A Generalized Framework for Multivariate Neutrosophic Random Vectors and Their Statistical Properties

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ABSTRACT

This paper discusses an aspect of the generalized neutrosophic concept of neutrosophic random variables, specifically for modeling uncertainty and indeterminacy in complex random systems. The study presents a precise and extended formulation of multivariable neutrosophic random vector constructs and their common structures, thereby providing a more comprehensive approach by discussing classical probability theories and reformulating them according to neutrosophic logic. Furthermore, the properties of multivariable common neutrosophic functions are investigated, and fundamental statistical measures extended according to neutrosophic logic are derived, such as the neutrosophic mathematical expectancy of multivariable neutrosophic vectors. Finally, the moment-generating function is examined from the perspective of extended neutrosophic logic and used as a powerful analytical tool for describing the linear structures of neutrosophic random variables. The aim of this study is to find a more flexible and accurate probabilistic environment for dealing with real-world phenomena where data are often unavailable, missing, or contain an aspect of indeterminacy, since neutrosophic logic relies in its study of phenomena on three aspects: the truth of the statement T, the falsity of the statement F, and the indeterminacy I.

1. Introduction

Classical probability theory assumes of precise inputs, numerical data, and clearly defined random processes. However, many phenomena in the real world around us are not governed solely by randomness but rather depend on dimensions that are more comprehensive and deeper than classical logic. The uncertainty aspect has been adopted as one of the basic dimensions in understanding the nature of the data, or the fact that part of it is missing or unknown, or the data is incomplete. These complexities are usually not able to be dealt with by classical logic and are outside the capabilities of traditional probability models. To address these cases and phenomena, neutrosophic logic emerged as a broader and more comprehensive framework than fuzzy logic. This logic introduces a third element: indeterminacy or uncertainty in understanding the nature of neutrosophic phenomena. This allows for a more precise and comprehensive approach to adapting to events, making the intuitive element influential in determining the nature of data [11]. Typically, more flexible than the classical mathematical model, neutrosophic logic has seen rapid development in many diverse fields, such as complex decision-making, machine learning, medical diagnostics, neural network analysis, communications, and social issues [9]. Its primary strength lies in its ability to quantify and simulate truth, falsehood, and indeterminacy, making it highly relevant and applicable to many systems where information is ambiguous, unobservable, or partially missing. In this context, neutrosophic probability is a broader and more comprehensive extension of classical theory, employing the TIF triad to provide a more accurate interpretation of random events [12, 2], where T represents the truth of the statement, F the falsity of the statement, and I the indeterminacy. Furthermore, several recent studies have

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successfully developed neutrosophic models of standard distributions—such as the normal distribution, Poisson, and Whipple distributions, etc. [1, 13, 10, 3]. This research focuses on the multivariate structure of neutrosophic random variables [9, 4]. It presents an in-depth study of neutrosophic random vectors and their common distribution functions. This study focuses primarily on the neutrosophic moment generation function as a unifying tool for assessing the independence of variables, marginal functions, and linear combinations [8, 5, 3, 6] to deal with uncertainties in stochastic phenomena and other issues.

2. Preliminary and basic definitions

Definition 2.1: Neutrosophic Random Vector: An n-dimensional neutrosophic random vector is defined as an ordered n-tuple of neutrosophic random variables, denoted by $\boldsymbol{X}_N = (X_{1N}, X_{2N}, \dots, X_{nN})$. The sample space of this neutrosophic random vector, denoted by $\boldsymbol{\mathcal{D}_N}$, is the set of all such n-tuples:

$$\boldsymbol{\mathcal{D}_N} = \{(X_{1N}, X_{2N}, \dots, X_{nN}) : X_{iN}(c) = x_{1_i} + I, for\ c \in \boldsymbol{C_N}, \boldsymbol{and\ \ i = 1\ \ \mathcal{TO}\ \ n}\}.$$

Furthermore, for any subset $A_N \subseteq \boldsymbol{\mathcal{D}_N}$, the neutrosophic probability is defined as

$$P[(X_{1N}, X_{2N}, \dots, X_{nN}) \in A_N] = P(\boldsymbol{C_{AN}}),$$

$$\boldsymbol{C_{AN}} = \{ c \in \boldsymbol{C_N} : X_{1N}(c), X_{2N}(c), \dots, X_{nN}(c) \in A_N\}.$$

Notation:

In this framework, we adopt the following vector notation:

I. The n-dimensional neutrosophic random vector is represented as a column vector: $\boldsymbol{X}_N = [X_{1N}, X_{2N}, \dots, X_{nN}]^{\mathcal{T}}$.

