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RESEARCH ARTICLE

New Generalizations of Pairwise Lindelöf Spaces

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ABSTRACT

This study is introducing two new different pairwise related to Lindelöf bitopological spaces. Lindelöf spaces are among of the most important topological spaces with important applications in various branches of mathematics. Therefore, many studies on Lindelöf topological spaces and pairwise Lindelöf bitopological spaces were presented in previous papers. However, those two new pairwise Lindelöf bitopological spaces are more general than what has been studied before. Therefore, the study defines a new pairwise interior operator and a new pairwise closure operator, furthermore, it reviews their characteristics where the researchers provide new strongly pairwise axioms (the strongly pairwise regular and the strongly pairwise normal axioms) in bitopological spaces. Moreover, many illustrative examples and counterexamples are reviewed, and a study was conducted on the relationship between the strongly pairwise regular axioms and the pairwise regular axioms, and the relationship between the strongly pairwise normal axioms and the pairwise normal axioms. In addition, this study presents some important results in strongly regular and strongly normal bitopological spaces. Finally, the findings showed some relationships between the new concepts of Lindelöf bitopological spaces.

Keywords: Bitopological spaces, Normal spaces, Pairwise Lindelf spaces, Regular spaces, Separation axioms

Introduction

In 2007–2009, Kilicman et al.[1,2] studied a new weaker form of the pairwise normal spaces and studied two types of pairwise Lindel.öf spaces. They also examined the concept of pairwise weakly Lindelöf spaces. Lindelöf space is a topological space in which every open cover has a countable subcover.

Many researchers have studied separation axioms in bitopological spaces.[3,4] In 2023, Binshahnah et al.[5] investigated new strongly separation axioms in the context of fuzzifying bitopological spaces. Similarly, in 2024, Bella et al.[6] studied subsets in an SDL space with Lindelöf closures.

In their study, Almuhur and Al-labadi[7] presented preopen sets in bitopological spaces to study pairwise strongly Lindelöf spaces. Also, Singh[8,9] explained that the class of set star-Lindelöf spaces lie between the class of Lindelöf spaces and the class of star-Lindelöf spaces and studied starcompact spaces. Some types of mappings in bitopological spaces have been studied.[10] Also, many properties in generalized pairwise Lindelöf spaces, pairwise sub-Lindelöf spaces and pairwise locally Lindelöf spaces have been reviewed.[11–14]

Accordingly, this study defines two strongly pairwise axioms (the strongly pairwise regular and the strongly pairwise normal axioms) in bitopological spaces. Likewise, it introduces two new pairwise Lindelöf bitopological spaces are more general than those that have been studied before.[1,2] Finally, the study focuses on some relationships and properties between the new concepts.

Preliminaries

Definition 1:[1,7] Let $\mathfrak{X}$ be a nonempty set and $(\mathfrak{X}, \gamma_1, \gamma_2)$ be a bitopological space.

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1. The family of all pairwise open sets "$\mathcal{O}^p$" is defined as $\mathcal{O}^p := \{\mathcal{U} : \mathcal{U} \in \gamma_1 \wedge \mathcal{U} \in \gamma_2\}$.
2. The family of all pairwise closed sets "$\mathcal{F}^p$" is defined as $\mathcal{F}^p := \{\mathcal{U} : \mathcal{U}^c \in \mathcal{O}^p\}$.

Theorem 1:[1] *In a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$. γ_i is regular with respect to γ_j iff for each point $y \in \mathfrak{X}$ and $H \in \gamma_i$ and $y \in H$, there exists $W \in \gamma_i$ such that $y \in W \subseteq cl_j(W) \subseteq H$, where $i, j \in \{1, 2\}$, $i \neq j$.*

A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is pairwise regular (p−regular) if γ_1 is regular with respect to γ_2 and vice versa.

Definition 2:[1] Let $(\mathfrak{X}, \gamma_1, \gamma_2)$ be a bitopological space. Then γ_1 is said to be p−normal iff for each $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ and $A \cap B = \emptyset$, there exist $\mathcal{U} \in \gamma_1$ and $\mathcal{V} \in \gamma_2$, s.t., $A \subseteq \mathcal{V}$, $B \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Definition 3:[1] Let $(\mathfrak{X}, \gamma_1, \gamma_2)$ be a bitopological space. Then γ_1 is said to be p_*−normal iff for each $A, B \in \mathcal{F}^p$ and $A \cap B = \emptyset$, there exist $\mathcal{U} \in \gamma_1$ and $\mathcal{V} \in \gamma_2$, s.t., $A \subseteq \mathcal{V}$, $B \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Theorem 2:[1] *A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be p_*−normal iff each $C \in \mathcal{F}^p$ and $D \in \mathcal{O}^p$, s.t., $C \subseteq D$, there exist $G \in \gamma_1$ and $F \in \mathcal{F}_2$, s.t., $C \subseteq G \subseteq F \subseteq D$.*

Definition 4:[1] A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be p−Lindelöf, if the topological spaces $(\mathfrak{X}, \gamma_1)$ and $(\mathfrak{X}, \gamma_2)$ are both Lindelöf.

