Pseudo Quasi-Dedekind Modules

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(Received: 19 / 12 / 2012 ---- Accepted: 19 / 5 / 2013)

Abstract

In this paper we introduce the concept of pseudo quasi-Dedekind module as a generalization of quasi-Dedekind module. A unitary R-module M is called pseudo quasi-Dedekind module if $\mathrm{Hom}\left(\frac{M}{N}, M\right) = 0$ for all weak-essential submodule N of M. Many basic properties, characterizations and example of pseudo quasi-Dedekind module are given.

Introduction

Let R be a commutative ring with identity, and M be a unitary R-module. An R-module M is called **quasi-Dedekind** if $\mathrm{Hom}\left(\frac{M}{N}, M\right) = 0$ for all non-zero submodule N of M [7]. An non-zero submodule N of M is called **weak essential** if $N \cap S \neq (0)$ for each non-zero semi-prime submodule S of M [2], where a submodule S of an R-module M is called semi-prime if for each $r \in R$, and $m \in M$, with $r^k x \in N$, $k \in Z$, then $rx \in N$ [3]. A non-zero R-module M is called **essentially quasi-Dedekind** R-module if $\mathrm{Hom}\left(\frac{M}{N}, M\right) = 0$ for all non-zero essential submodule N of M [5]. An R-module M is **uniform**, if every submodule of M is essential in M. A non-zero submodule of an R-module M is called **quasi-invertible** if $\mathrm{Hom}\left(\frac{M}{N}, N\right) = 0$ [7].

Definition 1: An R-module M is called pseudo quasi-Dedekind if $\mathrm{Hom}\left(\frac{M}{N}, N\right) = 0$ for all weakly essential submodule N of M.

Examples and Remarks 2:

1- Every quasi-Dedekind R-module is pseudo quasi-Dedekind module, but not conversely, since the Z-moduleZ_{36} is pseudo quasi-Dedekind module, but not quasi-Dedekind.

2- Z as a Z-module is pseudo quasi-Dedekind module.

3- $Z_{p^\infty}$is not pseudo quasi-Dedekind module.

4- Every semi-simple R-module is pseudo quasi-Dedekind R-module.

Proof: Suppose that M is a semi-simple R-module, and let $f \in \mathrm{End}(M), f \neq 0$. Assume that kerf is weak essential submodule in M. But M is a semi-simple, then M is the only weak essential submodule of M. That is $M = \ker f$, then $f = 0$which is a contradiction. Thus kerf is not weak essential submodule of M. Hence M is pseudo quasi-Dedekind R-module.

5- If M is a pseudo quasi-Dedekind R-module, then $\frac{M}{N}$ is not necessarily pseudo quasi- Dedekind R-module, where N is a submodule of M. For example: The Z-module Z is a pseudo quasi- Dedekind, and if $N = 4Z$, then $\frac{Z}{4Z} \cong Z_4$ is not pseudo quasi- Dedekind Z-module.

The following proposition give characterization of pseudo quasi- Dedekind module.

Proposition 3: Let M be an R-module. Then M is pseudo quasi- Dedekind R-module if and only if for each $f \in \mathrm{End}(M), f \neq 0$ implies that kerf is not weak essential in M.

Proof: Suppose that M is pseudo quasi- Dedekind R-module, and let $f \in \mathrm{End}(M), f \neq 0$. Assume that kerf is weak essential in M and define $h: \frac{M}{\ker f} \to M$ by $h(m + \ker f) = f(m)\ \forall m \in M$. It is clear that h is well defined and $h \neq 0$, Hence $\mathrm{Hom}(\frac{M}{\ker f}, M) \neq 0$, which is contradict our hypothesis. Hence kerf is not weak essential in M. Conversely; assume that there exists $g: \frac{M}{N} \to M, g \neq 0$ for some weak essential submodule N of M. Now, consider the following $M \xrightarrow{\pi} \frac{M}{N} \xrightarrow{g} M$, where π is natural projection mapping, then $\varphi = g o \pi \in \mathrm{End}(M)$ and $\varphi \neq 0$, since $N \subseteq \ker\varphi$ and N is weak essential in M then by [2: Remark 1.2] $\ker\varphi$ is weak essential in M which is a contradiction.

