المجلة العراقية للبحوث الانسانية والاجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN2790-1254

On the Stability of a Zero Solution of a Fractional Differential System

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Abstract

In this research, different types of stability of the zero solution for a specific fractional differential system were studied by giving the necessary conditions for each type of stability studied. Illustrative examples were also given showing the effectiveness of these conditions for determining stability.

Keywords: stability, zero solution, fractional differential system, Lipchitz, asymptotic stability, regular stability.

1. Introduction

Recently, the study of the stability of fractional differential systems has attracted much attention from authors. In 1996, [1] first studied the stability of linear fractional differential systems with the Caputo derivative. Since then, many researchers have carried out further studies on the stability of linear fractional differential systems [2], [3]. In nonlinear systems, the direct Lyapunov method paves the way for stability analysis as in Mittag-Leffler stability of a system without the explicit solution of the differential equations [4][5][6]. This method generalizes the idea that a system is stable if there are some filtered Lyapunov functions for the system. Lyapunov's direct method is a sufficient condition for demonstrating the stability of the system. Which means that systems can remain stable even if one does not find the Lyapunov function in order to deduce the stability property of the system. Stability analysis of nonlinear systems is more difficult and there are not many studies on this. Some researchers [7][8][9] studied a nonlinear fractional differential system. Then some work appeared on the stability of linear fractional differential systems of order 0<1<α [10][11]. However, not all fractional differential systems have fractional orders in the interval (0,1). There are fractional models that have fractional orders located in the interval (1,2), such as super diffusion [12]. Researchers in [13] studied the stability of n-dimensional fractional differential linear systems of order 1<α<2. In our research, we will study the fractional differential system.

$$D_{t_{0},t}^{\alpha} x(t) = Gx(t) + f\left(t, x(t), D_{t_{0},t}^{\beta} x(t)\right), t > t_0$$

$$x(t_0) = x_0, x'(t_0) = x_1 \ldots\ldots\ldots (1.1)$$

where $x(t) \in R^n,\ x_k \in R^n (k = 0,1), G \in R^{n\times n}, 1 < \alpha < 2,\ 0 < \beta < \alpha - 1$ and $f:[t_0, \infty) \times R^n \times R^n \to R^n$ a given continuous function. We will study different types of stability of solutions for system (1.1) using the constrained function f.

2. Preliminaries

Definition (2-1):[14] We call the function:

$$E_\alpha(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(\alpha k + 1)}, v \in \mathbb{C}, Re(\alpha) > 0 \ldots \ldots \ldots (2.1)$$

One-parameter Mittag-Leffler function. And the function

$$E_{\alpha,\beta}(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(\alpha k + \beta)}, v \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0 \ldots \ldots \ldots (2.2)$$

The Mittag-Leffler function with two parameters, and both (2.1) and (2.2) satisfy the conditions:

$$E_{\alpha,1}(v) = E_\alpha(v) \text{ and } E_1(v) = e^v$$

Definition (2-2):[14] If we derive the Mittag-Lefler function, the derivative will be in the following form:

$$\frac{d}{dv} E_{\alpha,1}(av^\alpha) = \sum_{k=1}^{\infty} \frac{a^k v^{\alpha k - 1}}{\Gamma(\alpha k)} = av^{\alpha-1} \sum_{k=0}^{\infty} \frac{(av^\alpha)^k}{\Gamma(\alpha k + \alpha)} = av^{\alpha-1} E_{\alpha,\alpha}(av^\alpha) \ldots \ldots \ldots (2.3)$$

And

$$\frac{d}{dv}\left(v^{\beta-1} E_{\alpha,\beta}(av^\alpha)\right) = v^{\beta-2} E_{\alpha,\beta-1}(av^\alpha) \ldots \ldots \ldots (2.4)$$

Definition (2-3):[14] Let $\alpha > 0$, for any function $f: [t\_0, T] \to R$, then the fractional integral of order α for the function f is known as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} f(s) ds \ldots \ldots \ldots (2.5)$$

Definition (2-4):[14] The Caputo derivative of the continuous function $f: [t\_0, T] \to R, n - 1 < \alpha \le n$ is given by the formula:

$$D_{t_0,t}^\alpha f(t) = \begin{cases} I^{n-\alpha} f^{(n)}(t), n - 1\alpha < n \\ f^{(n)}(t), \alpha = n \end{cases} \ldots \ldots \ldots (2.6)$$

We can write it in another form as follows:

$$D_{t_0,t}^\alpha f(t) = I^{n-\alpha}\left(D_{t_0,t}^n f(t)\right) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t - s)^{n-\alpha-1} f^{(n)}(s) ds \ldots \ldots \ldots (2.7)$$

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Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN2790-1254

Note (2-5): In our research, we will use the Caputo effect in all subsequent mathematical procedures.