II. A specific realization or observed value of the vector is denoted by

$$x_{\boldsymbol{N}} + I = [x_1 + I, x_2 + I, \dots, x_n + I]^{\mathcal{T}}.$$

Definition 2.2: Joint Neutrosophic Cumulative Distribution Function (JN- CDF) The Joint Neutrosophic Cumulative Distribution Function, denoted by $F_{X_N}(\boldsymbol{X_N})$, for a neutrosophic random vector $\boldsymbol{X}_N = (X_{1N}, X_{2N}, \dots, X_{nN})$ refer to the probability that each component X_{iN} is does not exceed a specific neutrosophic value $x_i + I$

$$F_{X_N}(x_1, x_2, \dots, x_n) = P(X_{1N} \le x_1 + I, X_{2N} \le x_2 + I, \dots, X_{nN} \le x_n + I).$$

The n-neutrosophic random variables are classified as discrete or continuous based on the representation of their (JN- CDF):

I. Discrete Case: The distribution is defined by the joint neutrosophic probability mass function (N-PMF):

$$F_{X_N}(x_{\boldsymbol{N}}) = \sum_{w_1 \le x_1 + I, w_2 \le x_2 + I, \dots, w_n \le x_n + I} P(w_1, w_2, \dots, w_n)$$

II. Continuous Case: The distribution is defined via the joint neutrosophic probability density function (N-PDF) through the integral:

$$F_{X_N}(X_N) = \int_{-\infty}^{X_1+I}\int_{-\infty}^{X_2+I}, \dots, \int_{-\infty}^{X_n+I} f(w_1, w_2, \dots, w_n)dw_1 dw_2 \dots dw_n.$$

Definition 2.3: Neutrosophic Probability Density Function (N-PDF)

For the continuous case, where the (JN-CDF) is partially differentiable, the Joint Neutrosophic Probability Density Function is obtained as:

$$f_{X_N}(X_N) = \frac{\partial^n}{\partial X_1 . \partial X_2 \dots \partial X_n}\left[F_{X_N}(X_N)\right]$$

The function f_{X_N} is considered a valid N-PDF if it satisfies non-negativity and yields a total measure that accounts for the inherent indeterminacy of the system. Except possibly at point where the neutrosophic probability is zero or the degree of indeterminacy is dominate. The functionf_{X_N} is regarded as a legitimate neutrosophic probability density function if it fulfills the following requirements. It must take non-negative values. It should be defined over the entire domine that yields an admissible total measure under uncertainty. Similarly, in the discrete case a point function is considered a neutrosophic joint probability mass function if it is defined and neutrosophically non-negative, and if the total sum of its value represents a non-crisp probabilistic mass, capturing the effect of indeterminacy inherent in the system. The nation of support of a random vector can also extended in the neutrosophic sense. For discrete random variable, the support consists of all points possessing a non-zero neutrosophic mass. For continuous random variable, the support includes all points that can be contained within open subsets having a positive neutrosophic probability. The support set is denoted by S, an acknowledging that its boundary may itself be indeterminate.

Example .2.4: *L*et x, y *and* z be independent continuous random variables with a common exponential distribution density function is define as:

$$f_N(x.y.z) = \begin{cases} e^{-[(x+I)(y+I)+(z+I)]} & \\ 0 \quad otherwise \end{cases} \qquad -I < x, y, z < \infty$$

The Joint Neutrosophic Cumulative Distribution Function is given by:

$$F_N(x, y, z) = P(X \le x - I, Y \le y - I, Z \le z - I)$$

$$= \int_{-I}^{z-I}\int_{-I}^{y-I}\int_{-I}^{x-I} e^{-u-v-w} dudvdw \qquad -I < x, y, z < \infty$$

$$F_N(x, y, z) = \left(1 - e^{-(x-I)}\right)\left(1 - e^{-(y-I)}\right)\left(1 - e^{-(z-I)}\right)$$

Definition .2.5: Let $(X_{1N}, X_{2N}, \dots, X_{nN})$ be a neutrosophic random vector and let

$Y_N = \mathcal{U}(X_{1N}, X_{2N}, \dots, X_{nN}) = u(\boldsymbol{X}_N)$ be a real-valued function defined on the neutrosophic sample space. The neutrosophic expected value of Y_N is said to exist if the corresponding neutrosophic integral (or sum) converges in the sense of neutrosophic measurements. For the continuous neutrosophic case if $f_N(\text{X}_{1N}, \text{X}_{2N}, \dots, \text{X}_{nN})$ denotes the neutrosophic joint probability density function, then the neutrosophic expectation of Y_N is defined by:

$$E(Y_N) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathcal{U}(\boldsymbol{X}_N)\, f_N(\boldsymbol{X}_N) d\text{x}_1 d\text{x}_2 \dots d\text{x}_n$$

For the discrete neutrosophic case if $P(\boldsymbol{X}_N)$ represents the neutrosophic joint probability mass function, then

$$E(Y_N) = \sum_{\text{x}_1}\sum_{\text{x}_2} \dots \sum_{\text{x}_n} u(\text{X}_{1N}, \text{X}_{2N}, \dots, \text{X}_{nN}) P_N(\text{X}_{1N}, \text{X}_{2N}, \dots, \text{X}_{nN})$$

The linear properties of expectation for n-dimensional cases remain value neutrosophic environments.