Proposition 4: Let M be a pseudo quasi- Dedekind R-module. Let $\bar{R} = \frac{R}{I}$ where I is an ideal in R such that $I \subseteq \mathrm{ann}_R(M)$. Then M is a pseudo quasi-Dedekind $\bar{R}$-module. Where $\mathrm{ann}_R(M)$={ $r \in R$,r m=0, $m \in M$ }

Proof: Since $\mathrm{Hom}_{\bar{R}}\left(\frac{M}{N}, M\right) = \mathrm{Hom}_R(\frac{M}{N}, M)$ for all submodule N of M [6], then the result follows easily.

The following proposition shows that pseudo quasi-Dedekind is an algebraic property.

Proposition 5: Let M and M' be R-modules such that $M \cong M'$. Then M is pseudo quasi- Dedekind R-module if and only if M' is a pseudo quasi- Dedekind R-module.

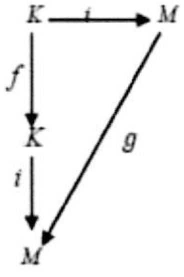
Proof:

⟹ Suppose that M is a pseudo quasi-Dedekind module, let $\varphi: M \to M'$ be an isomorphism, and let $f \in \mathrm{End}(M'), f \neq 0$. Now, $g = \varphi^{-1} o f o \varphi \in \mathrm{End}(M)$ and $g \neq 0$. Since M is pseudo quasi-Dedekind R-module, then kerg is not weak-essential in M, to prove that kerf is not weak-essential in M. We claim that $\ker f = \{y \in M' : \varphi^{-1}(y) \in \ker g\}$. To prove this, let $y \in \ker f$, then $f(y) = 0$, then $g(\varphi^{-1}(y)) = (\varphi^{-1} o f o \varphi)(\varphi^{-1}(y)) = \varphi^{-1}(f(y)) = \varphi^{-1}(0) = 0$. Then for all $y \in \ker f$, we have $\varphi^{-1}(y) \in \ker f$, so $\varphi^{-1}(\ker f) \subseteq \ker g$ not weak-essential in M which implies that $\varphi^{-1}(\ker f)$ is not weak-essential in M, so kerf is not weak essential in M'. Thus M' is a pseudo quasi- Dedekind R-module.

⟸The proof is similarly.

Proposition 6: Let M be a pseudo quasi- Dedekind R-module, and M is a quasi-injective. If N is weak-essential submodule of M such that N does not contained in any semi-prime submodule of M, then N is pseudo quasi- Dedekind .

Proof: Suppose that M is a pseudo quasi- Dedekind R-module, let $f \in End(N), f \neq 0$. Assume that kerf is weak-essential in N consider the following diagram.
Since M is a quasi-injective, then there exists $g \in End_R(M)$
such that $goi = iof$. then $g \neq 0$. Since M is a pseudo quasi-
Dedekind R-module, then kerg is not weak-essential in M. But $kerf \subseteq kerg$, then kerf is not weak-essential in M, and hence by assumption kerf is weak-essential in N, then by [2], kerf is weak-essential in M, which is a contradiction.



Thus kerf is not weak-essential in N, hence N is pseudo quasi- Dedekind R-module.
As a consequence of proposition 6 we have the following corollaries.

Corollary 7: Let M be an R-module such that M does not contained in any semi-prime submodules of $\bar{M}$. If $\bar{M}$ is pseudo quasi- Dedekind R-module, then M is a pseudo quasi- Dedekind R-module.