Lemma (2-6):[15] Let $G \in C^\wedge(n \times n)$,$0 < \alpha < 2$ and let ρ be an arbitrary real number such that $\pi\alpha/2 < \rho < min\{\pi, \pi\alpha\}$, then:

$$E_{\alpha,\beta}(G) \leq \frac{b}{1 + \|G\|} \dots \dots \dots (2.8)$$

Where b is a positive real number and $\rho \leq |\arg(spec(G))| \leq \pi$ here $spec(G)$ symbolizes the eigenvalues of the $matrix$ (G).

Lemma (2-7):[16] The system $D^{\alpha}_{t_{0,t}} x(t) = Gx(t)$ is asymptotically stable if $|arg(eig(G))| > \alpha\pi/2$ where $0 < \alpha < 2$ and $eig(G)$ are the eigenvalues of the $matrix$ G.

Note (2-8):[16] The turbulent system $D^{\alpha}_{t_{0,t}} x(t) = Ax(t) + f\big(t, x(t)\big)$is asymptotically stable if $|arg(eig)\ (G))| > \alpha\pi/2$ and the function f satisfies the inequality $\|f(t,x)\| \leq \theta\|x\|, t \geq t_0$ for some values of $\theta \in L^1$.

It can be observed that the behavior of the Mittag-Leffler function is relaxed for values of $\alpha < 1$ and exponential for values of $\alpha = 1$. It becomes submerged oscillator for values of $1 < \alpha < 2$ and oscillatory if $\alpha = 2$. The decay is very fast when t approaches $0^+$and very slow when t approaches ∞.

Definition (2-9):[17] that the constant x_{equ} represents an equilibrium of the fractional differential system

$$D^{\alpha} x(t) = f\big(x, x(t)\big), t \in [t, \infty) \dots \dots \dots (2.9)$$

If and only if $f\big(t, x_{equ}\big) = D^{\alpha} x(t)|_{x(t)=x_{equ}}$ for all values of factor $t > t_0$ where D^{α} means either the Caputo factor or the fractional differential Riemann-Liouville.

For ease, we will mention all the definitions and theorems related to stability when the equilibrium point is the origin of R^n. That is, x_{equ}. There is no loss of generality by doing this because any equilibrium point can be transformed into the origin by changing the variables. For confirmation, assume that the equilibrium point is y, then consider that $z = x - y$. Therefore, the Kabuto derivative of the variable z is:

$$D^{\alpha}_{t_{0,t}} z = D^{\alpha}_{t_{0,t}}(x - y) = f(t, x) = f(t, z + y) = g(t, y)$$

When $g(t, 0) =\ 0$, the system will have an equilibrium point at the origin of the new variable y.

Definition (2-10):[17] The zero-sum solution of the system (2.9) is said to be stable if $\epsilon > 0$ is found such that $\|x(t)\| \leq \epsilon$ for all values of $t > t_0$.

Definition (2-11):[17] The zero-sum solution of the system (2.9) is said to be asymptotically stable if $\|x(t)\| \to 0$ when $t \to \infty$.

Definition (2-12):[17] Let $H \subset R^n$ be a field containing the origin. The zero solution of system (2.9) is said to be stable according to Mittag-Leffler if

$$\|x(t)\| \leq \left(m(x(t_0)E_{\alpha,1}(-\gamma(t-t_0)^{\alpha})\right)^{\theta}, t \geq t_0 \dots\dots\dots (2.10)$$

Where $\gamma \geq 0, \theta > 0, m(0) = 0, m(x) \geq 0$ and m is a local Lebschitz function on the field H.

Definition (2-13):[17] Let $H \subset R^n$ be a field containing the origin. The zero solution of system (2.9) is said to be stable according to the generalized Mittag-Leffler if

$$\|x(t)\| \leq \left(m(x(t_0)(t-t_0)^{-\mu}E_{\alpha,1-\beta}(-\gamma(t-t_0)^{\alpha})\right)^{\theta}, t \geq t_0 \dots\dots\dots (2.11)$$

Where $\gamma \geq 0, \theta > 0,\ 0 < \delta < 1-\alpha, m(0) = 0, m(x) \geq 0$ and m is a local Beschitz function on the field H.

Note (2-14):[17] Both Mittag-Leffler stability and generalized stability according to Mittag-Leffler belong to algebraic stability, which also guarantees asymptotic stability.