Definition 5:[1] In a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$, γ_1 is said to be Lindelöf with respect to γ_2 if every γ_1−open cover of $\mathfrak{X}$ can be replaced by a countable γ_2−open cover.

A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be p_*−Lindelöf, if γ_1 is a Lindelöf with respect to γ_2 and vice versa.

Theorem 3:[1] *Every p−regular and p−Lindelöf bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is p_*−normal.*

Theorem 4:[1] *Every p−regular and p_*−Lindelöf bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is p_*−normal.*

Definition 6:[2] A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be ij−weakly Lindelöf (briefly, $w_{(i,j)}$−Lindelöf) if for every i−open cover $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ of $\mathfrak{X}$, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ, s.t., $\mathfrak{X} = cl_j(\cup_{n\in\mathbb{N}} \mathcal{U}_{\alpha_n})$.

A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be pairwise weakly Lindelöf (briefly, pw−Lindelöf) if it is both $w_{(1,2)}$−Lindelöf and $w_{(2,1)}$−Lindelöf.

Results and discussion

Definition 7: Let $(\mathfrak{X}, \gamma_1, \gamma_2)$ be a bitopological space and let $A \subseteq \mathfrak{X}$.

1. The pairwise closure operator "cl^p" is defined as $cl^p(A) = \cap\{B : A \subseteq B \wedge B^c \in \mathcal{O}^p\}$.
2. The pairwise interior operator "int^p" is defined as $int^p(A) = \cup\{\mathcal{U} : \mathcal{U} \subseteq A \wedge \mathcal{U} \in \mathcal{O}^p\}$.

Lemma 1: *In a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$.*

1. $\mathrm{cl_i(A) \subseteq cl^p(A)}$, $\forall \mathrm{A} \subseteq \mathfrak{X}$, $\mathrm{i} = 1, 2$.
2. $\mathrm{int^p(A) \subseteq int_i(A)}$, $\forall \mathrm{A} \subseteq \mathfrak{X}$, $\mathrm{i} = 1, 2$.

Proof:

1. $cl^p(A) = \cap\{B : A \subseteq B, B^c \in \mathcal{O}^p\}$

$$= \cap \left\{B : A \subseteq B, B^c \in \gamma_1 \wedge B^c \in \gamma_2\right\}$$
$$\supseteq \cap \left\{B : A \subseteq B, B^c \in \gamma_1\right\}$$
$$= cl_1(A).$$

Similarly, we can prove $cl_2(A) \subseteq cl^p(A)$.

2. $int^p(A) = \cup\{\mathcal{U} : \mathcal{U} \subseteq A, \mathcal{U} \in \mathcal{O}^p\}$

$$= \cup \{\mathcal{U} : \mathcal{U} \subseteq A, \mathcal{U} \in \gamma_1 \wedge \mathcal{U} \in \gamma_2\}$$
$$\subseteq \cup \{\mathcal{U} : \mathcal{U} \subseteq A, \mathcal{U} \in \gamma_1\}$$
$$= int_1(A).$$

Similarly, we can prove $int^p(A) \subseteq int_2(A)$.

Definition 8: A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be sp_1−regular iff for each point $y \in \mathfrak{X}$ and $H \in \gamma_1$ and $y \in H$, there exists $W \in \gamma_1$, s.t., $y \in W \subseteq cl^p(W) \subseteq H$.

A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be strongly pairwise regular (briefly, sp−regular) iff it is sp_1−regular and sp_2−regular.

Example 1: *Consider $\mathfrak{X} = \{s, m, l\}$ and $\gamma_1 = \{\emptyset, \mathfrak{X}, \{s\}, \{m, l\}\}$, $\gamma_2 = \{\emptyset, \mathfrak{X}, \{s\}, \{l\}, \{s, l\}, \{m, l\}\}$ be two topologies on $\mathfrak{X}$. Observe that $\mathcal{F}_1 = \{\emptyset, \mathfrak{X}, \{s\}, \{m, l\}\}$ and $\mathcal{F}_2 = \{\emptyset, \mathfrak{X}, \{m, l\}, \{s, m\}, \{m\}, \{s\}\}$.*

So, $\mathcal{O}^p = \{\emptyset, \mathfrak{X}, \{s\}, \{m, l\}\}$ and $\mathcal{F}^p = \{\emptyset, \mathfrak{X}, \{s\}, \{m, l\}\}$. From Definition 8, we have that $(\mathfrak{X}, \gamma_1, \gamma_2)$ is sp_1−regular space, but it is not sp_2−regular space (since, $l \in \{l\} \in \gamma_2$, $cl^p(\{l\}) = \{m, l\}$, $cl^p(\{s, l\}) = \mathfrak{X}$ and $cl^p(\{m, l\}) = \{m, l\}$).