Let $Y_{jN} = \mathcal{U}_j(\boldsymbol{X}_N)$ be a set of neutrosophic random variable, if the neutrosophic expectation $E(Y_{jN})$ exists for each j, then the linear operator is defined as: $E\left[\sum_{j=1}^{m} k_j\, \text{x}_{jN}\right] = \sum_{j=1}^{m} k_j\, E\left[\text{x}_{jN}\right]$, where $k_1, k_2 \dots k_n$ are neutrosophic constant that may contain an indeterminacy component such as: $(k = a + bI)$.

Definition 2.6: Neutrosophic marginal (PDF), consider a set of n-continuous neutrosophic random variables $\text{X}_{1N}, \text{X}_{2N}, \dots, \text{X}_{nN}$ with a neutrosophic joint PDF denoted by $f_N(\text{x}_1, \text{x}_2, \dots, \text{x}_n)$ to obtain the marginal density of single variable say x_1 we integrate over the indeterminacy -inclusive domain,

$$f_N(\text{X}_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_N(\text{x}_{1_1}, \text{x}_2, \dots, \text{x}_n) d\text{x}_2, d\text{x}_3 \dots, \text{x}_n.$$

In this framework, $f_{\text{X}_{1_N}}(\text{x}_1)$ represents the Neutrosophic marginal PDF of x_{1_N}.

Accordingly, the neutrosophic cumulative distribution function of x_{1_N} is given by

$$F_{\text{X}_{1_N}}(b) = P_{\text{X}_{1_N}}(\text{X} \le b) = \int_{-\infty}^{b} f_{\text{X}_{1_N}}(\text{x}_1) d\text{x}_1 .$$

This concept can be generalized as a subset of k variables. For instance, if $n = 6$, $k = 3$ and reselect the group x_2, x_4, x_5 the marginal PDF is derivative by integrating out the airing variables:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_N(\text{x}_2, \text{x}_3, \dots, \text{x}_6) d\text{x}_2, d\text{x}_3 \dots, \text{x}_6$$

3. The main results

Definition 3.1: The conditional probability is expanded to handle imprecise or independent data. If the marginal probability density $f_{\text{X}_{1_N}}(\text{x}_1) > 0$. The joint conditional PDF of $(\text{x}_2, \text{x}_3, \dots, \text{x}_n)$ given $\text{X}_1 = \text{x}_1$ is defined by the relation:

$$f_{\text{X}_N}(1, 2, \dots, n|1)(\text{x}_1, \text{x}_2, \dots, \text{x}_n|\text{x}_1) = \frac{f_{\text{X}_N}(\text{x}_1, \text{x}_2, \dots, \text{x}_n)}{f_{\text{X}_{1_N}}(\text{x}_1)}$$

The ratio captures the functional dependency between variables while preserving the neutrosophic uncertainty inherent in the joint distribution.

Definition 3.2: Neutrosophic Conditional Expectation

Let $\mathcal{U}(X_{1N}, X_{2N}, \dots, X_{nN})$ have a measurable neutrosophic function. The Neutrosophic Conditional Expectation of $u(\chi_{1N}, \chi_{2N}, \dots, \chi_{nN})$ given $X_{1N} = \chi_1$ is defined as:

$E\left[\left(\mathcal{U}(X_{1N}, X_{2N}, \dots, X_{nN})\middle|X_{1_N} = \chi_1\right)\right]$, provided the neutrosophic integral converges.

Accordingly, the induced neutrosophic random variable.

$E\left[\left(\mathcal{U}(X_{1N}, X_{2N}, \dots, X_{nN})\middle|\chi_{1_N}\right)\right] = h(X_{1_N})$ captures both randomness and indeterminacy.

