')\ \vartheta L\left(N(s'_1),\ a'\right) + (1 - \vartheta)\ l\left(N(s'_2), a'\right) \text{ holds.}$$

an almost quasi-convex map if for s'_1, $s'_2 \in K$, $a' \in M$ and $0 < \vartheta < 1$,

$$L\ (N(\vartheta\ s'_1 + (1 - \vartheta)s'_2), a') \leq max\left\{L\left(N(s'_1),\ a'\right),\ L\left(N(s'_2), a'\right)\right\} \text{ holds.}$$

Clearly, every almost affine map is almost quasi-convex. In the following, we will prove this fact and show that the converse is not true.

Results and discussion

Let us start with the following:

Proposition 1: Let (M, L) be VMS and $\emptyset \neq K \subset M$, if $K$is a closed convex subset with condition (c°). Then every almost affine map is almost quasi-convex.

Proof: Suppose that $N : K\ M$ be an almost affine map. To show that N is almost quasi-convex. Let $s'_1, s'_2 \in K$, $a' \in M$ and $0 < \vartheta < 1$, suppose that

$$L\left(N(\vartheta s'_1 + (1 - \vartheta)s'_2), a'\right) > max\left\{L\left(N(s'_1),\ a'\right), L\left(N(s'_2), a'\right)\right\}.$$

If $\{L(N(s'_1), a'), L(N(s'_2), a')\} = L(N(s'_1), a')$ then $L(N(s'_1), a') \geq L(N(s'_2), a')$ and $L(N(\vartheta s'_1 + (1 - \vartheta)s'_2), a') > L(N(s'_1), a')$

As N is an almost affine map, we have:

$$L\ (N(\vartheta\ s'_1 + (1 - \vartheta)s'_2), a')\ \vartheta L\left(N(s'_1),\ a'\right) + (1 - \vartheta)\ L\left(N(s'_2), a'\right)$$

then $L(N(s'_1), a') < \vartheta L(N(s'_1), a') + (1 - \vartheta)L(N(s'_2), a')$ and $(1 - \vartheta)\ L(N(s'_1),\ a') < (1 - \vartheta)\ L(N(s'_2), a')$ so, $L(N(s'_1),\ a') < L(N(s'_2), a')$ is a contradiction.

If max $\{L(N(s'_1),\ a'), L(N(s'_2),\ a')\} = L(N(s'_2),\ a')$ then $L(N(\vartheta s'_2 + (1 - \vartheta)s'_2),\ a') > L(N(s'_2),\ a')$. as N is an almost affine map, then

$$L\left(N(\vartheta s'_1 + (1 - \vartheta)s'_2),\ a'\right) \leq \vartheta\ L\left(N(s'_1),\ a'\right) + (1 - \vartheta)\ L\left(N(s'_2),\ a'\right)$$

so, $L(N(s'_2), a') < L(N(\vartheta s'_2 + (1 - \vartheta)s'_2), a') \leq \vartheta L(N(s'_1), a') + (1 - \vartheta)L(N(s'_2), a')$ thus $L(N(s'_2), a') < \vartheta L(N(s'_1), a') + (1 - \vartheta)L(N(s'_2), a')$ meaning that $L(N(s'_2),\ a') < L(N(s'_1),\ a')$, which is a contradiction. Therefore,

$$L\left(N(\vartheta s'_2 + (1 - \vartheta)s'_2), a'\right) \leq max\left\{L\left(N(s'_1), a'\right), L\left(N(s'_2), a'\right)\right\}$$

that is N is an almost quasi-convex map.

However, the converse is not true. Example 2 illustrates that an almost affine map may not be almost quasi-convex.

Example 2: *Let $M = R$, $K = [0, 1]$ with $L(s', n) = |s' - n|$ and $N : K M$ such that, for $' \in K$, $N(s') = \{0\ 0 \le s' < 1\ 5\ s' = 1$.*

Let $s'_1 = \frac{1}{2}$, $s'_2 = 1$, $a' = 5$ and $\vartheta = \frac{1}{2}$

$$L\left(N\left(\vartheta s'_2 + (1-\vartheta)s'_2\right), a'\right) = \left|N\left(\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\right) - 5\right|$$

$$= 5 = max\left\{L\left(N(s'_1), a'\right), L\left(N(s'_2), a'\right)\right\}$$

Thus, N is an almost quasi-convex map. But

$$\vartheta L\left(N(s'_1), a'\right) + (1-\vartheta)L\left(N(s'_2), a'\right) = \frac{1}{2}(5) + 0$$

$$= \frac{5}{2} < 5 = L\left(N\left(\vartheta s'_2 + (1-\vartheta)s'_2\right), a'\right)$$

then, N is not an almost affine map.

Now, we will present a new version of Prolla's Theorem in VMS.

Theorem 1: *Let (M, L) be VMS and $\emptyset \ne K \subset M$, if $K$is a compact convex subset with condition (c) and $N : K \to M$ be a continuous, onto, and almost affine map. Then, for each continuous map $\gamma : K M$, $\exists a' \in K$ such that*

$$L\left(N\left(a'\right), \gamma\left(a'\right)\right) = L\left(\gamma\left(a'\right), K\right). \tag{1}$$

where $L(\gamma(s'), K)$ =distance between $\gamma(s')$ and the set K.