Corollary 8: Let M be an R-module such that M does not contained in any semi-prime submodule of E(M). If E(M) is pseudo quasi- Dedekind R-module, then M is a pseudo quasi- Dedekind.

The converse of corollary 8 is not true in general as the following example says: Z_3 as a Z-module is a pseudo quasi- Dedekind , but $E(Z_3) \cong Z_{3^\infty}$ is not pseudo quasi- Dedekind R-module.
In the following proposition, we give a condition under which the quotient module of pseudo quasi- Dedekind R-module is pseudo quasi- Dedekind R-module.

Proposition 9: Let M be an R-module such that $\frac{M}{N}$ is a projective R-module for all weak-essential submodule N of M. If M is pseudo quasi- Dedekind R-module, then $\frac{M}{N}$ is a pseudo quasi- Dedekind R-module, for all weak-essential submodule N of M.

Proof: To prove that $Hom\left(\frac{\frac{M}{N}}{\frac{U}{N}}, \frac{M}{N}\right) = 0$ for all weak-essential submodule $\frac{U}{N}$ of an R-module $\frac{M}{N}$, where U is a submodule of M. By third isomorphism theorem $\frac{\frac{M}{N}}{\frac{U}{N}} \cong \frac{U}{N}$, then it is enough to prove that $Hom\left(\frac{M}{U}, \frac{M}{N}\right) = 0$. Let $f \in Hom\left(\frac{M}{U}, \frac{M}{N}\right)$, $f \neq 0$. Since $\frac{U}{N}$ is weak-essential submodule of $\frac{M}{U}$, then U is weak-essential in M. Now, by hypothesis $\frac{M}{U}$ is a projective R-module, hence there exists $g: \frac{M}{U} \to M$ such that $\pi o g = f$, where π is natural projection map. Since $f \neq 0$, then $g \neq 0$, thus $Hom(\frac{M}{U}, M) \neq 0$, U is weak-essential in M. That is M is not pseudo quasi- Dedekind R-module, which is a contradiction. Thus $\frac{M}{N}$ is a pseudo quasi-Dedekind R-module for all weak-essential submodule of M.

Proposition 10: If M is pseudo quasi- Dedekind R-modules, then $ann_R(M) = ann_R(N)$ for all weak-essential submodule N of M.

Proof: Suppose that M is pseudo quasi- Dedekind R-module, then $Hom\left(\frac{M}{N}, M\right) = 0$ for all weak-essential submodule N of M. Hence N is quasi-invertible submodule of M, then by [1, Prop. 1.4]$ann_R(M) = ann_R(N)$.
Recall that an R-module M is multiplication if each submodule N of M, N = IM for some ideal I of R. Equivalently M is multiplication if and only if each submodule N of M is of the form $N = [N:M]M$ [4]. Where $[N:M] = \{r \in R: rm \subseteq N\}$.

Proposition 11: Let M be a uniform R-module. Then the following statements are equivalent:
1- M is a is quasi- Dedekind R-module.
2- M is pseudo quasi- Dedekind R-module.
3- M is essentially quasi- Dedekind R-module.

Proof:
$(1) \Longrightarrow (2)$: Trivial.
$(2) \Longrightarrow (3)$: Since every essential submodule is weak-essential [2] then proof follows directly.
$(3) \Longrightarrow (1)$: Trivial.
Since every monomorphism R-module is uniform [1]. We have the following.

Corollary 12: Let M be a monomorphism R-module. Then the following are equivalents:
1- M is a is quasi- Dedekind R-module.
2- M is pseudo quasi- Dedekind R-module.
3- M is an essentially quasi- Dedekind R-module.

Proposition 13: Let M be a multiplication R-module with ann(M) is a prime ideal of R. Then M ispseudo quasi- Dedekind R-module.

