Numerical solution for non-linear two-dimensional Generalized BBM equation using the Haar wavelet method

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Abstract
In this paper, we applied the Haar wavelets method which is operational matrices of orthogonal functions to solve two dimensional Generalized Benjamin-Bona-Mahony equation numerically. It transforms the problem into a simple Lyapunov matrix equation. We compared this numerical result with the exact solution. The accuracy of the obtained solutions is quite high even if the number of calculation points is small, by increasing the number of collocation points the error of the solution rapidly decreases as shown by solving an example. We have been reducing the boundary conditions in the solution by using the finite differences method with respect to time and the notation (6) with respect to space .
Keywords : Generalized BBM equation, Haar wavelets ,Operational matrix, Lyapunov matrix equation .

Introduction
As a powerful mathematical tool, wavelet analysis has been widely used in image digital processing , quantum field theory, numerical analysis and many other fields in recent years .
Haar wavelets have been applied extensively for signal processing in communications and physics researches, and more mathematically focused on differential equations and even nonlinear problems . After discrediting.
the differential equation in a convential way like the finite difference approximation , wavelets can be used for algebraic manipulations in the system of equations obtained which may lead to better condition number of the resulting system [1] .Using the operational matrix of an orthogonal function to perform integration for solving, identifying and optimizing a linear dynamic system has several advantages :
(1) the method is computer oriented, thus solving higher order differential equation becomes a matter of dimension increasing .
(2) the solution is a multi-resolution type and large [2] .
(3) the answer is convergent, even the size of increment is very large .
The main characteristic of the operational method is to convert a differential equation into an algebraic one, and the core is the operational matrix for integration .
Lepik and Tamme [3] derived the solution of nonlinear Fredholm integral equations via the Haar wavelet method, they found that the main benefits of the Haar wavelet method are sparse representation, fast transformation, and possibility of implementation of fast algorithms especially if matrix representation is used. Also Lepik Uio [4] studied the application of the Haar wavelet transform to solve integral and differential equations. Zhi and Qing [1] established a clear procedure for finite-length beam problem and convection-diffusion equation solution via Haar wavelet technique. AL-Rawi and Qasem [5] found the numerical solution for nonlinear Murray equation by the operational matrices of Haar wavelet method and compared the results of this method with the exact solution, they transformed the nonlinear Murray equation into a linear algebraic equations that can be solved by Gauss-Jordan method. Hariharan and Kannan [6] developed an accurate and efficient Haar transform or Haar wavelet method for some of the well-known nonlinear parabolic partial differential equations. The equations include the Nowell-whitehead equation, Cahn-Allen equation , Fitz Hugh-Nagumo equation, and other equations .
Many authors have studied the solution for two dimensional Generalized Benjamin-Bona-Mahony equation .Adames et al. [7] studied the controllability of the Generalized Benjamin-Bona-Mahony equation (BBM) with homogeneous Dirichlet boundary conditions . Under some conditions they proved the system is approximately controllable.
Leiva et al. [8] proved the interior approximate controllability of the following Generalized Benjamin -Bona-Mahony equation with homogeneous Dirichlet boundary conditions. They proved that for all $\tau > 0$ and any nonempty open subset W of Ω the system is approximately controllable on $[0, \tau]$. YE [9] studied the initial-boundary value problem for a class of inhomogeneous BBM equations with nonlinear term $|u|^{\alpha-2}$ for some range of parameter α value, the decay estimates of the global solutions for this problem proved by means of Sobolev inequality. Changhong [10] studied the optimal decay rates of solutions for the Generalized Benjamin-Bona-Mahony equation in multi-dimensional space $(n \geq 3)$ by using Fourier transform and the energy method .Chueshov et al . [11] proved the Gevrey regularity at the global attractor of the dynamical system generated by the Generalized Benjamin-Bona-Mahony equation with periodic boundary conditions, this result means that elements of the attractor are real analytic functions in spatial variables .
In this paper, we study the numerical solution for nonlinear two-dimensional Generalized Benjamin-

Bona-Mahony equation by the operational matrices of Haar wavelet method and we compare the results of this method with the exact solution .

We organized our paper as follows. In section two, the Haar wavelet is introduced and an operational matrix is established .

Section three function approximation is presented. In

Section four we use Haar wavelets to solve nonlinear 2D-BBM equation . Section five numerical results are presented. Concluding remarks are given in section six .

Haar wavelets

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1) by [6] :

$$h_i(x) = \begin{cases} 1 & \frac{k}{m} \le x < \frac{k+1/2}{m} \\ -1 & \frac{k+1/2}{m} \le x < \frac{k+1}{m} \\ 0 & otherwise \end{cases} \quad (1)$$

Integer $m = 2^j$ $(j = 0,1,2,...., J)$ indicates the level of the wavelet; $k=0,1,2,...,m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i=m+k+1$, in the case of minimal values. $m=1,k=0$ we have $i=2$, the maximal value of i is $i = 2M = 2^{j+1}$. It is assumed that the value $i=1$ corresponds to the scaling function for which .

Let us define the collocation points $h_1 = 1$ in [0,1) $x_l = (l-0.5)/2M$, $(l = 1,2,...,2M)$ and discredits the Haar function $h_i(x)$; in this way we get the coefficient matrix $H(i,l) = (h_i(x_l))$, which has the dimension $2M*2M$. The operational matrix of integration P, which is a $2M$ square matrix, is defined by the equation: [4]

$$P_{i,1}(x) = \int_0^{x_l} h_i(x)\, dx \quad (2)$$

$$P_{i,v+1}(x) = \int_0^{x} P_{i,v}(x)\, dx \quad , \qquad v = 1,2,... \quad (3)$$

These integrals can be evaluated using equation (1) and first four of them are given by [4] as follows :

$$P_{i,1}(x) = \begin{cases} x - \alpha & for \;\; x \in [\alpha, \beta) \\ \gamma - x & for \;\; x \in [\beta, \gamma) \\ 0 & elsewhere \end{cases} \quad (4)$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & for \;\; x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & for \;\; x \in [\beta, \gamma) \\ \frac{1}{4m^2} & for \;\; x \in [\gamma, 1) \\ 0 & elsewhere \end{cases} \quad (5)$$

The integration of equation (3) from 0 to 1 is given in the following notation :

$$R_{i,v} = \int_0^1 P_{i,v}(x)\, dx \qquad (6)$$

such that :

$$R_{i,1}(x) = P_{i,2}(1) = \begin{cases} \frac{1}{2}(1-\alpha)^2 & for \;\; x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - 1)^2 & for \;\; x \in [\beta, \gamma) \\ \frac{1}{4m^2} & for \;\; x \in [\gamma, 1) \\ 0 & elsewhere \end{cases}$$