Definition 9: A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be strongly pairwise normal (briefly, sp−normal) iff for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$ with $S \cap T = \emptyset$, there exist $W, Q \in \mathcal{O}^p$, s.t., $S \subseteq W$, $T \subseteq Q$ and $W \cap Q = \emptyset$.

Example 2: *In Example 1, we have* $\{s\} \in \mathcal{F}_1$ *and* $\{m, l\} \in \mathcal{F}_2$. *There exist* $\{s\}$, $\{m, l\} \in \mathcal{O}^p$*(Similarly, for sets whose intersection is equal to the empty set, we can satisfy the condition). Thus* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* sp−*normal.*

Theorem 5: *If a bitopological space* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* sp−*regular, then it is* p−*regular.*

Proof: Let $(\mathfrak{X}, \gamma_1, \gamma_2)$ be sp−regular space, $x \in \mathfrak{X}$, $H \in \gamma_1$, and $x \in H$. Then, from Lemma 1 and since $(\mathfrak{X}, \gamma_1, \gamma_2)$ is sp_1−regular space. Then there exist an open set $\mathcal{U} \in \gamma_1$, s.t., $x \in \mathcal{U} \subseteq cl^p(\mathcal{U}) \subseteq H$.

So, $x \in \mathcal{U} \subseteq cl_2(\mathcal{U}) \subseteq H$. Thus γ_1 is regular with respect to γ_2. Also, from Lemma 1 and since $(\mathfrak{X}, \gamma_1, \gamma_2)$ is sp_2−regular space, we can prove that γ_2 is regular with respect to γ_1. Therefore, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is p−regular space.

The following example shows that the converse of Theorem 5 is not necessarily true.

Example 3: *Let* $\mathfrak{X} = \{s, m, l\}$, $\gamma_1 = \{\emptyset, \mathfrak{X}, \{m, l\}\}$ *and* $\gamma_2 = \{\emptyset, \mathfrak{X}, \{s\}\}$ *be two topologies on* $\mathfrak{X}$.

Observe that $\mathcal{F}_1 = \{\emptyset, \mathfrak{X}, \{s\}\}$ *and* $\mathcal{F}_2 = \{\emptyset, \mathfrak{X}, \{m, l\}\}$. *So,* $\mathcal{O}^p = \{\emptyset, \mathfrak{X}\}$ *and* $\mathcal{F}^p = \{\emptyset, \mathfrak{X}\}$.

Note that, from Theorem 1, γ_1 is regular with respect to γ_2 and γ_2 is regular with respect to γ_1. So, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is p−regular, but it is not sp−regular (since $m \in \{m, l\} \in \gamma_1$, but $cl^p(\{m, l\}) = \mathfrak{X}$).

Theorem 6: *A bitopological space* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* sp−*normal iff for each* $S \in \mathcal{F}_1$ *and* $D \in \gamma_2$ *s.t.* $S \subseteq D$, *there exist* $W \in \mathcal{O}^p$ *and* $F \in \mathcal{F}^p$ *s.t.* $S \subseteq W \subseteq F \subseteq D$.

Proof: Suppose that, a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is sp−normal, $S \in \mathcal{F}_1$, and $D \in \gamma_2$, s.t., $S \subseteq D$. Then $T = \mathfrak{X} - D \in \mathcal{F}_2$, with $S \cap T = \emptyset$. Since, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is sp−normal space. Then there exist $W, Q \in \mathcal{O}^p$, whereas $S \subseteq W$, $T \subseteq Q$ and $W \cap Q = \emptyset$. Hence $W \subseteq \mathfrak{X} - Q \subseteq \mathfrak{X} - T = D$. Take $\mathfrak{X} - Q = F \in \mathcal{F}^p$, thus $S \subseteq W \subseteq F \subseteq D$.

Conversely, assume that the condition holds. Let $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$, s.t., $S \cap T = \emptyset$. Then $\mathfrak{X} - T = D \in \gamma_2$ and $S \subseteq D$. So, there exists a pairwise open set $W \in \mathcal{O}^p$ and a pairwise closed set $F \in \mathcal{F}^p$, s.t., $S \subseteq W \subseteq F \subseteq D$. Thus $T = D - \mathfrak{X} \subseteq \mathfrak{X} - F$, $S \subseteq W$ and $\mathfrak{X} - F \cap W = \emptyset$. Take $Q = \mathfrak{X} - F \in \mathcal{O}^p$.

