False-Position Method For Estimating Reliability of Alpha Power Survival Transformation Exponential Distribution

Authors Names

Ahmed Ali Ahmed [a]

Makki A. Mohammed Salih [b]

Publication date: 19/6 /2026

Keywords: Alpha Power Survival Transformation Exponential Distribution, False Position Method, Estimation Methods ,Mean Squared Error.

ABSTRACT

Simulation was used to estimate the parameters and reliability function of the Alpha Power Survival Transformation Exponential Distribution using five estimation methods, including:

Four least squares methods, comparing their results with each other and with the maximum likelihood method by applying four parameter values to several sample sizes and repeating the sample 1000 times. The mean squared error was used to compare the obtained results, and it was found that: Estimators that use the maximum likelihood method are better than estimators in other methods at estimating the reliability of the distribution.

1. Introduction

One of the important distributions in statistical theory is (Exponential Distribution) which the probability density function (PDF) and cumulative density function (CDF) of it denoted respectively as:

$$f_e(x;\beta) = \beta e^{-\beta x}\,, \qquad x > 0\,, \beta > 0$$

$$F_e(x;\beta) = 1 - \beta e^{-\beta x}\,, \qquad x > 0\,, \beta > 0$$

and it's had many important statistical properties such as the lack of memory property [1]. Many studies and research papers have been written on this distribution to upgrade some statistical properties like the research papers by Nadarajah and Kotz in (2006) they are proposed a new distribution called beta exponential distribution hoping that will attract a wider range of applications in the field of reliability [2], after that Barreto-Souza et al. in (2010) they are proposed a new extension which called generalized exponential beta distribution and it's more flexible in analyzing positive data than the exponential beta distribution [3]. In (2011) Nadarajah and Haghighi they proposed a new extension of the exponential distribution it possesses several statistical properties and can be used as an alternative to the Weibull and gamma distributions [4]. In (2017) Elgarhy et al. they proposed a new expansion for the exponential distribution with four parameters, called exponentiated Weibull-exponential distribution [5]. One of generalizing statistical distributions is the new alpha power transformation method to find expanded statistical distribution with more flexible in representing data. In (2024), Madlool and Abdel AL-Kadim modified this method by using survival function of Exponential distribution and obtained a new statistical distribution called Alpha Power Survival Transformation Exponential distribution (APSTE), and this distribution will be the subject of study in this research paper, which have the following probability density function PDF, the cumulative density function CDF, reliability, and quantile respectively [6]:

[a]Department of Mathematics, College of Education, Mustansiriyah University-Baghdad-Iraq, E-Mail: ahmed.ali.ahmed.alalawy@gmail.com

[b]Department of Mathematics, College of Education, Mustansiriyah University-Baghdad-Iraq., E-Mail: dr_makki@uomustansiriyah.edu.iq

$$f_{APSTE}(x;\alpha,\beta)=\begin{cases}\frac{-ln\,(\alpha)\,\beta e^{-\beta x}\alpha^{e^{-\beta x}}}{1-\alpha}, & \alpha>0, \alpha\neq 1, \alpha\in R^{+}\\ \beta e^{-\beta x} & , \alpha=1\end{cases} \quad (1)$$

$$F_{APSTE}(x;\alpha,\beta)=\begin{cases}\frac{\alpha^{e^{-\beta x}}-\alpha}{(1-\alpha)}, & \alpha>0, \alpha\neq 1, \alpha\in R^{+}\\ e^{-\beta x} & , \alpha=1\end{cases} \quad (2)$$

$$R_{APSTE}(x;\alpha,\beta)=\begin{cases}\frac{1-\alpha^{e^{-\beta x}}}{1-\alpha}, & \alpha>0, \alpha\neq 1, \alpha\in R^{+}\\ 1-e^{-\beta x} & , \alpha=1\end{cases} \quad (3)$$

$$x_q=-\frac{ln\left(\frac{ln(q(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{\beta}\ ,0<q<1 \quad (4)$$

Where $x>0$ is a random variable and $\beta>0$ is a parameter.

Estimating distribution parameters, in addition to estimating the reliability function, is an important topic that has attracted the attention of many researchers, both past and present, due to its significance in the field of system failure probability or success, such as in systems engineering, industrial engineering, medicine, and other sciences. Therefore, this research focuses on estimating the reliability of the Alpha Power Survival Transformation Exponential distribution, as there is no previous study on the reliability of this distribution. Simulation was used to generate the data required for the estimation through three experiments. Parameter values were taken to be equal, and in two cases, one parameter was greater than the other. Data were generated that were characterized as small, medium, and large. Four methods were applied to achieve the research objective. We will discuss the methods used in the estimation in subsequent sections.

2. Estimation Methods

2.1. Initial Estimates of The Parameters

Since estimation methods rely on numerical methods to find estimates, and since these methods require initial values for the parameters, we will derive formulas for the initial values of the parameters using the median of distribution, as follows:

To find the initial value of parameters (α,β) , putting $(q=0.5)$ in equation (4) then getting: [7]

$$x_{med}=-\frac{ln\left(\frac{ln(0.5(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{\beta} \quad (5)$$

can be get the initial formulas of β as:

$$\hat{\beta}_0=-\frac{ln\left(\frac{ln(0.5(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{x_{med}} \quad (6)$$

From the equation (5) can be get the following:

$$-\beta x_{med}=ln\left(\frac{ln(0.5(1-\alpha)+\alpha)}{ln(\alpha)}\right) \quad (7)$$

Taking the exponential for two sides of the equation (7) getting on:

$$e^{-\beta x_{med}} = \frac{ln(0.5(1-\alpha)+\alpha)}{ln(\alpha)}$$

That's implies:

$$ln(\alpha) = \frac{ln(0.5(1-\alpha)+\alpha)}{e^{-\beta x_{med}}} \quad (8)$$

Taking the exponential for two sides of equation (8) getting:

$$e^{ln(\alpha)} = e^{\left(\frac{ln(0.5(1-\alpha)+\alpha)}{e^{-\beta x_{med}}}\right)}$$

That's implies:

$$\hat{\alpha}_0 = e^{\left(\frac{ln(0.5(1-\alpha)+\alpha)}{e^{-\beta x_{med}}}\right)} \quad (9)$$

The initial value of (α and β) are $\left(\hat{\alpha}_0 \; and \; \hat{\beta}_0\right)$ in equations (9) and (6) respectively.