Function approximation

Any square integrable function $u(x)$ in the interval *[0,1)* can be expanded by a Haar series of infinite terms [6] :

$$u(x) = \sum_{i=0}^{\infty} c_i\, h_i(x) \qquad i \in \{0\} \cup N \quad (7)$$

Where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 u(x) h_0(x) dx \quad , \quad c_n = 2^j \int_0^1 u(x) h_i(x) dx \quad (8)$$

$$i = 2^j + k, \quad j \ge 0, \quad 0 \le k < 2^j, \quad x \in [0,1)$$

Such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[u(x) - \sum_{i=0}^{m-1} c_i h_i(x) \right]^2 dx, \quad m = 2^j, \quad j \in \{0\} \cup N$$

Usually the series expansion of (7) contains infinite terms for smooth *u(x)*. If *u(x)* is piecewise constant by itself, or may be approximation as piecewise constant during each subinterval, then *u(x)* will be terminated at finite *m* terms, that is :

$$u(x) = \sum_{i=0}^{m-1} c_i\, h_i(x) = c_{(m)}^T\, h_{(m)}(x)$$

Where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as follows :

$c_{(m)}^T = [c_0, \ c_1, \ \ldots, \ c_{m-1}]$ And $h_{(m)}(x) = [h_0(x), h_1(x), \ldots\ldots, h_{m-1}(x)]^T$

Where T means transpose .

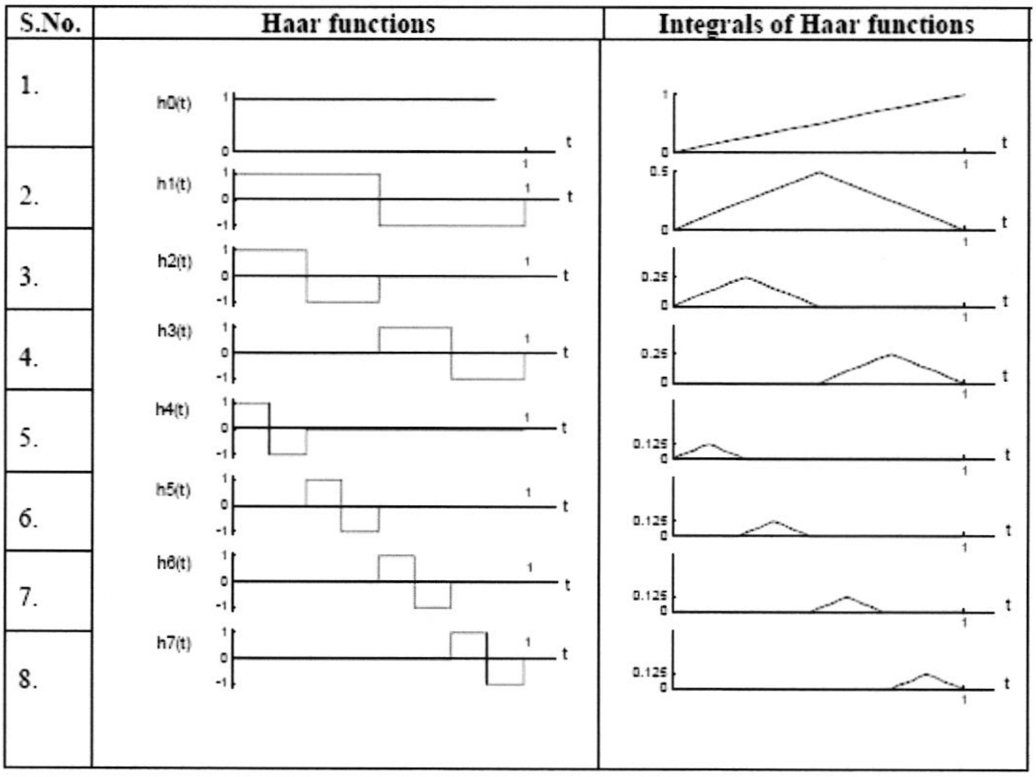


Fig. 1. First eight Haar functions

Similarly,a two dimensional function $u(x, y)$ which is square integrable in the interval $0 \le x < 1$ *and* $0 \le y < 1$ can be expanded into Haar series by [2] as follows :

$$u(x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} C_{i,j}\, h_i(x)\, h_j(y) \quad (9)$$

Where the coefficient matrix $C_{i,j}$ and the Haar function $h_i(x)$ *and* $h_j(y)$ are defined by :

$$C=\begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & \cdots & c_{1,m-1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & \cdots & c_{m-1,m-1} \end{bmatrix},\ H=\begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & \cdots & h_{1,m-1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & \cdots & h_{m-1,m-1} \end{bmatrix}$$

Equation (9) can be written in a discrete form as:

$$u(x,y)=H^T(x)\cdot C\cdot H(y) \tag{10}$$

Now, the integration with respect to a variable x of $u(x,y)$ by using equations (2) and (3) is [2] :

$$\begin{aligned}\int_0^x u(x,y)\,dx &= \int_0^x H^T(x)\cdot C\cdot H(y)\,dx \\ &= \int_0^x H^T(x)\,dx\ \cdot C\cdot H(y) \\ &= P_{i,1}^T(x)\cdot C\cdot H(y)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\int_0^y u(x,y)\,dy &= \int_0^y H^T(x)\cdot C\cdot H(y)\,dy \\ &= H^T(x)\cdot C\cdot \int_0^y H(y)\,dy \\ &= H^T(x)\cdot C\cdot P_{i,1}(y)\end{aligned} \tag{12}$$

performing the double integration, we obtain:

$$\int_0^y\int_0^x u(x,y)\,dx\,dy = P_{i,1}^T(x)\cdot C\cdot P_{i,1}(y) \tag{13}$$

and

$$\int_0^y\int_0^y\int_0^x\int_0^x u(x,y)\,dx\ \ dx\,dy\,dy = P_{i,2}^T(x)\cdot C\cdot P_{i,2}(y) \tag{14}$$

Mathematical Model

Let us consider the two- dimensional Generalized Benjamin-Bona-Mahony (GBBM) equation which has the form [7]:

$$u_{t_*} - a\,\Delta u_{t_*} - b\,\Delta u = b_1(X_*)u_1 + \cdots\cdots + b_m(X_*)u_m \qquad t_* \ge 0,\ X_* \in \Omega \quad (15)$$