Proof: For each $s' \in K$, define a set-valued $Ts' = \{a \in K : L(N(a'), \gamma(a')) \le L(N(s'), \gamma(a'))\}$. For each $s' \in K$, we show that Ts' is a closed set. Suppose that a is an accumulation point of. This implies a sequence $\langle s'_i \rangle \subseteq Ts'$ and $s'_i\ a$ such that $L(N(s'_i), \gamma(s'_i)) \le L(N(s'), \gamma(s'_i))$.

Then $lim_{i\to\infty} L(N(s'_i), \gamma(s'_i)) \le lim_{i\to\infty} L(N(s'), \gamma(s'_i))$.

As γ and N are continuous, $L(N(a'), \gamma(a')) \le L(N(s'), \gamma(a'))$ holds, which implies $a' \in Ts'$.

Meaning that Ts' is a closed set, $\forall s' \in K$. The compactness of K and closeness of Ts' make Ts' compact (because a closed subset of a compact set is compact).

Now, to prove that T is ($KKM-$map). Let $\{s'_0, s'_1, \ldots, s'_i\}$ be a finite subset of K; we suppose that $co\{s'_0, s'_1, \ldots, s'_i\} \subseteq \cup_{j=0}^{i} Ts'_j$.

If not, then $\exists u \in co\{s'_0, s'_1, \ldots, s'_i\}$ and $u \notin \cup_{j=0}^{i} Ts'_j$, where $u = \sum_{j=0}^{i} \vartheta_j s'_j$,0 ϑ_j 1 and $\sum_{j=0}^{i} \vartheta_j = 1$. As $u \notin \cup_{j=0}^{i} Ts'_j$ then

$$L(N(u), \gamma(u)) > L\left(N\left(s'_j\right), \gamma(u)\right), j = 0, 1, \ldots, i. \tag{2}$$

as N is an almost affine map, by Proposition 1, N is almost quasi-convex. Hence, by Definition 4, obtaining

$$L(N(u), \gamma(u)) = L(N\left(\sum_{j=0}^{i} \vartheta_j s'_j\right),$$

$$\gamma(u)) \le max\{L(N(s'_0), \gamma(u)), \ldots, L(N(s'_i), \gamma(u))\}$$
$$< L(N(u), \gamma(u))$$

(By Eq. (2)), this is a contradiction. So $co\{s'_0, s'_1, \ldots, s'_i\} \subseteq \cup_{j=0}^{i} Ts'_j$.

That is T is (KKM-map). By extension of (KKM-map) Theorem 1, $\cup_{j=0}^{i} Ts'_j \ne \emptyset$ means there is a point $a \in K$ such that $a \in \cup_{j=0}^{i} Ts'_j$, that is $a' \in Ts'$, $\forall s' \in K$. Therefore, $L(N(a'), \gamma(a')) = L(\gamma(a'), K)$.

Corollary 1: *Let $(\Pi, \| \cdot \|)$ be a normed space and $\emptyset \ne \Theta \subset \Pi$. If Θ is a compact convex, γ and N have the same hypotheses as in Theorem 1. Suppose that $\gamma(r') \in \Theta$, $\forall r' \in \Theta$, then $\exists s' \in \Theta$ such that $N(s') = \gamma(s')$. Moreover, if $N = I$, then γ has a fixed point.*

Proof: By Theorem 1, there is a $s' \in \Theta$ such that $||N(s') - \gamma(s')|| = d(\gamma(s'), \Theta)$.

But $\gamma(r') \in \Theta$ for all $r' \in \Theta$, so $|| N(s') - \gamma(s') || = d(\gamma(s'), \Theta) = 0$. Thus $N(s') = \gamma(s')$.

An intuitive conclusion to the result of the previous one is to put $N = I$, (I is an identity map), in Corollary 1, to note that γ has a fixed point (meaning, $\exists s' \in \Theta$, $\gamma(s') = s'$).

Corollary 2 (Fixed-Point Result): *If $\emptyset \ne \Theta \subseteq \Pi$ and γas in Corollary 1, where $\gamma(r) \in \Theta$, $\forall r \in \Theta$ then $\exists s' \in \Theta$, $N(s') = s'$*

Proof: It is clear, as $||s' - \gamma(s')|| = d(\gamma(s'), \Theta) = 0$.

Also, if $N = I$ in Theorem 1, then a special case of Ky Fan's theorem holds as follows.

Corollary 3: *Let $R^l = \{s' = (s'_j), j = 1, 2, \ldots, l\}$ be the Euclidean space with $\| s' \| = \sum_{j=1}^{l} |s'_j|$. If $\emptyset \ne< \Theta \subset R^l$, Θ is a closed, bounded, convex subset of R^l and $\gamma : \Theta R^l$ is a continuous function. Then$\exists s' \in \Theta$, $||s' - \gamma(s')|| = d(\gamma(s'), \Theta)$.*

Now, another application of approximation and fixed-point results is given to find the zero of a polynomial equation.