2.2. False Position Method

One method of numerically solving nonlinear equations as a form $h(x) = 0$, is the false position method. This method depending on the interval $[x_1, x_2]$ which containing the root x, and can getting it by using the following form: [8]

$$x_{j+1} = x_j - \frac{h(x_j)(x_j - x_{j-1})}{h(x_j) - h(x_{j-1})}, j = 2,3, \ldots \quad (10)$$

In this section, this method can be used to find the value of the estimator in a nonlinear equation. The estimators value $(\hat{\alpha}, \hat{\beta})$ will be contained in intervals that can be defined by taking some neighbors for the estimated initial values (α_0, β_0) as the following:

$$\hat{\alpha} \in [\hat{\alpha}_0 - \varepsilon, \hat{\alpha}_0 + \varepsilon], \text{ and } \hat{\beta} \in [\hat{\beta}_0 - \varepsilon, \hat{\beta}_0 + \varepsilon] \text{ where } 0 < \varepsilon < 1$$

therefore $\left(\hat{\alpha} \; and \; \hat{\beta}\right)$ can be evaluated by the false position method from equation (10) as the form:

$$\hat{\alpha} = (\hat{\alpha}_0 + \varepsilon) - \frac{h(\hat{\alpha}_0+\varepsilon)((\hat{\alpha}_0+\varepsilon)-(\hat{\alpha}_0-\varepsilon))}{h(\hat{\alpha}_0+\varepsilon)-h(\hat{\alpha}_0-\varepsilon)} \quad (11)$$

$$\hat{\beta} = \left(\hat{\beta}_0 + \varepsilon\right) - \frac{h\left(\hat{\beta}_0+\varepsilon\right)\left((\hat{\beta}_0+\varepsilon)-(\hat{\beta}_0-\varepsilon)\right)}{h\left(\hat{\beta}_0+\varepsilon\right)-h\left(\hat{\beta}_0-\varepsilon\right)} \quad (12)$$

2.3. Least Square Method Type 1 (LS1)

Minimizing the value of quantile function is the basic idea of this method, which depending on making the quantile function to the general form of linear regression as follows: [9]

Take the plotting position formula as:

$$p_i = \frac{i}{k+1}, i = 1,2, \ldots, k \quad \text{, where k is the size of sample} \quad (13)$$

From equation (4) we have:

$$x_q = -\frac{ln\left(\frac{ln(q(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{\beta} \quad ,0 < q < 1$$

Replacing q from equation (4) by (p_i) of equation (13)

$$x_{(p_i)} = -\frac{ln\left(\frac{ln((p_i)(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{\beta} \tag{14}$$

$$-\beta x_{(p_i)} = ln\left(\frac{ln((p_i)(1-\alpha)+\alpha)}{ln(\alpha)}\right)$$

$$-\beta x_{(p_i)} = ln(ln((p_i)(1-\alpha) + \alpha)) - ln(ln(\alpha))$$

$$ln(ln((p_i)(1-\alpha) + \alpha)) = ln(ln(\alpha)) - \beta x_{(p_i)} \tag{15}$$

The linear regressions formula is:

$$\gamma_i = \delta + \mu\varphi_i + \varepsilon_i \tag{16}$$

By comparing between equations (15) and (16), getting:

$$\gamma_i = ln(ln((p_i)(1-\alpha) + \alpha)) \tag{17}$$

$$\delta = ln(ln(\alpha)) \quad \text{implies} \quad \alpha = e^{e^{\delta}} \tag{18}$$

$$\mu = \beta \tag{19}$$

$$\varphi_i = -x_{(p_i)} \tag{20}$$

From equation (16) getting:

$$\varepsilon_i = \gamma_i - \delta - \mu\varphi_i \tag{21}$$

Taking the sum for equation (21) after squaring both sides

$$\sum_{i=1}^{k} \varepsilon_i^2 = \sum_{i=1}^{k}(\gamma_i - \delta - \mu\varphi_i)^2 \tag{22}$$

Let $$\omega = \sum_{i=1}^{k} \varepsilon_i^2 \tag{23}$$

Substituting equation (23) in equation (22) getting:

$$\omega(\delta, \mu) = \sum_{i=1}^{k}(\gamma_i - \delta - \mu\varphi_i)^2 \tag{24}$$

Taking the partial derivative for equation (24) respect to δ, then equaling to zero, getting:

$$\frac{\partial\omega}{\partial\delta} = 2\sum_{i=1}^{k}(\gamma_i - \delta - \mu\varphi_i)\,(1)$$

$$\sum_{i=1}^{k}(\gamma_i - \delta - \mu\varphi_i) = 0$$

Implies $$\sum_{i=1}^{k}\gamma_i = k\delta + \mu\sum_{i=1}^{k}\varphi_i \tag{25}$$

Also taking the partial derivative for equation (23) respect to μ, then equaling to zero, getting:

$$\frac{\partial\omega}{\partial\mu} = 2\sum_{i=1}^{k}(\gamma_i - \delta - \mu\varphi_i)(-\varphi_i)$$

$$\sum_{i=1}^{k}(\gamma_i - \delta - \mu\varphi_i)(-\varphi_i) = 0$$

Implies $$\sum_{i=1}^{k}\gamma_i\varphi_i = \delta\sum_{i=1}^{k}\varphi_i + \sum_{i=1}^{k}\mu\varphi_i \quad (26)$$

Right multiplication equation (25) by $\sum_{i=1}^{k}\varphi_i$ and left multiplication equation (26) by k respectively as:

$$\sum_{i=1}^{k}\gamma_i \sum_{i=1}^{k}\varphi_i = k\delta\sum_{i=1}^{k}\varphi_i + \mu\left(\sum_{i=1}^{k}\varphi_i\right)^2 \quad (27)$$

$$k\sum_{i=1}^{k}\gamma_i\varphi_i = k\delta\sum_{i=1}^{k}\varphi_i + k\sum_{i=1}^{k}\mu\varphi_i \quad (28)$$

By subtracting equation (28) form (27), will get:

$$\sum_{i=1}^{k}\gamma_i \sum_{i=1}^{k}\varphi_i - k\sum_{i=1}^{k}\gamma_i\varphi_i = \mu\left(\left(\sum_{i=1}^{k}\varphi_i\right)^2 - k\sum_{i=1}^{k}\mu\varphi_i\right)$$

$$\hat{\mu}_{LS1} = \frac{\sum_{i=1}^{k}\gamma_i \sum_{i=1}^{k}\varphi_i - k\sum_{i=1}^{k}\gamma_i\varphi_i}{\left(\sum_{i=1}^{k}\varphi_i\right)^2 - k\sum_{i=1}^{k}\mu\varphi_i} \quad (29)$$

From equation (25), getting:

$$\sum_{i=1}^{k}\gamma_i - \mu\sum_{i=1}^{k}\varphi_i = k\delta$$

Implies $$\hat{\delta}_{LS1} = \frac{\sum_{i=1}^{k}\gamma_i - \hat{\mu}_{LS1}\sum_{i=1}^{k}\varphi_i}{k} \quad (30)$$

By substituting value of $\hat{\mu}_{LS1}$ from equation (29) into equation (19) getting on:

$$\hat{\beta}_{LS1} = \frac{\sum_{i=1}^{k}\gamma_i \sum_{i=1}^{k}\varphi_i - k\sum_{i=1}^{k}\gamma_i\varphi_i}{\left(\sum_{i=1}^{k}\varphi_i\right)^2 - k\sum_{i=1}^{k}\mu\varphi_i} \quad (31)$$

Where the values of γ_i and φ_i were defined in equations (17) and (20) respectively.