With the initial and homogeneous Dirichlet boundary Conditions :

$$u(X_*,t_*)=0 \qquad t_* \ge 0 \quad , \quad X_* \in \partial\Omega \quad (16)$$

Where a and b are positive parameters . $b_i \in L^2(\Omega, IR^2)$, the control functions $u_i \in L^2(0,t_1; IR)$ $i=1,2,3,....,m.$ $X_* = [x_*, y_*]$, Ω is a bounded domain in IR^2.Since the Haar wavelets are defined for $x \in [0,1)$, we must first normalize equation (15) and initial-boundary conditions(16) in regard to $X_* = [x_*, y_*]$ .We changing the variables:

$$X = \frac{1}{L}(X_* - a), \qquad t = t_* - 0, \qquad L = b - a \quad X = [x, y]$$

Then equations (15) and (16) become:

$$u_t - \frac{a}{L^2}\Delta u_t - \frac{b}{L^2}\Delta u = b_1(LX)u_1 + \cdots\cdots + b_m(LX)u_m \qquad t \geq 0, X \in \Omega \quad (17)$$

With the initial and boundary conditions:

$$u(X,t) = 0 \quad t \geq 0 \quad , \quad X \in \partial\Omega \quad (18)$$

Let us divide the interval *(0,T]* into *N* equal parts of length $\Delta t = T/N$ and denote to $t_s = (s-1)\Delta t$, *s=1,2,...,N.*

We assume that $u_{xxyyt}(x,y,t)$ can be expanded in terms of Haar wavelets as follows:

$$u_{xxyyt}(x,y,t) = H_m^T(x)\, C_m\, H_m(y) \qquad t \in (t_s, t_{s+1}] \quad (19)$$

Where the matrix $C_{(m)}$ is constant matrix in the subinterval $t \in (t_s, t_{s+1}]$. Integrating (19) with respect to t from (t_s) to t and double integrating with respect to x from 0 to x , and double integrating with respect to y from 0 to y ,we obtain :

$$u_{xxyy}(x,y,t) = (t-t_s)\, H_m^T(x)\, C_m\, H_m(y) + u_{xxyy}(x,y,t_s) \quad (20)$$

$$\begin{aligned} u_{xxy}(x,y,t) &= (t-t_s)\, H_m^T(x)\, C_m\, P_{i,1}(y) + u_{xxy}(x,y,t_s) \\ &\quad + \left[u_{xxy}(x,0,t) - u_{xxy}(x,0,t_s)\right] \end{aligned} \quad (21)$$

$$\begin{aligned} u_{xx}(x,y,t) &= (t-t_s)\, H_m^T(x)\, C_m\, P_{i,2}(y) + u_{xx}(x,y,t_s) \\ &\quad + y\left[u_{xxy}(x,0,t) - u_{xxy}(x,0,t_s)\right] + \left[u_{xx}(x,0,t) - u_{xx}(x,0,t_s)\right] \end{aligned} \quad (22)$$

$$\begin{aligned} u_x(x,y,t) &= (t-t_s)\, P_{i,1}^T(x)\, C_m\, P_{i,2}(y) + u_x(x,y,t_s) \\ &\quad + y\left[u_{xy}(x,0,t) - u_{xy}(x,0,t_s)\right] - y\left[u_{xy}(0,0,t) - u_{xy}(0,0,t_s)\right] \\ &\quad + \left[u_x(x,0,t) - u_x(x,0,t_s)\right] + \left[u_x(0,y,t) - u_x(0,y,t_s)\right] \\ &\quad - \left[u_x(0,0,t) - u_x(0,0,t_s)\right] \end{aligned} \quad (23)$$

$$\begin{aligned} u(x,y,t) &= (t-t_s)\, P_{i,2}^T(x)\, C_m\, P_{i,2}(y) + u(x,y,t_s) \\ &\quad + y\left[u_y(x,0,t) - u_y(x,0,t_s)\right] - y\left[u_y(0,0,t) - u_y(0,0,t_s)\right] \\ &\quad - x\left[u_x(0,0,t) - u_x(0,0,t_s)\right] + x\left[u_x(0,y,t) - u_x(0,y,t_s)\right] \\ &\quad - xy\left[u_{xy}(0,0,t) - u_{xy}(0,0,t_s)\right] + \left[u(x,0,t) - u(x,0,t_s)\right] \\ &\quad + \left[u(0,y,t) - u(0,y,t_s)\right] - \left[u(0,0,t) - u(0,0,t_s)\right] \end{aligned} \quad (24)$$

We can be reduce the order of the boundary conditions used in equations (19)-(24) by using the boundary condition at *x=1* and notation (6) instead of the derivatives $u_x(0,y,t), u_x(0,y,t_s), u_x(0,0,t)$ and $u_x(0,0,t_s)$.

The values of unknown terms $u_x(0,y,t), u_x(0,y,t_s), u_x(0,0,t)$ and $u_x(0,0,t_s)$ can be calculated by integrating equation (23) from *0* to *1* and we get:

$$\int_0^1 u_x(x,y,t)\,dx = \int_0^1 (t-t_s)\,P_{i,1}^T(x)\,C_m\,P_{i,2}(y)\,dx + \int_0^1 u_x(x,y,t_s)\,dx$$
$$+\, y\int_0^1 \left[u_{xy}(x,0,t) - u_{xy}(x,0,t_s)\right]dx - y\int_0^1 \left[u_{xy}(0,0,t) - u_{xy}(0,0,t_s)\right]dx$$
$$+\int_0^1 \left[u_x(x,0,t) - u_x(x,0,t_s)\right]dx + \int_0^1 \left[u_x(0,y,t) - u_x(0,y,t_s)\right]dx$$
$$-\int_0^1 \left[u_x(0,0,t) - u_x(0,0,t_s)\right]dx$$

This implies that

$$\begin{aligned}
&\left[u_x(0,y,t) - u_x(0,y,t_s)\right] - \left[u_x(0,0,t) - u_x(0,0,t_s)\right] = \\
&-(t-t_s)\,P_{i,2}^T(1)\,C_m\,P_{i,2}(y) + \left[u(1,y,t) - u(1,y,t_s)\right] \\
&- y\left[u_y(1,0,t) - u_y(1,0,t_s)\right] + y\left[u_y(0,0,t) - u_y(0,0,t_s)\right] \\
&+ y\left[u_{xy}(0,0,t) - u_{xy}(0,0,t_s)\right] - \left[u(1,0,t) - u(1,0,t_s)\right] \\
&- \left[u(0,y,t) - u(0,y,t_s)\right] + \left[u(0,0,t) - u(0,0,t_s)\right]
\end{aligned} \tag{25}$$