Theorem 7: *If a bitopological space* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* sp−*normal, then it is* p−*normal.*

Proof: Let a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ be sp−normal and let $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$ s.t. $S \cap T = \emptyset$. Then there exist $W, V \in \mathcal{O}^p$, s.t., $S \subseteq W$, $T \subseteq V$ and $W \cap V = \emptyset$. Thus $W \in \gamma_2$ and $V \in \gamma_1$ (by Definition 1(1)). Therefore, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is p−normal.

The converse of Theorem 7 is not always true. The following example shows that.

Example 4: *Consider* $\mathfrak{X} = \{s, m, l\}$ *and* $\gamma_1 = \{\emptyset, \mathfrak{X}, \{m\}, \{l\}, \{m, l\}\}$, $\gamma_2 = \{\emptyset, \mathfrak{X}, \{s\}, \{m\}, \{s, m\}, \{m, l\}\}$ *be two topologies on* $\mathfrak{X}$. *Observe that* $\mathcal{F}_1 = \{\emptyset, \mathfrak{X}, \{s\}, \{s, l\}, \{s, m\}\}$ *and* $\mathcal{F}_2 = \{\emptyset, \mathfrak{X}, \{l\}, \{s\}, \{m, l\}, \{s, l\}\}$.

So, $\mathcal{O}^p = \{\emptyset, \mathfrak{X}, \{m\}, \{m, l\}\}$. *From Definition 2, the bitopological space* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* p−*normal. Note that* $\{s\} \in \mathcal{F}_1$ *and* $\{l\} \in \mathcal{F}_2$, *but there are no pairwise open sets* $W, Q \in \mathcal{O}^p$, *s.t.,* $\{s\} \subseteq W$, $\{l\} \subseteq Q$ *and* $W \cap Q = \emptyset$. *Therefore,* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is not* sp−*normal space.*

Definition 10: A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be sp_i−Lindelöf iff for every γ_i−open cover of $\mathfrak{X}$ can be replaced with a countable pairwise open cover.

A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be sp−Lindelöf iff it is sp_1−Lindelöf and sp_2−Lindelöf.

Example 5: *Consider* $\mathfrak{X} = \{s, m, l\}$, $\gamma_1 = \{\emptyset, \mathfrak{X}, \{m\}, \{l\}, \{m, l\}\}$ *and* $\gamma_2 = \{\emptyset, \mathfrak{X}, \{s\}, \{m\}, \{s, m\}\}$ *be two topologies on* $\mathfrak{X}$. *So,* $\mathcal{O}^p = \{\emptyset, \mathfrak{X}, \{m\}\}$. *Thus* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* sp−*Lindelöf space.*

Example 6: *Consider* $\mathfrak{X} = \{s, m, l\}$ *and* $\gamma_1 = \{\emptyset, \mathfrak{X}, \{m\}, \{m, l\}\}$ *be a topology on* $\mathfrak{X}$. *Let* γ_2 *be a discrete topology on* $\mathfrak{X}$. *So,* $\mathcal{O}^p = \{\emptyset, \mathfrak{X}, \{m\}, \{m, l\}\}$. *The family* $\{\{s\}, \{m\}, \{l\}\}$ *is an* γ_2*-open the cover of* $\mathfrak{X}$ *cannot be replaced with a countable* $\mathcal{O}^\text{p}$−*open cover. Thus,* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is not* sp−*Lindelöf space.*

Definition 11: A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be sw_i−Lindelöf iff for every γ_i−open cover $\{\mathcal{U}_\alpha : \alpha \in \Lambda\}$ of $\mathfrak{X}$, there exists a countable subset $\{\mathcal{U}_{\alpha_m} : m \in \mathbb{N}\}$, s.t., $\mathfrak{X} = cl^p(\cup_{m\in\mathbb{N}} \mathcal{U}_{\alpha_m})$.

A bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is said to be spw−Lindelöf iff it is both sw_1−Lindelöf and sw_2−Lindelöf.