Also substituting value of $\hat{\delta}_{LS1}$ from equation (30) into equation (18) respectively will get on

$$\hat{\alpha}_{LS1} = e^{e^{\hat{\delta}_{LS1}}} \quad (32)$$

By substituting equations (31) and (32) in equation (3) getting the estimation of reliability function as:

$$\hat{R}_{LS1}(x) = \frac{1-(\hat{\alpha}_{LS1})^{e^{-(\hat{\beta}_{LS1})x}}}{1-\hat{\alpha}_{LS1}} \quad (33)$$

2.4. Weighted Least Square Method (WLS)

In (1988) Swain, Venkatraman and Wilson suggested this method [7] using the following technique:[10]

From equation (21) we have $\varepsilon_i = \gamma_i - \delta - \mu\varphi_i$

Multiplying both sides of the equation (21) by $\frac{1}{\gamma_i}$, then squaring both sides and taking the sum getting:

$$\sum_{i=1}^{k}(\frac{\varepsilon_i}{\gamma_i})^2 = \sum_{i=1}^{k}\left(1-\frac{\delta}{\gamma_i}-\frac{\mu\varphi_i}{\gamma_i}\right)^2 \tag{34}$$

Let
$$w = \sum_{i=1}^{k}(\frac{\varepsilon_i}{\gamma_i})^2 \tag{35}$$

$$t_i = \frac{1}{\gamma_i} \tag{36}$$

$$\zeta_i = \frac{\varphi_i}{\gamma_i} \tag{37}$$

By substituting the equations (35), (36) and (37) in equation (34) getting:

$$w = \sum_{i=1}^{k}(1 - t_i\delta - \mu\zeta_i)^2 \tag{38}$$

Taking the partial derivative for equation (38) respect to δ then equaling to zero, getting:

$$\frac{\partial w}{\partial \delta} = 2\sum_{i=1}^{k}(1 - t_i\delta - \mu\zeta_i)\,(-t_i) = 0$$

$$\sum_{i=1}^{k}(t_i + t_i^2\delta + t_i\mu\zeta_i) = 0$$

$$\delta\sum_{i=1}^{k} t_i^2 = \sum_{i=1}^{k} t_i - \mu\sum_{i=1}^{k} t_i\zeta_i$$

Implies
$$\delta = \frac{\sum_{i=1}^{k} t_i - \mu\sum_{i=1}^{k} t_i\zeta_i}{\sum_{i=1}^{k} t_i^2} \tag{39}$$

Taking the partial derivative for equation (38) respect to μ then equaling to zero, getting:

$$\frac{\partial w}{\partial \mu} = 2\sum_{i=1}^{k}(1 - t_i\delta - \mu\zeta_i)\,(-\zeta_i) = 0$$

$$\sum_{i=1}^{k}(-\zeta_i + \zeta_i t_i\delta + \mu\zeta_i^2) = 0$$

$$-\sum_{i=1}^{k}\zeta_i + \delta\sum_{i=1}^{k}\zeta_i t_i + \mu\sum_{i=1}^{k}\zeta_i^2 = 0$$

$$\delta\sum_{i=1}^{k}\zeta_i t_i = \sum_{i=1}^{k}\zeta_i - \mu\sum_{i=1}^{k}\zeta_i^2$$

$$\delta = \frac{\sum_{i=1}^{k}\zeta_i - \mu\sum_{i=1}^{k}\zeta_i^2}{\sum_{i=1}^{k}\zeta_i t_i} \tag{40}$$

Since the equation (39) equal to equation (40) that's implies:

$$\frac{\sum_{i=1}^{k} t_i - \mu\sum_{i=1}^{k} t_i\zeta_i}{\sum_{i=1}^{k} t_i^2} = \frac{\sum_{i=1}^{k}\zeta_i - \mu\sum_{i=1}^{k}\zeta_i^2}{\sum_{i=1}^{k}\zeta_i t_i}$$

$$\sum_{i=1}^{k}\zeta_i t_i\sum_{i=1}^{k} t_i - \mu\left(\sum_{i=1}^{k} t_i\zeta_i\right)^2 = \sum_{i=1}^{k} t_i^2\sum_{i=1}^{k}\zeta_i - \mu\sum_{i=1}^{k} t_i^2\sum_{i=1}^{k}\zeta_i^2$$

$$\mu\sum_{i=1}^{k} t_i^2\sum_{i=1}^{k}\zeta_i^2 - \mu\left(\sum_{i=1}^{k} t_i\zeta_i\right)^2 = \sum_{i=1}^{k} t_i^2\sum_{i=1}^{k}\zeta_i - \sum_{i=1}^{k}\zeta_i t_i\sum_{i=1}^{k} t_i$$

Implies
$$\hat{\mu}_{WLS} = \frac{\sum_{i=1}^{k} t_i^2 \sum_{i=1}^{k} \zeta_i - \sum_{i=1}^{k} \zeta_i t_i \sum_{i=1}^{k} t_i}{\sum_{i=1}^{k} t_i^2 \sum_{i=1}^{k} \zeta_i^2 - \left(\sum_{i=1}^{k} t_i \zeta_i\right)^2} \quad (41)$$

By substituting equation (41) in equation (39) getting:

$$\hat{\delta}_{WLS} = \frac{\sum_{i=1}^{k} t_i - (\hat{\mu}_{WLS}) \sum_{i=1}^{k} t_i \zeta_i}{\sum_{i=1}^{k} t_i^2} \quad (42)$$

By substituting the value of $\hat{\mu}_{WLS}$ from equation (41) into equation (19), then

$$\hat{\beta}_{WLS} = \frac{\sum_{i=1}^{k} t_i^2 \sum_{i=1}^{k} \zeta_i - \sum_{i=1}^{k} \zeta_i t_i \sum_{i=1}^{k} t_i}{\sum_{i=1}^{k} t_i^2 \sum_{i=1}^{k} \zeta_i^2 - \left(\sum_{i=1}^{k} t_i \zeta_i\right)^2} \quad (43)$$

Where the values of t_i and ζ_i were defined in equations (36) and (37) respectively.