Such that

$$P_{i,2}(1) = R_{i,1}(x) = \begin{cases} \frac{1}{2}(1-\alpha)^2 & for\ x\in[\alpha,\beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma-1)^2 & for\ x\in[\beta,\gamma) \\ \frac{1}{4m^2} & for\ x\in[\gamma,1) \\ 0 & elsewhere \end{cases} \tag{26}$$

By substituting equation (25) into equation (24) and with simplification, we get:

$$\begin{aligned}
u(x,y,t) &= (t-t_s)\,P_{i,2}^T(x)\,C_m\,P_{i,2}(y) + u(x,y,t_s) + y\left[u_y(x,0,t) - u_y(x,0,t_s)\right] \\
&- y\left[u_y(0,0,t) - u_y(0,0,t_s)\right] + \left[u(x,0,t) - u(x,0,t_s)\right] \\
&+ \left[u(0,y,t) - u(0,y,t_s)\right] - \left[u(0,0,t) - u(0,0,t_s)\right] \\
&+ x\,\{\ -(t-t_s)\,R_{i,1}^T(x)\,C_m\,P_{i,2}(y) + \left[u(1,y,t) - u(1,y,t_s)\right] \\
&- y\left[u_y(1,0,t) - u_y(1,0,t_s)\right] + y\left[u_y(0,0,t) - u_y(0,0,t_s)\right] \\
&- \left[u(1,0,t) - u(1,0,t_s)\right] - \left[u(0,y,t) - u(0,y,t_s)\right] + \left[u(0,0,t) - u(0,0,t_s)\right]\ \}
\end{aligned} \tag{27}$$

Similarly by using the boundary condition at $y=1$ and notation (6), we get:

$$\begin{aligned}
u(x,1,t) &= (t-t_s)\,P_{i,2}^T(x)\,C_m\,R_{i,1}(y) + u(x,1,t_s) + \left[u_y(x,0,t) - u_y(x,0,t_s)\right] \\
&- \left[u_y(0,0,t) - u_y(0,0,t_s)\right] + \left[u(x,0,t) - u(x,0,t_s)\right] \\
&+ \left[u(0,1,t) - u(0,1,t_s)\right] - \left[u(0,0,t) - u(0,0,t_s)\right] \\
&+ x\,\{\ -(t-t_s)\,R_{i,1}^T(x)\,C_m\,R_{i,1}(y) + \left[u(1,1,t) - u(1,1,t_s)\right] \\
&- \left[u_y(1,0,t) - u_y(1,0,t_s)\right] + \left[u_y(0,0,t) - u_y(0,0,t_s)\right] \\
&- \left[u(1,0,t) - u(1,0,t_s)\right] - \left[u(0,1,t) - u(0,1,t_s)\right] + \left[u(0,0,t) - u(0,0,t_s)\right]\ \}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &\left[u_y(x,0,t) - u_y(x,0,t_s)\right] - \left[u_y(0,0,t) - u_y(0,0,t_s)\right] \\
&- x\left[u_y(1,0,t) - u_y(1,0,t_s)\right] + x\left[u_y(0,0,t) - u_y(0,0,t_s)\right] = \\
&-(t-t_s)\,P_{i,2}^T(x)\,C_m\,R_{i,1}(y) + \left[u(x,1,t) - u(x,1,t_s)\right] \\
&- \left[u(x,0,t) - u(x,0,t_s)\right] - \left[u(0,1,t) - u(0,1,t_s)\right] + \left[u(0,0,t) - u(0,0,t_s)\right] \\
&+ x\,(t-t_s)\,R_{i,1}^T(x)\,C_m\,R_{i,1}(y) - x\left[u(1,1,t) - u(1,1,t_s)\right] \\
&+ x\left[u(1,0,t) - u(1,0,t_s)\right] + x\left[u(0,1,t) - u(0,1,t_s)\right] - x\left[u(0,0,t) - u(0,0,t_s)\right]
\end{aligned} \tag{28}$$

By substituting equation (28) into equation (27), we get :

$$\begin{aligned}u(x,y,t) &= (t-t_s)\, P_{i,2}^T(x)\, C_m\, P_{i,2}(y) + u(x,y,t_s)\\
&+ [u(x,0,t)-u(x,0,t_s)] + [u(0,y,t)-u(0,y,t_s)] - [u(0,0,t)-u(0,0,t_s)]\\
&- x(t-t_s)\, R_{i,1}^T(x)\, C_m\, P_{i,2}(y) + x[u(1,y,t)-u(1,y,t_s)]\\
&- x[u(1,0,t)-u(1,0,t_s)] - x[u(0,y,t)-u(0,y,t_s)] + x[u(0,0,t)-u(0,0,t_s)]\\
&- y(t-t_s)\, P_{i,2}^T(x)\, C_m\, R_{i,1}(y) + y[u(x,1,t)-u(x,1,t_s)]\\
&- y[u(x,0,t)-u(x,0,t_s)] - y[u(0,1,t)-u(0,1,t_s)] + y[u(0,0,t)-u(0,0,t_s)]\\
&+ xy(t-t_s)\, R_{i,1}^T(x)\, C_m\, R_{i,1}(y) - xy[u(1,1,t)-u(1,1,t_s)]\\
&+ xy[u(1,0,t)-u(1,0,t_s)] + xy[u(0,1,t)-u(0,1,t_s)]\\
&- xy[u(0,0,t)-u(0,0,t_s)]\end{aligned} \tag{29}$$

Now the derivatives of equation (29) with respect to t, x and y, give us:

$$\begin{aligned}u_t(x,y,t) &= P_{i,2}^T(x)\, C_m\, P_{i,2}(y) + u_t(0,y,t) + u_t(x,0,t) - u_t(0,0,t)\\
&- x\, R_{i,1}^T(x)\, C_m\, P_{i,2}(y) + x u_t(1,y,t)\\
&- x u_t(1,0,t) - x u_t(0,y,t) + x u_t(0,0,t)\\
&- y\, P_{i,2}^T(x)\, C_m\, R_{i,1}(y) + y u_t(x,1,t)\\
&- y u_t(0,1,t) - y u_t(x,0,t) + y u_t(0,0,t)\\
&+ xy\, R_{i,1}^T(x)\, C_m\, R_{i,1}(y) - xy u_t(1,1,t)\\
&+ xy u_t(1,0,t) + xy u_t(0,1,t) - xy u_t(0,0,t)\end{aligned} \tag{30}$$