Example 7: *Consider* $\mathfrak{X} = \{s, m, l\}$. *Suppose that* $\gamma_1 = \{\emptyset, \mathfrak{X}, \{m\}, \{s, m\}, \{m, l\}\}$ *and* $\gamma_2 = \{\emptyset, \mathfrak{X}, \{s\}, \{m\}, \{s, m\}\}$ *be two topologies on* $\mathfrak{X}$. *So,* $\mathcal{O}^p = \{\emptyset, \mathfrak{X}, \{m\}, \{m, s\}\}$. *Thus,* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is* spw−*Lindelöf space. Note that the family* $\{\{s, m\}, \{m, l\}\}$ *is an* γ_1−*open cover cannot be replaced with a countable* $\mathcal{O}^\text{p}$−*open cover. So,* $(\mathfrak{X}, \gamma_1, \gamma_2)$ *is not* sp−*Lindelöf space.*

Example 8: *Consider $\mathfrak{X} = \{s, m, l\}$. Suppose that $\gamma_1 = \{\emptyset,\ \mathfrak{X}, \{s\}, \{m\}, \{s, m\}, \{m, l\}\}$ and $\gamma_2 = \{\emptyset,\ \mathfrak{X}, \{s\}, \{m\}, \{s, m\}\}$ be two topologies on $\mathfrak{X}$. So, $\mathcal{O}^p = \{\emptyset,\ \mathfrak{X}, \{s\}, \{m\}, \{s, m\}\}$ and $\mathcal{F}^p = \{\emptyset,\ \mathfrak{X}, \{m, l\}, \{s, l\}, \{l\}\}$. Since the family $\{\{s\}, \{m, l\}\}$ is an γ_1-open cover, $\{s\} \in \mathcal{O}^p$ and $cl^p(\{s\}) = \{s, l\} \neq \mathfrak{X}$. Thus, $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is not* spw$-$*Lindelöf space.*

Theorem 8: *If a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* sp$-$*Lindelöf, then it is* p$-$*Lindelöf.*

Proof: By Definition 1, it is obvious.

The converse of Theorem 8 is not always true. The following example explains that.

Example 9: *[see Example 6] It is clear $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* p$-$*Lindelöf space. But $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is not* sp$-$*Lindelöf space.*

Theorem 9: *If a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* sp$-$*Lindelöf, then it is* p_*-*Lindelöf.*

Proof: Follows using Definition 1.

The converse of Theorem 9 is not always true. The following example explains that.

Example 10: *[see Example 4][1] Let γ_1 be the upper limit topology and γ_2 be the lower limit topology on $\mathbb{R}$.*

Where $\mathcal{B}_1 = \{(a, b] : a, b \in \mathbb{R},\ a < b\}$ and $\mathcal{B}_2 = \{[a, b) : a, b \in \mathbb{R},\ a < b\}$ are bases of γ_1 and γ_2 respectively.

Then $(\mathbb{R},\ \gamma_1,\ \gamma_2)$ is p_*-Lindelöf space. The family $\mathcal{B}^p = \{(a, b) : a, b \in \mathbb{R},\ a < b\}$ is a base for $\mathcal{O}^p$. Thus $(\mathbb{R},\ \gamma_1,\ \gamma_2)$ is not sp$-$Lindelöf space (since, $\{(n, n+1] : n \in \mathbb{Z}\}$ is an γ_1- open cover of $\mathbb{R}$ cannot be replaced with a countable $\mathcal{O}^{\mathrm{p}}-$open cover).

Theorem 10: *If a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* p$-$*regular and* sp$-$*Lindelöf, then it is* p_*-*normal.*

Proof: From Definition 1 and Theorem 3, the result follows.

Lemma 2: *If a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* sp$-$*Lindelöf and* $\mathrm{B} \in \mathcal{F}_{\mathrm{i}}$*, s.t., $i = 1, 2$. Then B is also* sp$-$*Lindelöf.*

Proof: Let a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ be sp$-$Lindelöf and $B \in \mathcal{F}_i$, if $i = 1$ or 2. Then $\mathfrak{X} - B \in \gamma_i$. Now, if $\{W_\delta : \delta \in \Delta\}$ is an γ_i-open cover of B, then $\mathfrak{X} = (\cup_{\delta \in \Delta} W_\delta) \cup (\mathfrak{X} - B)$. Hence, $\{\mathfrak{X} - B, W_\delta : \delta \in \Delta\}$ is an γ_i-open cover of $\mathfrak{X}$. Since, $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is sp$-$Lindelöf space, then we have two cases.

Case 1: If $\mathfrak{X} - B \in \mathcal{O}^p$, then $\{\mathfrak{X} - B, W_\delta : \delta \in \Delta\}$ is γ_i-open cover of $\mathfrak{X}$ can be replaced with a countable pairwise open cover $\{\mathfrak{X} - B, W_{\delta_1}, W_{\delta_2}, W_{\delta_3}, \cdots\}$, but B and $\mathfrak{X} - B$ are disjoint. Therefore, $\{W_{\delta_m} : m \in \mathbb{N}\}$ is a countable pairwise open cover of B.