Definition 3.3: Neutrosophic Mutual Independence

The neutrosophic random variables $\chi_{1N}, \chi_{2N}, \dots, \chi_{nN}$ are said to be Mutually Neutrosophically Independent if and only if

$f_N(\chi_1, \chi_2, \dots, \chi_n) = f_{\chi_{1_N}}(\chi_1).f_{\chi_{2_N}}(\chi_2)\dots f_{\chi_{n_N}}(\chi_n)$, where the multiplication is interpreted in the neutrosophic sense. In the discrete case, this condition reduces to $P_N(\chi_1, \chi_2, \dots, \chi_n) = P_{X_{1_N}}(\chi_1).P_{X_{2_N}}(\chi_2)\dots P_{X_{i_N}}(\chi_n)$.

Theorem 3.4: Let $X_{1N}, X_{2N}, \dots, X_{nN}$ neutrosophic random variables defined on the same neutrosophic sample space $\boldsymbol{C_N}$. These variables are said to be stochastically independent if and only if their joint neutrosophic cumulative distribution function (JN-CDF) can be factorized into the product of their respective marginal neutrosophic cumulative distribution functions. Formally:

$F_{\chi_N}(\boldsymbol{X_N}) = F_{\chi_N}(X_{1N}, X_{2N}, \dots, X_{nN}) = \prod_{i=1}^{n} F_{\chi_{iN}}(\chi_{iN})$.

For all neutrosophic real numbers $\chi_{iN} \in R(I)$.

Proof:

Assume that the neutrosophic random variables $X_{1N}, X_{2N}, \dots, X_{nN}$ are independent. By the definition of neutrosophic independence for events, the joint probability of their occurrences is the product of their individual probabilities. The joint JN-CDF is defined as:

$F_{\chi_N}(X_{1N}, X_{2N}, \dots, X_{nN}) = P(X_{1N} \le \chi_{1\mathrm{N}}, X_{2N} \le \chi_{2\mathrm{N}}, \dots, X_{nN} \le \chi_{n\mathrm{N}})$

Due to the independence property, the joint event

$\{X_{1N} \le \chi_{1\mathrm{N}} \cap X_{2N} \le \chi_{2\mathrm{N}} \cap \dots \cap X_{nN} \le \chi_{n\mathrm{N}}\}$ can be partitioned into the product of marginal probabilities:

$P(\cap_{i=1}^{n}\{X_{iN} \le \chi_{i\mathrm{N}}\}) = \prod_{i=1}^{n} P(X_{iN} \le \chi_{i\mathrm{N}})$

Since each $P(X_{iN} \le x_{i\mathrm{N}})$ is the marginal neutrosophic distribution function $F_{\chi_N}(x_{i\mathrm{N}})$ it follows that:

$F_{\chi_N}(X_{1N}, X_{2N}, \dots, X_{nN}) = F_{\chi_{1N}}(\chi_{1\mathrm{N}}).F_{\chi_{2N}}(\chi_{2\mathrm{N}})\dots F_{\chi_{nN}}(\chi_{n\mathrm{N}})$.

Definition 3.5: Neutrosophic Joint Moment Generated Function (N-JMGF) of $\mathrm{X}_{1_N}, \mathrm{X}_{2_N}, \dots, \mathrm{X}_{n_N}$ is defined as

$M_{\mathrm{X}_{n_N}}(t_1, t_2, \dots, t_n) = E\left[e^{\left(t_1 \mathrm{X}_{1_N}, t_2 \mathrm{X}_{2_N}, \dots, t_n \mathrm{X}_{n_N}\right)}\right]$, provided that the neutrosophic expectation exists for $-h_i < t_i < h_i$, $i = 1,2, \dots, n, h_i > 0$.

The Neutrosophic Marginal Moment Generated Function of single neutrosophic random variable X_{i_N}, $i = 1,2, \dots, n$ is obtained by:

$M_{\mathrm{X}_{i_N}}(t_i) = M_{\mathrm{X}_{i_N}}(0,0,0, t_i, 0,0,0)$. This formulation allows the extension of marginal neutrosophic form joint structure.

Theorem 3.6: Neutrosophic Independent via MGF of $\mathrm{X}_{1N}, \mathrm{X}_{2N}, \dots, \mathrm{X}_{nN}$ are said to be mutually Neutrosophically Independent if and only if their joint neutrosophic MGF satisfies: $M_{\mathrm{X}_{n_N}}(t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} M_{\mathrm{X}_{i_N}}(t_i)$.

Neutrosophic independence is completely charaterized by factorization condition, extending the classical probabilistic result to environments involving uncertainty and indeterminacy.

4. Conclusions

This research established a generalized framework for multivariate neutrosophic random vectors. We successfully derived the joint neutrosophic cumulative distribution function and the joint moment-generating function (NJ-MGF). The results demonstrate that this framework provides a superior ability to model stochastic systems characterized by indeterminacy, offering a more flexible alternative to classical probability model

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