It can be used the finite difference formula with respect to time in equation (30) then after a simplification, we get :

$$\begin{aligned}u_t(x,y,t) &= P_{i,2}^T(x)\, C_m\, P_{i,2}(y) + (1-x)\left[\frac{u(0,y,t)-u(0,y,t_s)}{(t-t_s)}\right]\\
&+ (1-y)\left[\frac{u(x,0,t)-u(x,0,t_s)}{(t-t_s)}\right] - (1-x-y+xy)\left[\frac{u(0,0,t))-u(0,0,t_s)}{(t-t_s)}\right]\\
&- x\, R_{i,1}^T(x)\, C_m\, P_{i,2}(y) + x\left[\frac{u(1,y,t)-u(1,y,t_s)}{(t-t_s)}\right] - (x-xy)\left[\frac{u(1,0,t)-u(1,0,t_s)}{(t-t_s)}\right]\\
&- y\, P_{i,2}^T(x)\, C_m\, R_{i,1}(y) + y\left[\frac{u(x,1,t)-u(x,1,t_s)}{(t-t_s)}\right] - (y-xy)\left[\frac{u(0,1,t)-u(0,1,t_s)}{(t-t_s)}\right]\\
&+ xy\, R_{i,1}^T(x)\, C_m\, R_{i,1}(y) - xy\left[\frac{u(1,1,t)-u(1,1,t_s)}{(t-t_s)}\right]\end{aligned} \tag{31}$$

Note that $\Delta t = t - t_s$.

$$\begin{aligned}u_x(x,y,t) &= (t-t_s)\, P_{i,1}^T(x)\, C_m\, P_{i,2}(y) + u_x(x,y,t_s) + [u_x(x,0,t)-u_x(x,0,t_s)]\\
&- (t-t_s)\, R_{i,1}^T(x)\, C_m\, P_{i,2}(y) + [u(1,y,t)-u(1,y,t_s)]\\
&- [u(1,0,t)-u(1,0,t_s)] - [u(0,y,t)-u(0,y,t_s)] + [u(0,0,t)-u(0,0,t_s)]\\
&- y(t-t_s)\, P_{i,1}^T(x)\, C_m\, R_{i,1}(y) + y[u_x(x,1,t)-u_x(x,1,t_s)]\\
&- y[u_x(x,0,t)-u_x(x,0,t_s)]\\
&+ y(t-t_s)\, R_{i,1}^T(x)\, C_m\, R_{i,1}(y) - y[u(1,1,t)-u(1,1,t_s)]\\
&+ y[u(1,0,t)-u(1,0,t_s)] + y[u(0,1,t)-u(0,1,t_s)]\\
&- y[u(0,0,t)-u(0,0,t_s)]\end{aligned} \tag{32}$$

$$
\begin{aligned}
u_{xx}(x,y,t) &= (t-t_s)\, H_m^T(x)\, C_m\, P_{i,2}(y) + u_{xx}(x,y,t_s) \\
&+ \left[u_{xx}(x,0,t) - u_{xx}(x,0,t_s)\right] - y\,(t-t_s)\, H_m^T(x)\, C_m\, R_{i,1}(y) \\
&+ y\left[u_{xx}(x,1,t) - u_{xx}(x,1,t_s)\right] - y\left[u_{xx}(x,0,t) - u_{xx}(x,0,t_s)\right]
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
u_{xxt}(x,y,t) &= H_m^T(x)\, C_m\, P_{i,2}(y) - y\, H_m^T(x)\, C_m\, R_{i,1}(y) \\
&+ \left[u_{xxt}(x,0,t)\right] + y\left[u_{xxt}(x,1,t) - u_{xxt}(x,0,t)\right]
\end{aligned}
$$

By using the finite difference method , we get :

$$
\begin{aligned}
u_{xxt}(x,y,t) &= H_m^T(x)\, C_m\, P_{i,2}(y) - y\, H_m^T(x)\, C_m\, R_{i,1}(y) \\
&+ (1-y)\left[\frac{u_{xx}(x,0,t) - u_{xx}(x,0,t_s)}{(t-t_s)}\right] + y\left[\frac{u_{xx}(x,1,t) - u_{xx}(x,1,t_s)}{(t-t_s)}\right]
\end{aligned} \tag{34}
$$

Also, we get :

$$
\begin{aligned}
u_y(x,y,t) &= (t-t_s)\, P_{i,2}^T(x)\, C_m\, P_{i,1}(y) + u_y(x,y,t_s) + \left[u_y(0,y,t) - u_y(0,y,t_s)\right] \\
&- x\,(t-t_s)\, R_{i,1}^T(x)\, C_m\, P_{i,1}(y) + x\left[u_y(1,y,t) - u_y(1,y,t_s)\right] \\
&- x\left[u_y(0,y,t) - u_y(0,y,t_s)\right] \\
&- (t-t_s)\, P_{i,2}^T(x)\, C_m\, R_{i,1}(y) + \left[u(x,1,t) - u(x,1,t_s)\right] \\
&- \left[u(x,0,t) - u(x,0,t_s)\right] - \left[u(0,1,t) - u(0,1,t_s)\right] \\
&+ \left[u(0,0,t) - u(0,0,t_s)\right] \\
&+ x\,(t-t_s)\, R_{i,1}^T(x)\, C_m\, R_{i,1}(y) - x\left[u(1,1,t) - u(1,1,t_s)\right] \\
&+ x\left[u(1,0,t) - u(1,0,t_s)\right] + x\left[u(0,1,t) - u(0,1,t_s)\right] \\
&- x\left[u(0,0,t) - u(0,0,t_s)\right]
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
u_{yy}(x,y,t) &= (t-t_s)\, P_{i,2}^T(x)\, C_m\, H_m(y) + u_{yy}(x,y,t_s) \\
&+ \left[u_{yy}(0,y,t) - u_{yy}(0,y,t_s)\right] \\
&- x\,(t-t_s)\, R_{i,1}^T(x)\, C_m\, H_m(y) + x\left[u_{yy}(1,y,t) - u_{yy}(1,y,t_s)\right] \\
&- x\left[u_{yy}(0,y,t) - u_{yy}(0,y,t_s)\right]
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
u_{yyt}(x,y,t) &= P_{i,2}^T(x)\, C_m\, H_m(y) - x\, R_{i,1}^T(x)\, C_m\, H_m(y) \\
&+ \left[u_{yyt}(0,y,t)\right] + x\left[u_{yyt}(1,y,t) - u_{yyt}(0,y,t)\right]
\end{aligned} \tag{37}
$$