Corollary 4: *Let $\gamma : \Theta \to R^l$ be a continuous function on a closed, bounded subset Θ of R^l. If $s' - \sigma\gamma(s') \in \Theta$, for all $s' \in \Theta$, for some $\sigma > 0$. Then $\gamma(s') = 0$ has a solution, i.e., $\exists\theta \in \Theta$ such that $\gamma(\theta) = 0$.*

Proof: Let $h(s') = s' - \sigma\gamma(s')$, for all $s' \in \Theta$. Then $h : \Theta \to R^l$is a continuous function. Therefore, there is a $\exists\theta \in \Theta$ such that $\| \theta - h(\theta) \| = d(h(\theta), \Theta)$, by Corollary 3. As for all $s' \in \Theta$, $h(s') \in \Theta$, so, h g has a fixed point, say $\theta = h(\theta)$. Consequently, $\theta - \sigma\gamma(\theta) = \theta$, that is, $\gamma(\theta) = 0$ as $\sigma > 0$.

Discussion

The results obtained in this work systematically relax the structural assumptions and extend the classical Prolla's theorem in approximation from normed linear spaces to VMS for continuous, onto, and almost affine maps. This appears in Theorem 1 under hypotheses on the background subset (compact, convex with condition (c')) and Corollary 1 as a special case. And Corollary 2 represents the relation between finding a fixed point and the availability of the best approximation.

As future work, firstly, it is possible to return to Balaj[27] and adopt the concept of weak-KKM to extend the results of the paper, and then apply them to equilibrium problems with finite values and minimum and maximum inequalities. Secondly, also for a set-valued map, we can use this idea in,[28] where the closedness and coercivity are relaxed. This may give new results about the minimum-maximum inequality and the variational relations, enabling us to solve such problems even when the usual closedness and coercivity conditions are not satisfied. Since approximation theory has applications in various fields, we hope to apply our results in approximation journals, as in Zayed et al.[29] and Albundi.[30]

Conclusions

This theoretical framework has been utilized in instances that align with the best approximation approach outlined in prior studies. Through the application of the KKM-map framework, the validity of the best approximation theorem has been proved. This search generalizes the original theorem within this case. By using notions of vector metric space, the connection between best approximation and fixed points is explored and elaborated upon. The method used could be adapted to asymptotically nonexpansive or quasi-contractive maps or applied in more general topological vector spaces. Also, the conditions involving compactness or starshapedness might be further weakened by using alternative fixed point theorems or introducing measures of noncompactness.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Furthermore, any Figures and images that are not ours have been included with the necessary permission for republication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at the University of Baghdad.

Authors' contributions statement

All authors contributed equally to the conception and design of the study, and the work was conducted collaboratively among them. S.S.A. developed the original research idea, while N.M.J.I. and Sh.S.A. verified and carried out the work.

Journal declaration

The first author (N.M.J.I.) is an editor for the journal but did not participate in the peer review process other than as an author. The authors declare no other conflict of interest.

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تعميم مبرهنة برولا في التقريب

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الخلاصة

العامل الرئيسي في إثبات العديد مما يُسمى بنتائج التقاطع ومبرهنات أفضل تقريب مرتبطة بها (مثل مبرهنة بروير الشهيرة للنقطة الثابتة) هي نظرية كناستر-كوراتوفسكي-مازوركيفيتش التي وُسِّعت من فضاءات متجهات إلى فضاءات هاوسدورف بواسطة كي فان. يتطلب هذا النوع من المبرهنات تقديم نوع من بنية التحدب المجردة للفضاء التبولوجي. في كلٍّ من الرياضيات البحتة والتطبيقية، تلعب نظرية التقريب دورًا محوريًا. يُهيئ ربط أفضل تقريب بنظرية النقطة الصامدة أداةً فعّالة لتحليل التطبيقات غير الخطية، وتقارب العمليات التكرارية، وحل مسائل التباين. هذا البحث مكرس لتقديم توسيع جديد مبرهنة برولا المعروفة في مجال التقريب الثابت من الفضاء المعياري إلى نوع من الفضاءات المترية. أولا، يتم ذكر المفاهيم الأساسية للفضاء المتري المتجه، ويتم تقديم تعريف أفضل تقريب، ويتم تعريف ووصف العلاقة مع النقاط الصامدة. ثانيا، تم إثبات نظرية التقريب الافضل باستخدام حالة تطبيق KKM)حيث ينتمي التركيب المحدب لأي عدد منتهي من العناصر في $\Theta\subset\Pi$ بالكامل إلى اتحاد صورها تحت تطبيق $N: \Theta \rightarrow 2^{2\Pi}$ N ،KKM.(تم تطبيق هذا البناء النظري على حالات خاصة والتي ظهرت كطريقة للتقريب الافضل في الأوراق السابقة.

الكلمات المفتاحية: أفضل تقريب، نقطة صامدة، تطبيق KKM ، الفضاءات المترية، مبرهنة برولا.