Also substituting value of $\hat{\delta}_{WLS}$ from equation (42) into equation (18) respectively getting on:

$$\hat{\alpha}_{WLS} = e^{e^{\hat{\delta}_{WLS}}} \quad (44)$$

By substituting equations (61) and (62) in equation (3) getting the estimation of reliability function as:

$$\hat{R}_{WLS}(x) = \frac{1 - (\hat{\alpha}_{WLS})^{e^{-(\hat{\beta}_{WLS})x}}}{1 - \hat{\alpha}_{WLS}} \quad (45)$$

2.5. Least Square Method Type 2 (LS2)

In (1983) Kinderman and Lariccia suggested this method which depend on Based on the inverse distribution equation and its partial derivative with respect to the two parameters, and then using Taylor's series as follows: [11]

From equation (14) we have $x_{(p_i)} = -\frac{ln\left(\frac{ln\left((p_i)(1-\alpha) + \alpha\right)}{ln(\alpha)}\right)}{\beta}, i = 1,2,3, \dots, k$

Taking the partial derivatives for equation (14) with respect to (α) as the following:

$$\frac{\partial x_{(p_i)}}{\partial \alpha} = -\frac{1}{\beta} \frac{\left(\frac{ln(\alpha)\left(\frac{-p_i+1}{p_i(1-\alpha)+\alpha}\right) - (ln(p_i(1-\alpha)+\alpha))\frac{1}{\alpha}}{(ln\,\alpha)^2}\right)}{\left(\frac{ln(p_i(1-\alpha)+\alpha)}{ln(\alpha)}\right)} = -\frac{1}{\beta} \frac{ln(\alpha)\left(\frac{-p_i+1}{p_i(1-\alpha)+\alpha}\right) - (ln(p_i(1-\alpha)+\alpha))\frac{1}{\alpha}}{(ln\,\alpha)^2 \left(\frac{ln(p_i(1-\alpha)+\alpha)}{ln(\alpha)}\right)}$$

$$\frac{\partial x_{(p_i)}}{\partial \alpha} = -\frac{ln(\alpha)\left(\frac{-p_i+1}{p_i(1-\alpha)+\alpha}\right) - \left(\frac{ln(p_i(1-\alpha)+\alpha)}{\alpha}\right)}{\beta\, ln(\alpha)\, ln(p_i(1-\alpha)+\alpha)} \quad (46)$$

Taking the partial derivatives for equation (17) with respect to (α) as the following:

$$\frac{\partial x_{(p_i)}}{\partial \beta} = -\frac{\beta(0) - ln\left(\frac{ln(p_i(1-\alpha)+\alpha)}{ln(\alpha)}\right)(1)}{\beta^2} = -\frac{-ln\left(\frac{ln(p_i(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{\beta^2}$$

$$\frac{\partial x_{(p_i)}}{\partial \beta} = \frac{ln\left(\frac{ln(p_i(1-\alpha)+\alpha)}{ln(\alpha)}\right)}{\beta^2} \quad (47)$$

Now, the two parameters are estimated using the first-order Taylor series formula, as follows:

$$g(t+r) = g(t) + r_1 g_1'(t) + r_2 g_2'(t) \quad (48)$$

Where

$$\left.\begin{aligned} r_1 = \beta - \hat{\beta}_0\,,\ \ r_2 = \alpha - \hat{\alpha}_0 \\ g(t+r) = E\left(x_{(p_i)}\right) \\ g(t) = -\frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0} \\ g_1'(t) = \frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0^2} \\ g_2'(t) = -\frac{ln(\hat{\alpha}_0)\left(\frac{-p_i+1}{p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0}\right)-\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{\hat{\alpha}_0}\right)}{\hat{\beta}_0\, ln(\hat{\alpha}_0)\, ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)} \end{aligned}\right\} \quad (49)$$

If substitute equation (49) in equation (48) then will get on:

$$E\left(x_{(p_i)}\right) = -\frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0} + \left(\beta - \hat{\beta}_0\right)\frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0^2}$$

$$+(\alpha - \alpha_0)\left(-\frac{ln(\hat{\alpha}_0)\left(\frac{-p_i+1}{p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0}\right)-\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{\hat{\alpha}_0}\right)}{\hat{\beta}_0\, ln(\hat{\alpha}_0)\, ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}\right)$$

Let,
$$M = \begin{pmatrix} -\frac{ln\left(\frac{ln(p_1(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0} & \frac{ln\left(\frac{ln(p_1(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0^2} & -\frac{ln(\hat{\alpha}_0)\left(\frac{-p_1+1}{p_1(1-\hat{\alpha}_0)+\hat{\alpha}_0}\right)-\left(\frac{ln(p_1(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{\hat{\alpha}_0}\right)}{\hat{\beta}_0\, ln(\hat{\alpha}_0)\, ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)} \\ \vdots & \vdots & \vdots \\ -\frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0} & \frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0^2} & -\frac{ln(\hat{\alpha}_0)\left(\frac{-p_i+1}{p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0}\right)-\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{\hat{\alpha}_0}\right)}{\hat{\beta}_0\, ln(\hat{\alpha}_0)\, ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)} \end{pmatrix} \quad (50)$$

Let
$$\psi(p_i) = -\frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0} \quad (51)$$

$$\eta(p_i) = \frac{ln\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{ln(\hat{\alpha}_0)}\right)}{\hat{\beta}_0^2} \quad (52)$$

$$\Omega(p_i) = -\frac{ln(\hat{\alpha}_0)\left(\frac{-p_i+1}{p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0}\right)-\left(\frac{ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)}{\hat{\alpha}_0}\right)}{\hat{\beta}_0\, ln(\hat{\alpha}_0)\, ln(p_i(1-\hat{\alpha}_0)+\hat{\alpha}_0)} \quad (53)$$

Substituting equations (51), (52) and (53) in equation (50) getting:

$$M = \begin{pmatrix} \psi(p_1) & \eta(p_1) & \Omega(p_1) \\ \vdots & \vdots & \vdots \\ \psi(p_i) & \eta(p_i) & \Omega(p_i) \end{pmatrix} \quad (54)$$

To estimate the parameters, use the following formula:

$$S^T = (M^T M)^{-1} M^T X_i \tag{55}$$

Where $$S^T = \begin{pmatrix} g(t) \\ \beta - \hat{\beta}_0 \\ \alpha - \alpha_0 \end{pmatrix} \text{ and } X_i = \begin{pmatrix} x_{(1)} \\ \vdots \\ x_{(n)} \end{pmatrix} \tag{56}$$