When using the finite difference method , we get :

$$
\begin{aligned}
u_{yyt}(x,y,t) &= P_{i,2}^T(x)\, C_m\, H_m(y) - x\, R_{i,1}^T(x)\, C_m\, H_m(y) \\
&+ (1-x)\left[\frac{u_{yy}(0,y,t) - u_{yy}(0,y,t_s)}{(t-t_s)}\right] + x\left[\frac{u_{yy}(1,y,t) - u_{yy}(1,y,t_s)}{(t-t_s)}\right]
\end{aligned} \tag{38}
$$

Now , substitute equations (31)-(38) into equation (17), to get :

$$
\begin{aligned}
&P_{i,2}^{T}(x)\, C_m\, P_{i,2}(y) - x\; R_{i,1}^{T}(x)\, C_m\, P_{i,2}(y) - y\; P_{i,2}^{T}(x)\, C_m\, R_{i,1}(y) \\
&+ x\, y\; R_{i,1}^{T}(x)\, C_m\, R_{i,1}(y) - \left(\frac{a}{L^2}\right) H_m^{T}(x)\, C_m\, P_{i,2}(y) + \left(\frac{a}{L^2}\right) y\; H_m^{T}(x)\, C_m\, R_{i,1}(y) \\
&- \left(\frac{a}{L^2}\right) P_{i,2}^{T}(x)\, C_m\, H_m(y) + \left(\frac{a}{L^2}\right) x\; R_{i,1}^{T}(x)\, C_m\, H_m(y) \\
&- \left(\frac{b}{L^2}\right) \Delta t\; H_m^{T}(x)\, C_m\, P_{i,2}(y) + \left(\frac{b}{L^2}\right) y\, \Delta t\; H_m^{T}(x)\, C_m\, R_{i,1}(y) \\
&- \left(\frac{b}{L^2}\right) \Delta t\; P_{i,2}^{T}(x)\, C_m\, H_m(y) + \left(\frac{b}{L^2}\right) x\, \Delta t\; R_{i,1}^{T}(x)\, C_m\, H_m(y) = G
\end{aligned}
\tag{39}
$$

such that :

$$
\begin{aligned}
G &= b_1(LX)\,u_1 + \cdots\cdots + b_m(LX)\,u_m \\
&- (1-x)\left[\frac{u(0,y,t) - u(0,y,t_s)}{\Delta t}\right] - (1-y)\left[\frac{u(x,0,t) - u(x,0,t_s)}{\Delta t}\right] \\
&+ (1-x-y+x\,y)\left[\frac{u(0,0,t)) - u(0,0,t_s)}{\Delta t}\right] - x\left[\frac{u(1,y,t) - u(1,y,t_s)}{\Delta t}\right] \\
&+ (x - x\,y)\left[\frac{u(1,0,t) - u(1,0,t_s)}{\Delta t}\right] - y\left[\frac{u(x,1,t) - u(x,1,t_s)}{\Delta t}\right] \\
&+ (y - x\,y)\left[\frac{u(0,1,t) - u(0,1,t_s)}{\Delta t}\right] + x\,y\left[\frac{u(1,1,t) - u(1,1,t_s)}{\Delta t}\right] \\
&+ \left(\frac{a}{L^2\,\Delta t} - \frac{a}{L^2\,\Delta t}\,y + \frac{b}{L^2} - \frac{b}{L^2}\,y\right)\left[u_{xx}(x,0,t) - u_{xx}(x,0,t_s)\right] \\
&+ \left(\frac{a}{L^2\,\Delta t} + \frac{b}{L^2}\right) y\left[u_{xx}(x,1,t) - u_{xx}(x,1,t_s)\right] + \left(\frac{b}{L^2}\right) u_{xx}(x,y,t_s) \\
&+ \left(\frac{a}{L^2\,\Delta t} - \frac{a}{L^2\,\Delta t}\,x + \frac{b}{L^2} - \frac{b}{L^2}\,x\right)\left[u_{yy}(0,y,t) - u_{yy}(0,y,t_s)\right] \\
&+ \left(\frac{a}{L^2\,\Delta t} + \frac{b}{L^2}\right) x\left[u_{yy}(1,y,t) - u_{yy}(1,y,t_s)\right] + \left(\frac{b}{L^2}\right) u_{yy}(x,y,t_s)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\; &P_{i,2}^{T}(x)\, C_m\, P_{i,2}(y) + x\,y\; R_{i,1}^{T}(x)\, C_m\, R_{i,1}(y) \\
&- x\; R_{i,1}^{T}(x)\, C_m\, P_{i,2}(y) - y\; P_{i,2}^{T}(x)\, C_m\, R_{i,1}(y) \\
&- \left(\frac{a+b\,\Delta t}{L^2}\right) H_m^{T}(x)\, C_m\, P_{i,2}(y) - \left(\frac{a+b\,\Delta t}{L^2}\right) P_{i,2}^{T}(x)\, C_m\, H_m(y) \\
&+ \left(\frac{a+b\,\Delta t}{L^2}\right) y\; H_m^{T}(x)\, C_m\, R_{i,1}(y) + \left(\frac{a+b\,\Delta t}{L^2}\right) x\; R_{i,1}^{T}(x)\, C_m\, H_m(y) = G
\end{aligned}
\tag{40}
$$

equation (40) can be written in the form :

$$
\begin{aligned}
&\left[P_{i,2}^{T}(x) - x\,R_{i,1}^{T}(x) - \left(\frac{a+b\,\Delta t}{L^2}\right) H_m^{T}(x)\right]\cdot C_m \cdot \left[P_{i,2}(y) - y\,R_{i,1}(y) - \left(\frac{a+b\,\Delta t}{L^2}\right) H_m(y)\right] \\
&- \left(\frac{a+b\,\Delta t}{L^2}\right)^2 H_m^{T}(x)\, C_m\, H_m(y) = G
\end{aligned}
\tag{41}
$$