Case 2: If $\mathfrak{X} - B \notin \mathcal{O}^p$, then $\{\mathfrak{X} - B, W_\delta : \delta \in \Delta\}$ is γ_i-open cover of $\mathfrak{X}$ can be replaced with a countable pairwise open cover $\{W_{\delta_1}, W_{\delta_2}, W_{\delta_3}, \cdots\}$. Also, it is a countable pairwise open cover of B.

Theorem 11: *If a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* sp$-$*regular and* sp$-$*Lindelöf, then it is* sp$-$*normal space.*

Proof: Let $A \in \mathcal{F}_1$ and $\in \mathcal{F}_2$, s.t., $A \cap B = \emptyset$. Since $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is sp$-$regular space and $\mathfrak{X} - A \in \gamma_1$. Then $\forall x \in B$, we have $x \in \mathfrak{X} - A$, there exists an γ_1-open set $\mathrm{T_x}$, s.t., $x \in T_x \subseteq cl^p(T_x) \subseteq \mathfrak{X} - A$.

We have $cl^p(T_x) \cap A = \emptyset$ and $\{T_x : x \in B\}$ forms an γ_1-open cover of B. Since $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is sp$-$Lindelöf space, then B is sp$-$Lindelöf by Lemma 2. So, there exists a countable pairwise open cover of B and assume that $\{T_{x_j} : j \in \mathbb{N}\}$. Similarly, $\forall$y $\in$ A and $y \in \mathfrak{X} - B$, there is an γ_2-open set O_y, s.t., $y \in O_y \subseteq cl^p(O_y) \subseteq \mathfrak{X} - B$.

We have $cl^p(O_y) \cap B = \emptyset$ and $\{O_y : y \in A\}$ forms an γ_2-open cover of A. Since $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is sp$-$Lindelöf space, then A is sp$-$Lindelöf by Lemma 2. So, there exists a countable pairwise open cover of A and assume that $\{O_{y_j} : j \in \mathbb{N}\}$. Suppose that $\mathcal{W}_s = O_{y_j} - \cup\{cl^p(\mathcal{V}_i) : i < s\}$ and $\mathcal{V}_s = T_{x_s} - \cup\{cl^p(\mathcal{W}_i) : i \leq s\}$.

Since, $\mathcal{W}_s \cap cl^p(\mathcal{V}_t) = \emptyset$ for $t < s$, it follows that $\mathcal{W}_s \cap \mathcal{V}_t = \emptyset$ for $t < s$.

Similarly, $\mathcal{V}_t \cap cl^p(\mathcal{W}_s) = \emptyset$ for $s \leq t$, it follows that $\mathcal{V}_t \cap \mathcal{W}_s = \emptyset$ for $s \leq$ t. Thus $\mathcal{W}_s \cap \mathcal{V}_t = \emptyset$ for all s and t.

Now, we set $\mathcal{W} = \cup\{\mathcal{W}_s : s \in \mathbb{N}\}$ and $\mathcal{V} = \cup\{\mathcal{V}_s : \mathrm{s} \in \mathbb{N}\}$.

We have $\mathcal{W} \cap \mathcal{V} = \emptyset$, $cl^p(\mathcal{V}_i) \cap A = \emptyset$ and $cl^p(\mathcal{W}_i) \cap B = \emptyset$, for all i. Therefore, the pairwise open set $\mathcal{W}$ contains A. Also, the pairwise open set $\mathcal{V}$ contains B.

The converse of Theorem 11 is not always true. The following example explains that.

Example 11: *In Examples 1 and 2, we have a bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* sp$-$*normal. But, it is not* sp$-$*regular space.*

Theorem 12: *A bitopological space $(\mathfrak{X},\ \gamma_1,\ \gamma_2)$ is* $\mathrm{sw_i}-$*Lindelöf iff every family $\{\mathcal{C}_\alpha : \alpha \in \Lambda\}$ of γ_i-closed, s.t., $\cap_{\alpha \in \Lambda} \mathcal{C}_\alpha = \emptyset$, can be reduced to a countable subfamily $\{\mathcal{C}_{\alpha_m} : m \in \mathbb{N}\}$, s.t. $int^p(\cap_{m \in \mathbb{N}} \mathcal{C}_{\alpha_m}) = \emptyset$.*