Substituting equations (56) and (54) in equation (55) getting:

$$\begin{pmatrix} g(t) \\ \beta - \beta_0 \\ \alpha - \alpha_0 \end{pmatrix} = \left[\begin{pmatrix} \psi(p_1) & \dots & \psi(p_i) \\ \eta(p_1) & \dots & \eta(p_i) \\ \Omega(p_1) & \dots & \Omega(p_i) \end{pmatrix} \begin{pmatrix} \psi(p_1) & \eta(p_1) & \Omega(p_1) \\ \vdots & \vdots & \vdots \\ \psi(p_i) & \eta(p_i) & \Omega(p_i) \end{pmatrix} \right]^{-1} \begin{pmatrix} \psi(p_1) & \dots & \psi(p_i) \\ \eta(p_1) & \dots & \eta(p_i) \\ \Omega(p_1) & \dots & \Omega(p_i) \end{pmatrix} \begin{pmatrix} x_{(1)} \\ \vdots \\ x_{(n)} \end{pmatrix}$$

Then getting on:

$$\begin{pmatrix} g(t) \\ \beta - \hat{\beta}_0 \\ \alpha - \hat{\alpha}_0 \end{pmatrix} = \begin{pmatrix} S_1(p) \\ S_2(p) \\ S_3(p) \end{pmatrix} \tag{57}$$

From equation (57) can get the following:

$$\beta - \hat{\beta}_0 = S_2(p)$$

Implies $$\hat{\beta}_{LS2} = S_2(p) + \hat{\beta}_0 \tag{58}$$

Also $$\alpha - \hat{\alpha}_0 = S_3(p)$$

Implies $$\hat{\alpha}_{LS2} = S_3(p) + \hat{\alpha}_0 \tag{59}$$

By substituting equations (58) and (59) in equation (3) getting the estimation of reliability function as:

$$\hat{R}_{LS2}(x) = \frac{1 - (\hat{\alpha}_{LS2})^{e^{-(\hat{\beta}_{LS2})x}}}{1 - \hat{\alpha}_{LS2}} \tag{60}$$

2.6. Ordinary Least Square Method (OLS)

This method relies on minimizing the difference between the distribution function and its default value by taking its partial derivatives for the two parameters, as follows: [12]

Let, $$U = \sum_{i=1}^{k} \left(F\left(x_{(i)}\right) - \frac{i}{1+k} \right)^2 , \; i = 1,2, \dots, k \tag{61}$$

By substituting equation (2) into equation (61) getting:

$$U = \sum_{i=1}^{k} \left(\frac{\alpha^{e^{-\beta x_{(i)}}} - \alpha}{(1-\alpha)} - \frac{i}{1+k} \right)^2 \tag{62}$$

Taking the partial derivative for equation (63) respect to (α) then equaling to zero, getting:

$$\frac{\partial U}{\partial \alpha} = 2\sum_{i=1}^{k}\left(\frac{\alpha^{e^{-\beta x_{(i)}}}-\alpha}{(1-\alpha)}-\frac{i}{1+k}\right)\left(\frac{(1-\alpha)\left(e^{-\beta x_{(i)}}\,\alpha^{\left(e^{-\beta x_{(i)}}-1\right)}-1\right)-\left(\alpha^{e^{-\beta x_{(i)}}}-\alpha\right)(-1)}{(1-\alpha)^2}\right)$$

$$\frac{\partial U}{\partial \alpha} = 2\sum_{i=1}^{k}\left(\frac{\alpha^{e^{-\beta x_{(i)}}}-\alpha}{(1-\alpha)}-\frac{i}{1+k}\right)\left(\frac{(1-\alpha)\left(e^{-\beta x_{(i)}}\,\alpha^{\left(e^{-\beta x_{(i)}}-1\right)}-1\right)+\left(\alpha^{e^{-\beta x_{(i)}}}-\alpha\right)}{(1-\alpha)^2}\right)=0$$

Let, $h_1(\alpha)=\frac{\partial U}{\partial a}=0$ that's implies

$$h_1(\alpha)=\sum_{i=1}^{k}\frac{\left(\frac{\alpha^{e^{-\beta x_{(i)}}}-\alpha}{(1-\alpha)}-\frac{i}{1+k}\right)\left((1-\alpha)\left(e^{-\beta x_{(i)}}\,\alpha^{\left(e^{-\beta x_{(i)}}-1\right)}-1\right)+\alpha^{e^{-\beta x_{(i)}}}-\alpha\right)}{(1-\alpha)^2}=0 \quad (63)$$

To get the numerically solution for equation (63) by false position method, take the two intervals

$$[\hat{\alpha}_0-\varepsilon,\hat{\alpha}_0+\varepsilon],\left[\hat{\beta}_0-\varepsilon,\hat{\beta}_0+\varepsilon\right]$$

Then:

$$h_1(\hat{\alpha}_0-\varepsilon)=\sum_{i=1}^{k}\frac{\left(\frac{(\hat{\alpha}_0-\varepsilon)^{e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}}-(\hat{\alpha}_0-\varepsilon)}{(1-(\hat{\alpha}_0-\varepsilon))}-\frac{i}{1+k}\right)\left(\begin{array}{c}(1-(\hat{\alpha}_0-\varepsilon))\left(e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}\,(\hat{\alpha}_0-\varepsilon)^{\left(e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}-1\right)}-1\right)\\+(\hat{\alpha}_0-\varepsilon)^{e^{-(\beta_0-\varepsilon)x_{(i)}}}-(\hat{\alpha}_0-\varepsilon)\end{array}\right)}{(1-(\hat{\alpha}_0-\varepsilon))^2}=0$$

$$h_1(\hat{\alpha}_0+\varepsilon)=\sum_{i=1}^{k}\frac{\left(\frac{(\hat{\alpha}_0+\varepsilon)^{e^{-(\hat{\beta}_0+\varepsilon)\,x_{(i)}}}-(\hat{\alpha}_0+\varepsilon)}{(1-(\hat{\alpha}_0+\varepsilon))}-\frac{i}{1+k}\right)\left(\begin{array}{c}(1-(\hat{\alpha}_0+\varepsilon))\left(e^{-(\hat{\beta}_0+\varepsilon)x_{(i)}}\,(\hat{\alpha}_0+\varepsilon)^{\left(e^{-(\hat{\beta}_0+\varepsilon)x_{(i)}}-1\right)}-1\right)\\+(\hat{\alpha}_0+\varepsilon)^{e^{-(\hat{\beta}_0+\varepsilon)x_{(i)}}}-(\hat{\alpha}_0+\varepsilon)\end{array}\right)}{(1-(\hat{\alpha}_0+\varepsilon))^2}=0$$