Multiplying by $\left[P_{i,2}^{T}(x) - x\,R_{i,1}^{T}(x) - \left(\frac{a+b\,\Delta t}{L^2}\right) H_m^{T}(x)\right]^{-1}$ the right hand side and by $\left[H_m(y)\right]^{-1}$ the left hand side of each term in equation (41), we obtain :

$$
\begin{aligned}
&-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{a+b\,\Delta t}{L^{2}}\right)H_{m}^{T}(x)\right]^{-1}\cdot\left(\frac{a+b\,\Delta t}{L^{2}}\right)^{2}H_{m}^{T}(x)\cdot C_{m}\\
&+C_{m}\cdot\left[P_{i,2}(y)-y\,R_{i,1}(y)-\left(\frac{a+b\,\Delta t}{L^{2}}\right)H_{m}(y)\right]\cdot\left[H_{m}(y)\right]^{-1}\\
&-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{a+b\,\Delta t}{L^{2}}\right)H_{m}^{T}(x)\right]^{-1}\cdot G\cdot\left[H_{m}(y)\right]^{-1}=0
\end{aligned}
\qquad (42)
$$

Equation (42) is the Lyapunov matrix equation which can be solved by one of the packages [12] or by using MATLAB Language :

X=Lyap (A,B,C)

To solve the equation AX+XB+C=0

Such that the matrices A,B and C must have compatible dimensions but need not be square .

Finally, The solution of the problem is found according to (29) .

Numerical results

In this section, we have solved two-dimensional BBM equation (17) with the initial and homogeneous Dirichlet boundary conditions (18) by using the equation (42) such that [7] :

$$
\left.\begin{aligned}
&u(x,y,0)=0\\
&u(0,y,t)=u(1,y,t)=0\\
&u(x,0,t)=u(x,1,t)=0
\end{aligned}\right\}\quad t\geq 0
$$

$$
\begin{aligned}
&b_{1}(x,y)=\sin(2\pi x)\sin(2\pi y)\\
&b_{2}=b_{3}=\cdots=b_{m}=0\\
&u_{1}(t)=(1+4\,a\pi^{2}+4\,b\pi^{2})e^{t}\\
&u_{2}=u_{3}=\cdots=u_{m}=0
\end{aligned}
\qquad (31)
$$

The exact solution is given by:

$$u(x,y,t)=\sin(2\pi x)\sin(2\pi y)e^{t}$$

Numerical results of the computer are presented in table (1) where $m=32$, here. $a=1, b=1, L=1, \Delta t=0.0001, t=0.01$

Table (1) Comparison between the numerical solution and the exact solution When *m=32*.

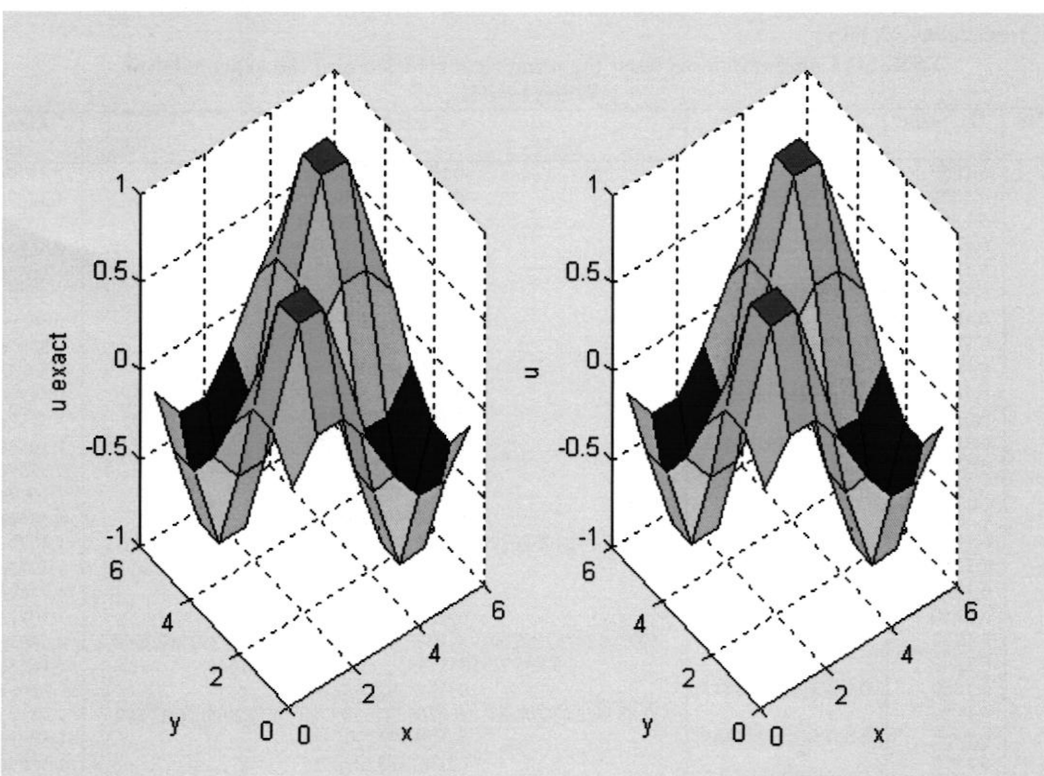
The value x	The value y	Wavelet solution	Exact solution	Absolute error
0.0982	6.1850	-	-0.00970391536960	1.4779e-007
0.2945	6.1850	0.00970376757952	-0.02873883008315	4.3769e-007
0.4909	6.1850	-	-0.04666932767354	7.1077e-007
0.6872	6.1850	0.02873839239241	-0.06280634917389	9.5654e-007
0.8836	6.1850	-	-0.07652975789769	1.1655e-006
1.0799	6.1850	0.04666861690233	-0.08731217094387	1.3298e-006
1.2763	6.1850	-	-0.09473922622590	1.4429e-006
1.4726	6.1850	0.06280539263676	-0.09852550617445	1.5005e-006
1.6690	6.1850	-	-0.09852550617445	1.5005e-006
1.8653	6.1850	0.07652859235383	-0.09473922622590	1.4429e-006
2.0617	6.1850	-	-0.08731217094387	1.3298e-006
2.2580	6.1850	0.08731084118447	-0.07652975789769	1.1655e-006
2.4544	6.1850	-	-0.06280634917389	9.5654e-007
2.6507	6.1850	0.09473778335288	-0.04666932767354	7.1077e-007
2.8471	6.1850	-	-0.02873883008315	4.3769e-007
3.0434	6.1850	0.09852400563660	-0.00970391536960 0.00970391536960	1.4779e-007
3.2398	6.1850	-	0.02873883008315	1.4779e-007
3.4361	6.1850	0.09852400563660	0.04666932767354	4.3769e-007
3.6325	6.1850	-	0.06280634917389	7.1077e-007
3.8288	6.1850	0.09473778335288	0.07652975789769 0.08731217094387 0.09473922622590	9.5654e-007
4.0252	6.1850	-	0.09852550617445 0.09852550617445	1.1655e-006
4.2215	6.1850	0.08731084118447	0.09473922622590	1.3298e-006
4.4179	6.1850	-	0.08731217094387 0.07652975789769 0.06280634917389	1.4429e-006
4.6142	6.1850	0.07652859235383	0.04666932767354	1.5005e-006
4.8106	6.1850	-	0.02873883008315	1.5005e-006
5.0069	6.1850	0.06280539263676	0.00970391536960	1.4429e-006
5.2033	6.1850	-		1.3298e-006
5.3996	6.1850	0.04666861690233		1.1655e-006
5.5960	6.1850	-		9.5654e-007
5.7923	6.1850	0.02873839239241		7.1077e-007