Proof: Let the condition be holds and let $\{\mathcal{U}_\alpha : \alpha \in \Lambda\}$ be a family γ_i-open cover of $\mathfrak{X}$. Then the family $\{\mathfrak{X} - \mathcal{U}_\alpha : \alpha \in \Lambda\}$ is γ_i-closed subset of $\mathfrak{X}$. Since $\mathfrak{X} = \cup_{\alpha\in\Lambda}\mathcal{U}_\alpha$, then $\mathfrak{X} - \cup_{\alpha\in\Lambda}\mathcal{U}_\alpha = \emptyset$. Therefore $\cap_{\alpha\in\Lambda}(\mathfrak{X} - \mathcal{U}_\alpha) = \emptyset$. By hypothesis, we have a countable subfamily $\{\mathfrak{X} - \mathcal{U}_{\alpha_m} : m \in \mathbb{N}\}$, s.t. $int^p(\cap_{m\in\mathbb{N}}(\mathfrak{X} - \mathcal{U}_{\alpha_m})) = \emptyset$. So

$$\mathfrak{X} = \mathfrak{X} - int^p\left(\underset{m\in\mathbb{N}}{\cap}(\mathfrak{X} - \mathcal{U}_{\alpha_m})\right)$$

$$= cl^p\left(\mathfrak{X} - \underset{m\in\mathbb{N}}{\cap}(\mathfrak{X} - \mathcal{U}_{\alpha_m})\right) = cl^p\left(\underset{m\in\mathbb{N}}{\cup}\mathcal{U}_{\alpha_m}\right).$$

Therefore, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is $\mathrm{sw_i}-$Lindelöf space.

Conversely, let $\{\mathcal{C}_\alpha : \alpha \in \Lambda\}$ be a family of γ_i-closed subsets of $\mathfrak{X}$, s.t.,$\cap_{\alpha\in\Lambda}\mathcal{C}_\alpha = \emptyset$.

Then $\mathfrak{X} = \mathfrak{X} - \cap_{\alpha\in\Lambda}\mathcal{C}_\alpha = \cup_{\alpha\in\Lambda}(\mathfrak{X} - \mathcal{C}_\alpha)$. Thus the family $\{\mathfrak{X} - \mathcal{C}_\alpha : \alpha \in \Lambda\}$ is $\gamma_\mathrm{i}-$open cover of $\mathfrak{X}$. Since, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is $\mathrm{sw_i}-$Lindelöf space, then we have a subfamily $\{\mathfrak{X} - \mathcal{C}_{\alpha_m} : m \in \mathbb{N}\}$, s.t.,

$$\mathfrak{X} = cl^p\left(\underset{m\in\mathbb{N}}{\cup}(\mathfrak{X} - \mathcal{C}_{\alpha_m})\right). \text{ Therefore,}$$

$$\emptyset = \mathfrak{X} - cl^p\left(\underset{m\in\mathbb{N}}{\cup}(\mathfrak{X} - \mathcal{C}_{\alpha_m})\right)$$

$$= int^p\left(\mathfrak{X} - \underset{m\in\mathbb{N}}{\cup}(\mathfrak{X} - \mathcal{C}_{\alpha_m})\right) = int^p\left(\underset{m\in\mathbb{N}}{\cap}\mathcal{C}_{\alpha_m}\right).$$

Theorem 13: *If a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ is $\mathrm{w_{(i,j)}}-$Lindelöf, then it is $\mathrm{sw_i}-$Lindelöf space.*

Proof: Let a bitopological space $(\mathfrak{X}, \gamma_1, \gamma_2)$ be $\mathrm{w_{(i,j)}}-$Lindelöf space and let $\{\mathcal{U}_\alpha : \alpha \in \Lambda\}$ be γ_i-open cover of $\mathfrak{X}$. Then we have a countable subfamily $\{\mathcal{U}_{\alpha_m} : m \in \mathbb{N}\}$, s.t., $\mathfrak{X} = cl_j(\cup_{m\in\mathbb{N}}\mathcal{U}_{\alpha_m})$.

From Lemma 1, we have $\mathfrak{X} = cl_j(\cup_{m\in\mathbb{N}}\mathcal{U}_{\alpha_m}) \subseteq cl^p(\cup_{m\in\mathbb{N}}\mathcal{U}_{\alpha_m})$. Therefore, $(\mathfrak{X}, \gamma_1, \gamma_2)$ is $\mathrm{sw_i}-$Lindelöf space.

The converse of Theorem 13 is not always true. The following examples explain that.