The value of $\hat{\alpha}$ can be obtained by using the false position method from equation (14) as a form:

$$\hat{\alpha}_{OLS}=(\hat{\alpha}_0+\varepsilon)-\frac{h_1(\hat{\alpha}_0+\varepsilon)((\hat{\alpha}_0+\varepsilon)-(\hat{\alpha}_0-\varepsilon))}{h_1(\hat{\alpha}_0+\varepsilon)-h_1(\hat{\alpha}_0-\varepsilon)} \quad (64)$$

Taking the partial derivative for equation (62) respect to (β) then equaling to zero, getting:

$$\frac{\partial U}{\partial \beta}=2\sum_{i=1}^{k}\left(\frac{\alpha^{e^{-\beta x_{(i)}}}-\alpha}{1-\alpha}-\frac{i}{1+k}\right)\frac{\left(\alpha^{e^{-\beta x_{(i)}}}\,e^{-\beta x_{(i)}}\,(-1)x_{(i)}\ln(\alpha)\right)}{(1-\alpha)}$$

$$2\sum_{i=1}^{k}\left(\frac{\alpha^{e^{-\beta x_{(i)}}}-\alpha}{1-\alpha}-\frac{i}{1+k}\right)\frac{\left(-x_{(i)}\alpha^{e^{-\beta x_{(i)}}}e^{-\beta x_{(i)}}\ln(\alpha)\right)}{(1-\alpha)}=0$$

Let $h_2(\beta)=\frac{\partial U}{\partial \beta}=0$, that's implies

$$h_2(\beta)=\sum_{i=1}^{k}\frac{\left(\frac{\alpha^{e^{-\beta x_{(i)}}}-\alpha}{1-\alpha}-\frac{i}{1+k}\right)\left(-x_{(i)}\alpha^{e^{-\beta x_{(i)}}}e^{-\beta x_{(i)}}\ln(\alpha)\right)}{(1-\alpha)}=0 \quad (65)$$

To get the numerically solution for equation (65) by false position method, take the two intervals

$$[\hat{\alpha}_0-\varepsilon,\hat{\alpha}_0+\varepsilon],[\hat{\beta}_0-\varepsilon,\hat{\beta}_0+\varepsilon]$$

$$h_2(\hat{\beta}_0-\varepsilon)=\sum_{i=1}^{k}\frac{\left(\frac{(\hat{\alpha}_0-\varepsilon)^{e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}}-(\hat{\alpha}_0-\varepsilon)}{(1-(\hat{\alpha}_0-\varepsilon))}-\frac{i}{1+k}\right)\left(-x_{(i)}(\hat{\alpha}_0-\varepsilon)^{e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}}e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}\ln(\hat{\alpha}_0-\varepsilon)\right)}{(1-(\hat{\alpha}_0-\varepsilon))}=0$$

$$h_2(\hat{\beta}_0+\varepsilon)=\sum_{i=1}^{k}\frac{\left(\frac{(\hat{\alpha}_0+\varepsilon)^{e^{-(\hat{\beta}_0+\varepsilon)x_{(i)}}}-(\hat{\alpha}_0+\varepsilon)}{(1-(\hat{\alpha}_0+\varepsilon))}-\frac{i}{1+k}\right)\left(-x_{(i)}(\hat{\alpha}_0-\varepsilon)^{e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}}e^{-(\hat{\beta}_0-\varepsilon)x_{(i)}}\ln(\hat{\alpha}_0+\varepsilon)\right)}{(1-(\hat{\alpha}_0+\varepsilon))}=0$$

The value of $\hat{\beta}$ can be obtained by using the false position method from equation (12) as a form:

$$\hat{\beta}_{OLS}=(\hat{\beta}_0+\varepsilon)-\frac{h_2(\hat{\beta}_0+\varepsilon)((\hat{\beta}_0+\varepsilon)-(\hat{\beta}_0-\varepsilon))}{h_2(\hat{\beta}_0+\varepsilon)-h_2(\hat{\beta}_0-\varepsilon)} \quad (66)$$

By substituting equations (64) and (66) in equation (3) getting the estimation of reliability function as:

$$\hat{R}_{OLS}(x)=\frac{1-(\hat{\alpha}_{OLS})^{e^{-(\hat{\beta}_{OLS})x}}}{1-\hat{\alpha}_{OLS}} \quad (67)$$

2.7. Maximum Likelihood Method (MLE)

In (1921) Fisher proposed this method. The general idea is depending on finding the parameter which maximize the likelihood function, and can get it by some technique as follows: [13]

Take the product for PDF function in equation (1) to get the likelihood function

$$L(\alpha,\beta;x_1,x_2,\dots,x_k)=\prod_{i=1}^{k}f(x_i)=\left(\frac{-\ln(\alpha)\beta}{1-\alpha}\right)^{k}e^{-\beta\sum x_i}\ \alpha^{\sum_{i=1}^{k}e^{-\beta x_i}}$$

For some simplification take the natural logarithm, getting

$$\ln(L(\alpha,\beta;x_1,x_2,\dots,x_k))=k\ln(-\ln(\alpha)\,\beta)-k\ln(1-\alpha)-\left(\beta\sum_{i=1}^{k}x_i\right)+\ln(\alpha)\sum_{i=1}^{k}e^{-\beta x_i} \quad (68)$$

Taking the partial derivative for equation (68) respect to α then equaling to zero, getting:

$$\frac{\partial\ln\prod_{i=1}^{k}f(x_i)}{\partial\alpha}=\frac{k(\frac{-1}{\alpha})}{-\ln(\alpha)}-\frac{k(-1)}{1-\alpha}+\frac{1}{\alpha}\sum_{i=1}^{k}e^{-\beta x_i}=0$$

$$\frac{k}{\alpha\ln(\alpha)}+\frac{k}{1-\alpha}+\frac{1}{\alpha}\sum_{i=1}^{k}e^{-\beta x_i}=0$$

Let,

$$h_3(\alpha)=\frac{k}{\alpha\ln(\alpha)}+\frac{k}{1-\alpha}+\frac{1}{\alpha}\sum_{i=1}^{k}e^{-\beta x_i}=0 \quad (69)$$

To get the numerically solution for equation (69) by false position method, take the two intervals

$$[\hat{\alpha}_0-\varepsilon,\hat{\alpha}_0+\varepsilon],\left[\hat{\beta}_0-\varepsilon,\hat{\beta}_0+\varepsilon\right]$$