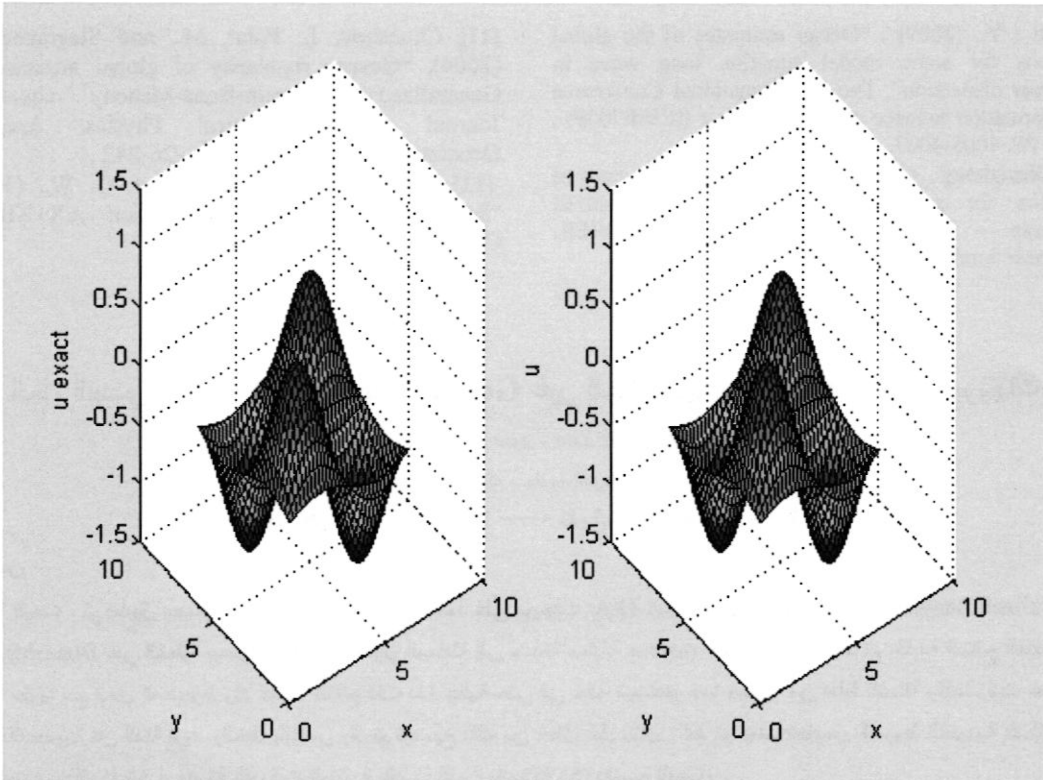
5.9887	6.1850	-		4.3769e-007
6.1850	6.1850	0.00970376757952 0.00970376757952 0.02873839239241 0.04666861690233 0.06280539263676 0.07652859235383 0.08731084118447 0.09473778335288 0.09852400563660 0.09852400563660 0.09473778335288 0.08731084118447 0.07652859235383 0.06280539263676 0.04666861690233 0.02873839239241 0.00970376757952		1.4779e-007

Table (2) the mean square error of numerical solution when

$$a=1, b=1, L=1, \Delta t=0.0001$$

m	$\lvert u_{ex} - u \rvert$
8	*1.1873e-005*
16	*3.1003e-006*
32	*7.6554e-007*
64	*1.7321e-007*
128	*2.4588e-008*





Figures (2) and (3) Comparison of the numerical solutions when *m=8* and *m=32* with the exact solution

Conclusions

In this paper, solving the nonlinear two-dimensional Generalized Benjamin-Bona-Mahony equation by using Haar wavelet method was discussed .

The fundamental idea of Haar wavelet method is to convert the differential equation into a simple Lyapunov matrix equation .

We found that Haar wavelet has a good approximation effect by comparing with the exact solution of 2D-GBBM equation at the same time. The bigger resolution J is obtained more accurately in the solution, as it is seen in table (1) when $m=32$. Also when $m=64$, $m=128$, …, we can obtain the results closer to the exact values as it is seen in table (2). Figure (2) shows the Comparison between numerical solution when $m=8$ and $m=32$ with exact solution . We also reduced the boundary conditions used in the solution by using the finite differences method with respect to time and by using the notation (6) when x=L with respect to space and the results were a high resolution. Matlab language is used to finding the results and figure drawing, it's characteristic at high accuracy and large speed .

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الحل العددي لمعادلة Generalized BBM غير الخطية ببعدين باستخدام طريقة مويجات

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الملخص

في هذا البحث ، تم تطبيق مصفوفة العوامل للتكاملات التي تعتمد على مويجات Haar لإيجاد الحل العددي لمعادلة-Generalized Benjamin Bona-Mahony غير الخطية ببعدين ، حيث تم تحويل المعادلة إلى صيغة معادلة Lyapunov matrix . وقد تم مقارنة النتائج العددية التي حصلنا عليها مع الحل المضبوط وقد كانت النتائج ذات دقة عالية حتى في حالة استخدام عدد صغير من نقاط الشبكة وكلما زادت عدد نقاط الشبكة المحسوبة فان الدقة تزداد والخطأ يتناقص وقد تم توضيح ذلك من خلال حل مثال . لقد تم أيضا تخفيض الشروط الحدودية المطلوبة في الحل العددي وذلك باستخدام طريقة الفروقات المنتهية بالنسبة للزمن والعلاقة (6) بالنسبة للفضاء .