Example 12: *Let γ_1 be the usual topology and γ_2 be the discrete topology on $\mathbb{R}$, where $\mathcal{B}_1 = \{(\mathrm{a}, \mathrm{b}) : \mathrm{a}, \mathrm{b} \in \mathbb{R}, \ \mathrm{a} < \mathrm{b}\}$ is a base of γ_1. We have, $\mathcal{O}^\mathrm{p} = \gamma_1$ and the family $\{\{\mathrm{x}\} : \mathrm{x} \in \mathbb{R}\}$ is the γ_2-open cover of $\mathbb{R}$ cannot be replaced by a countable subcover $\{\mathcal{U}_{\alpha_\mathrm{m}} : \mathrm{m} \in \mathbb{N}\}$, s.t., $\mathbb{R} = \mathrm{cl}_1(\cup_{\mathrm{m}\in\mathbb{N}}\mathcal{U}_{\alpha_\mathrm{m}})$. Then $(\mathbb{R}, \gamma_1, \gamma_2)$ is not $\mathrm{w_{(2,1)}}-$Lindelöf space. But, $(\mathbb{R}, \gamma_1, \gamma_2)$ is $\mathrm{sw_1}-$Lindelöf space.*

Example 13: *Let Ω be the set of ordinals which are less than or equal to the first uncountable ordinal ω_1. Let γ_1 be the order topology on $\Omega_0 = \Omega/\{\omega_1\}$ and γ_2 be the discrete topology on Ω_0, where the collection β of the form $(\mathrm{a}, \mathrm{b}) = \{\mathrm{x} \in \Omega_0 : \mathrm{a} < \mathrm{x} < \mathrm{b}\}$, $(\mathrm{s}, \omega_1) = \{\mathrm{x} \in \Omega_0 : \mathrm{s} < \mathrm{x} < \omega_1\}$ and $[1, \mathrm{t}) = \{\mathrm{x} \in \Omega_0 : 1 \le \mathrm{x} < \mathrm{t}\}$ is the base of γ_1. Then, $(\mathbb{R}, \gamma_1, \gamma_2)$ is not $\mathrm{w_{(1,2)}}-$Lindelöf space. We have $\mathcal{O}^\mathrm{p} = \gamma_1$, therefore, it is $\mathrm{sw_1}-$Lindelöf space.*

Conclusion

This study has deeply investigated the importance of the results that are strongly regular and strongly normal bitopological spaces. Additionally, it has introduced two new pairwise Lindelöf spaces. Actually, these two spaces are generalizations of Lindelo double dot f spaces, which have been widely investigated before.[1,2] Moreover, the study presents several important results that are related to these spaces. Therefore, the researchers believe that the important results will be, positively, utilized in future work on fuzzifying bitopological spaces, L-fuzzifying bitopological spaces, and (L, M)-fuzzy bitopological spaces.

Authors' declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Al-Azhar University, Egypt.

Authors' contributions statement

The authors worked together on this work, with H.M.B. providing the main idea and A.K.M. offering important suggestions that contributed to improving the research.

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تعميمات جديدة لفضاءات ليندلوف الثنائية

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الخلاصة

تقدم هذه الدراسة فضاءين ثنائيين جديدين مختلفين من النوع ليندلوف. تُعد فضاءات ليندلوف من أهم الفضاءات التوبولوجية ولها تطبيقات واسعة في مختلف فروع الرياضيات. ولذلك نشرت العديد من الدراسات حول فضاءات ليندلوف وفضاءات ليندلوف التوبولوجية الثنائية في ابحاث سابقة. ومع ذلك فإن هذين الفضاءين الجديدين من النوع ليندلوف هما في الواقع أكثر عمومية مما تمت دراسته من قبل. لذلك, تعرف الدراسة مؤثر ثنائي داخلي جديدا ومؤثر ثنائي إغلاقي جديدا, علاوة على ذلك, فإنها تستعرض خصائصهما حيث يقدم الباحثون مسلمات انفصال ثنائية قوية جديدة (مسلمات الانفصال الثنائية القوية المنتظمة ومسلمات الانفصال الثنائية القوية العادية) في الفضاءات التوبولوجية الثنائية. علاوة على ذلك، تم استعرض العديد من الأمثلة التوضيحية والأمثلة العكسية, كما تم إجراء دراسة حول العلاقة بين مسلمات الانفصال الثنائية القوية المنتظمة ومسلمات الانفصال الثنائية المنتظمة والعلاقة بين مسلمات الانفصال الثنائية القوية العادية ومسلمات الانفصال الثنائية العادية. بالإضافة إلى ذلك، تقدم هذه الدراسة بعض النتائج المهمة لمسلمات الانفصال الثنائية القوية المنتظمة ومسلمات الانفصال الثنائية القوية العادية في الفضاءات التوبولوجية الثنائية. وأخيرا، أظهرت النتائج بعض العلاقات بين المفاهيم الجديدة لفضاءات ليندلوف الثنائية.

الكلمات المفتاحية: الفضاءات التوبولوجية الثنائية، الفضاءات العادية، فضاءات ليندلوف الثنائية، الفضاءات المنتظمة، مسلمات الانفصال.