Then:

$$h_3(\hat{\alpha}_0-\varepsilon)=\frac{k}{(\hat{\alpha}_0-\varepsilon)\,ln\left((\hat{\alpha}_0-\varepsilon)\right)}+\frac{k}{1-(\hat{\alpha}_0-\varepsilon)}+\frac{1}{(\hat{\alpha}_0-\varepsilon)}\sum_{i=1}^{k}e^{-(\hat{\beta}_0-\varepsilon)x_i}=0$$

$$h_3(\hat{\alpha}_0+\varepsilon)=\frac{k}{(\hat{\alpha}_0+\varepsilon)\,ln\left((\hat{\alpha}_0+\varepsilon)\right)}+\frac{k}{1-(\hat{\alpha}_0+\varepsilon)}+\frac{1}{(\hat{\alpha}_0+\varepsilon)}\sum_{i=1}^{k}e^{-(\hat{\beta}_0+\varepsilon)x_i}=0$$

The value of $\hat{\alpha}$ can be obtained by using the false position method from equation (11) as a form:

$$\hat{\alpha}_{MLE}=(\hat{\alpha}_0+\varepsilon)-\frac{h_3(\hat{\alpha}_0+\varepsilon)\left((\hat{\alpha}_0+\varepsilon)-(\hat{\alpha}_0-\varepsilon)\right)}{h_3(\hat{\alpha}_0+\varepsilon)-h_3(\hat{\alpha}_0-\varepsilon)} \quad (70)$$

Taking the partial derivative for equation (68) respect to (β) then equaling to zero, getting:

$$\frac{\partial \ln\prod_{i=1}^{k}f(x_i)}{\partial\beta}=\frac{k(-\,ln(\alpha)(1))}{-\,ln(\alpha)\beta}-(1)\sum_{i=1}^{k}x_i+ln(\alpha)\sum_{i=1}^{k}(-x_i)e^{-\beta x_i}=0$$

Let,
$$h_4(\beta)=\frac{k}{\beta}-\sum_{i=1}^{k}x_i-ln(\alpha)\sum_{i=1}^{k}x_ie^{-\beta x_i} \quad (71)$$

To get the numerically solution for equation (71) by false position method, take the two intervals

$$[\hat{\alpha}_0-\varepsilon,\hat{\alpha}_0+\varepsilon],\left[\hat{\beta}_0-\varepsilon,\hat{\beta}_0+\varepsilon\right]$$

$$h_4\left(\hat{\beta}_0-\varepsilon\right)=\frac{k}{(\hat{\beta}_0-\varepsilon)}-\sum_{i=1}^{k}x_i-ln(\hat{\alpha}_0-\varepsilon)\sum_{i=1}^{k}x_ie^{-(\hat{\beta}_0-\varepsilon)x_i}=0$$

$$h_4\left(\hat{\beta}_0+\varepsilon\right)=\frac{k}{(\hat{\beta}_0+\varepsilon)}-\sum_{i=1}^{k}x_i-ln(\hat{\alpha}_0+\varepsilon)\sum_{i=1}^{k}x_ie^{-(\hat{\beta}_0+\varepsilon)x_i}=0$$

The value of $\hat{\beta}$ can be obtained by using the false position method from equation (12) as a form:

$$\hat{\beta}_{MLE}=\left(\hat{\beta}_0+\varepsilon\right)-\frac{h_4\left(\hat{\beta}_0+\varepsilon\right)\left((\hat{\beta}_0+\varepsilon)-(\hat{\beta}_0-\varepsilon)\right)}{h_4\left(\hat{\beta}_0+\varepsilon\right)-h_4\left(\hat{\beta}_0-\varepsilon\right)} \quad (72)$$

By substituting equations (70) and (71) in equation (3) getting the estimation of reliability function as:

$$\hat{R}_{MLE}(x)=\frac{1-(\hat{\alpha}_{MLE})^{e^{-(\hat{\beta}_{MLE})x}}}{1-\hat{\alpha}_{MLE}} \quad (73)$$

3. Simulation Experiments

Simulation is an important method for generating the data required to estimate the parameters and reliability of a specific distribution. Therefore, we used this method in this research to estimate the reliability of the Alpha Power Survival Transformation Exponential distribution through four experiments (E_1,E_2,E_3,E_4), as shown in Table 1 for the actual parameter values. as follows:

Table 1. - The real values of the parameters

Experiments→ Parameters ↓	E_1	E_2	E_3	E_4
α	2	1.5	0.5	2.5
β	2	0.5	1.5	3.5

To make a suitable comparison, four sample size values were taken, described as small, medium, and large (k=10,40,70,100).

To make a comparison using the mean squared error criterion $MSE(\tau) = \frac{\sum_{j=1}^{k}(\hat{\tau}_j-\tau)^2}{N}$, sample replicated were used, with N=1000 replicate per sample.

The following equations: (31), (32), (33), (43), (44), (45), (58), (59), (60), (64), (66), (67), (70), (72) and (73) were used for parameters estimation and reliability function estimation.

The results were computed using MATLAB language version R2013a

Table 2. - Mean squared error values for estimators in E_1

Estimators	k	MLE	LS1	LS2	OLS	WLS	Best
$\hat{\alpha}$	10	1.961832	1.201561	0.222552	1.886030	1.450716	LS2
$\hat{\beta}$		0.557520	0.915523	2.157701	0.657297	0.780350	MLE
$\hat{R}$		0.005278	0.010799	0.045225	0.006797	0.006630	MLE
$\hat{\alpha}$	40	1.008101	0.139564	0.042156	0.969020	0.340963	LS2
$\hat{\beta}$		0.202802	0.276271	1.952364	0.230226	0.232816	MLE
$\hat{R}$		0.001479	0.003619	0.043939	0.001843	0.002714	MLE
$\hat{\alpha}$	70	0.931972	0.084967	0.024765	0.896664	0.256809	LS2
$\hat{\beta}$		0.173048	0.186007	2.469092	0.196802	0.151097	WLS
$\hat{R}$		0.000937	0.002428	0.044615	0.001260	0.001535	MLE
$\hat{\alpha}$	100	0.872794	0.045861	0.014101	0.848590	0.223121	LS2
$\hat{\beta}$		0.153717	0.128145	2.538804	0.181387	0.110471	WLS
$\hat{R}$		0.000669	0.001662	0.0451047	0.000971	0.000935	MLE

Table 3. - Mean squared error values for estimators in E_2

Estimators	k	MLE	LS1	LS2	OLS	WLS	Best
$\hat{\alpha}$	10	0.302666	0.269702	0.070847	0.386923	0.319714	LS2
$\hat{\beta}$		0.028878	0.054883	15.93881	0.032570	0.042526	MLE
$\hat{R}$		0.004433	0.011207	0.112929	0.004709	0.009714	MLE
$\hat{\alpha}$	40	0.158672	0.019484	0.010005	0.162550	0.027278	LS2
$\hat{\beta}$		0.006828	0.014743	7.712051	0.007850	0.010711	MLE