Proof: Assume that M is a multiplication R-module with ann(M) is a prime ideal of R, then M is faithful multiplication $\bar{R}$ -module, where $\bar{R} = \frac{R}{ann_R(M)}$. But ann(M) is a prime ideal of R, then $\bar{R}$ is an integral domain. Hence by [5] every non-zero ideal of $\bar{R}$ is a quasi-invertible, thus for any non-zero submodule N of M we have [N: M] is a quasi-invertible ideal of $\bar{R}$. Hence by [7, Cor. 2.3], [N: M] is an essential ideal of $\bar{R}$. Thus by [4, Th. 2.13], $N = [N:M]M$ is essential $\bar{R}$-submodule of M. Hence N is an essential submodule of M and so M is uniform module. But by [5, Prop. 2.1.5], M is essentially quasi- Dedekind R-module.

Hence by Prop.11 M is a pseudo quasi- Dedekind R-module.

Proposition 14: Let M be a multiplication R-module, with $End_R(M)$ is an integral domain, then M is a pseudo quasi- Dedekind R-module.

Proof: Let $f \in End_R(M), f \neq 0$. Since $End_R(M)$ is an integral domain, then f is a non-zero devisor. But M is a multiplication, then [1, lemma 2.2] f is a monomorphism. Hence M is a quasi- Dedekind R-module, which implies that M is pseudo quasi-Dedekind R-module.

Recall that an R-module M is called scalar module if, for all $f \in End_R(M)$ there exists $r \in R$ such that $f(x) = rx \ \forall x \in M$ [9].

Proposition 15: Let M be a scalar quasi-prime R-module, then M is a pseudo quasi- Dedekind R-module.

Proof: Let $f \in End_R(M), f \neq 0$. Since M is scalar R-module, then there exists a non-zero $r \in R$ such that $f(x) = rx$ for all $x \in M$. Now suppose that kerf is not weak-essential submodule of M, then for any semi-prime submodule S of M, there exists a non-zero $m \in S$ and there exists a non-zero element $l \in R$ such that $0 \neq lm \in kerf$ [2, Prop. 1.2]. But M is a quasi-prime R-module. Hence $f(lm) = 0$, that is $r(lm) = 0$, then $rl \in ann_R(M)$. But M is a quasi-prime, then $ann_R(M)$ is a prime ideal. Hence either $r \in ann_R(M)$ or $l \in ann_R(M)$. That is either $rm = 0$ or $lm = 0$. But $lm \neq 0$ for any m. Hence $f = 0$ which is a contradiction.Therefore kerf is not weak-essential of M. Hence M is a pseudo quasi-Dedekind R-module.

Proposition 16: Let M be scalar R-module, with $ann_R(M)$ is a prime ideal of R, then $End_R(M)$ is a pseudo quasi-Dedekind R-module.

Proof: Since M is a scalar R-module, then by [8, lemma. 6.2], $End_R(M) \cong \frac{R}{ann_R(M)}$. But $ann_R(M)$ is prime ideal of R, then $\frac{R}{ann_R(M)}$ is an integral domain. Therefore $End_R(M)$ is an integral domain, Hence by [5] $End_R(M)$ is a quasi-Dedekind R-module. Hence $End_R(M)$ is a pseudo quasi-Dedekind R-module.

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المقاسات شبه الديدكاتية الكاذبة

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(تاريخ الاستلام: 19 / 12 / 2012 ---- تاريخ القبول: 19 / 5 /2013)

الملخص

في هذا البحث قدمنا مفهوم المقاسات شبه الديدكاتية الكاذبة كتعميم للمقاسات شبة الديدكاتية. المقاس الاحادي على الحلقة R ، M يدعى مقاسا شبه ديدكاتيآ كاذبآ اذا كان $Hom\left(\frac{M}{N}, M\right) = 0$ لكل مقاس جزئي ضعيف جوهري N من M . العديد من الصفات الأساسية و المكافئات و الأمثلة لمفهوم المقاسات شبه الديدكاتية الكاذبة اعطيت.