$\hat{R}$		0.001011	0.003578	0.082404	0.001216	0.002255	MLE
$\hat{\alpha}$	70	0.141204	0.010463	0.004681	0.143006	0.012875	LS2
$\hat{\beta}$		0.004740	0.009502	4.737795	0.005620	0.005938	MLE
$\hat{R}$		0.000604	0.002291	0.067067	0.000748	0.001205	MLE
$\hat{\alpha}$	100	0.137614	0.007055	0.002944	0.138148	0.004769	LS2
$\hat{\beta}$		0.004064	0.006372	4.333723	0.004823	0.009675	MLE
$\hat{R}$		0.000439	0.001459	0.062434	0.000531	0.000941	MLE

Table 4. - Mean squared error values for estimators in E_3

Estimators	k	MLE	LS1	LS2	OLS	WLS	Best
$\hat{\alpha}$	10	0.018707	Complex	0.013673	0.019876	Complex	LS2
$\hat{\beta}$		0.166618	Complex	3.667073	0.181103	Complex	MLE
$\hat{R}$		0.008588	Complex	0.055136	0.009372	Complex	MLE
$\hat{\alpha}$	40	0.007507	Complex	0.004176	0.008145	Complex	LS2
$\hat{\beta}$		0.053591	Complex	4.427411	0.058367	Complex	MLE
$\hat{R}$		0.002767	Complex	0.058978	0.002962	Complex	MLE
$\hat{\alpha}$	70	0.005299	Complex	0.002293	0.005879	Complex	LS2
$\hat{\beta}$		0.040606	Complex	5.304081	0.045104	Complex	MLE
$\hat{R}$		0.001737	Complex	0.057049	0.001914	Complex	MLE
$\hat{\alpha}$	100	0.004332	Complex	0.001575	0.004956	Complex	LS2
$\hat{\beta}$		0.036733	Complex	6.883006	0.043242	Complex	MLE
$\hat{R}$		0.001386	Complex	0.063683	0.001666	Complex	MLE

Table 4. - Mean squared error values for estimators in E_4

Estimators	k	MLE	LS1	LS2	OLS	WLS	Best
$\hat{\alpha}$	10	3.968687	2.617454	0.894149	4.202471	2.552492	LS2
$\hat{\beta}$		1.578163	2.523613	4.569890	1.678148	2.496931	MLE
$\hat{R}$		0.004571	0.009310	0.048049	0.005994	0.011629	MLE
$\hat{\alpha}$	40	2.742382	0.469293	0.106051	2.702209	0.807104	LS2
$\hat{\beta}$		0.719441	0.970131	3.901852	0.812613	1.046098	MLE
$\hat{R}$		0.001300	0.003604	0.041960	0.001731	0.004122	MLE
$\hat{\alpha}$	70	2.501331	0.229475	0.055683	2.412568	0.642070	LS2
$\hat{\beta}$		0.640474	0.644814	4.006823	0.744069	0.718566	MLE
$\hat{R}$		0.000872	0.002505	0.041029	0.001209	0.002586	MLE
$\hat{\alpha}$	100	2.399860	0.186928	0.036090	2.314989	0.580176	LS2
$\hat{\beta}$		0.610614	0.499585	3.954475	0.699047	0.524395	LS1
$\hat{R}$		0.000724	0.001827	0.040553	0.000969	0.001622	MLE

4. Conclusions

- In the all experiments E_1, E_2, E_3 and E_4, where the values of the two parameters are equal and one is greater than the other, the following becomes clear:

Across all sample size values, we observe that the LS2 estimator for α parameter is the best compared to the other estimators for the estimating methods. For the β parameter, we observe that the MLE estimator is the best among the other methods. For the reliability estimator, we observe that the maximum probability estimator is the best.

- Analysis of the above values and results shows that the effect of estimating the β for calculating reliability advantage is greater than the effect of estimating the α parameter.

- In Table 3 (E_3), we observe that when the value of α is less than one, both the LS1 and WLS produce complex estimated values. This is due to the presence of the algebraic term ln(ln(α)), resulting in a negative value for ln(α). Consequently, the value of ln(*), for a negative value is complex, which is a drawback of both methods

References

[1] N. Balakrishnan, A. P. Basu, The Exponential Distribution: Theory, Methods and Applications, Taylor & Francis Group, (1995).

[2] S. Nadarajah, S. Kotz, The beta exponential distribution, Reliability Engineering and System Safety, 91 (2006), 689–697.

[3] W. Barreto-Souza, A. H. Santos, G. M. Cordeiro, The beta generalized exponential distribution, Journal of Statistical Computation and Simulation, 80 (2) (2010), 159–172.

[4] S. Nadarajah, F. Haghighi, An extension of the exponential distribution, Statistics, 45 (6) (2011), 543–558.

[5] M. Elgarhy, M. Shakil, G. Kibria, Exponentiated Weibull-exponential distribution with applications, Applications and Applied Mathematics: An International Journal (AAM), 12 (2) (2017), 5.

[6] K. Abdel Al-Kadim, H. Madlool, Alpha power survival transformation exponential distribution, Journal of Computational Analysis and Applications, 33 (2) (2024).

[7] J. J. Swain, S. Venkatraman, J. R. Wilson, Least-squares estimation of distribution functions in Johnson's translation system, Journal of Statistical Computation and Simulation, 29 (4) (1988), 271–297.

[8] W. Gautschi, Numerical Analysis, 2nd ed., Springer, (2011).

[9] D. K. Ibrahim, M. A. M. Salih, Comparison of reliability estimates for maximum likelihood and least squares methods of the exponentiated inverse Rayleigh distribution, AIP Conference Proceedings, 3282 (1) (2025), 020015-(1–8).

[10] M. A. M. Salih, M. M. Hasan, Linear formula estimators for the reliability of transmuted Pareto distribution, Journal of College of Education, (2) (2021).

[11] Z. S. Noori, M. A. M. Salih, Comparison of some methods for estimating parameters and reliability of the generalized inverse Weibull distribution, AIP Conference Proceedings, 2845 (1) (2023), 060011-(1–11).

[12] K. H. Habib, M. A. Khaleel, H. Al-Mofleh, P. E. Oguntunde, S. J. Adeyeye, Parameters estimation for the [0,1] truncated Nadarajah–Haghighi Rayleigh distribution, Scientific African, 23 (2024), e02105.

[13] J. Aldrich, R. A. Fisher and the making of maximum likelihood 1912–1922, Statistical Science, 12 (3) (1997), 162–176.