Definition (2-15) [18] It is said that the zero solution of the system (2.9) is uniformly stable if $J > 0$ and $\varepsilon > 0$ exist such that

$$\|x(t)\| \leq J\|x(t_0)\|, t \geq t_0 \dots\dots\dots (2.12)$$

Whenever $\|x(t_0)\| \leq \varepsilon$

Lemma (2-16):[19] (Grönohl-Bellman inequality) Suppose that:

$$x(t) \leq q(t) + \int_{t_0}^{t} k(s)x(s)\,ds, t \in [t_0, T)$$

When all bounding functions are continuous on the domain $[t_0, T), T \leq \infty$ and $k(t) \geq 0$, then $x(t)$ satisfies:

$$x(t) \leq q(t) + \int_{t_0}^{t} k(s)q(s)e^{\int_s^t k(y)dy}\,ds, t \in [t_0, T)$$

If $q(t)$ is a non-increasing function, and in addition to the above, then

$$x(t) \leq q(t)e^{\int_s^t k(s)ds}, t \in [t_0, T)$$

Lemma (2-17): For any $1 < \alpha < 2$, the solution to the system (1.1) is:

$$x(t) = E_{\alpha,1}(A(t-t_0)^{\alpha})x_0 + (t-t_0)E_{\alpha,2}(A(t-t_0)^{\alpha})x_1 + \int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f\left(\tau, x(\tau), D_{t_0,t}^{\beta}x(\tau)\right)d\tau$$

3. The Stability

Print ISSN 2710-0952 Electronic ISSN2790-1254

In the next section, we will study classical stability and asymptotic stability if the function is restricted to the system (1.1), assuming that the function $f:[t_0,\infty)\times R^n\times R^n\to R^n$ is continuous such that $f(t,0,0)=\ 0$ and the Beschitz condition is met. In addition to that, we need the following assumptions:

a) Let G be a matrix such that $|spec(G)|\neq 0$ and $|arg(spec(G))|\geq \alpha\pi/2$ where $spec(G)$ denotes the eigenvalues and $1<\alpha<2$.
b) Assume that the critical eigenvalues of the $matrix\ G$ satisfying $|arg(spec(G))|=\alpha\pi/2$ have the same algebraic geometric multiplicities.

4. Main Results

Theorem (4-1): Let both (a) and (b) be true. Assume that there is a positive function $\theta(t)$ that satisfies the following conditions:

1) The integral $\int_{t_0}^{\infty}(t-\tau)^{\alpha-1}\theta(\tau)d\tau.t\geq t_0$ is bounded.
2) $\left\|f\left(t,x,D_{t_0,t}^{\beta}x\right)\right\|\leq\theta(t)\|x(t)\|$

Then the zero solution of system (1.1) is stable and bounded.

Proof: Using (2-17) we get:

$$\|x(t)\|\leq\|x_0\|\|E_\alpha(G(t-t_0)^\alpha)\|+\|x_1\|\left\|(t-t_0)E_{\alpha,2}(G(t-t_0)^\alpha)\right\| \\ +\int_{t_0}^{t}\left\|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(G(t-\tau)^\alpha)f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau\ \ldots\ldots\ldots(4.1)$$

Since the matrix $E_{\alpha,\alpha-i}(G(t-t_0)^\alpha)$ is bounded to values $i=0,1,2$ and $t\in[t_0,\infty)$ and $1<\alpha<2$, there are positive constants such as W_0, W_1, and W_2so that:

$$\|E_\alpha(G(t-t_0)^\alpha)\|\leq W_0$$

$$\left\|(t-t_0)E_{\alpha,2}(G(t-t_0)^\alpha)\right\|\leq W_1$$

$$\left\|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(G(t-\tau)^\alpha)\right\|\leq W_2$$

Then, equation (4.1) becomes as:

$$\|x(t)\|\leq W_0\|x_0\|+W_1\|x_1\|+W_2\int_{t_0}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau$$

Applying the Gronol-Bellman inequality to equation (4.2) we get:

$$\|x(t)\|\leq(W_0\|x_0\|+W_1\|x_1\|)e^{W_2\int_{t_0}^{t}(t-\tau)^{\alpha-1}\theta(\tau)d\tau}\ \ldots\ldots\ldots(4.3)$$

From condition (a) there is a positive constant W_3 such that

$$\int_{t_0}^{t}(t-\tau)^{\alpha-1}\theta(\tau)\,d\tau<\int_{t_0}^{\infty}(t-\tau)^{\alpha-1}\theta(\tau)\,d\tau<W_3$$

Print ISSN 2710-0952 Electronic ISSN2790-1254

Then equation (4,3) becomes:

$$\|x(t)\| \leq (W_0\|x_0\| + W_1\|x_1\|)e^{W_2W_3} := W$$

Where $\|x(t)\| \leq \mathrm{W},\ \mathrm{W} > 0$, so that the zero solution for the system (1,1) is stable and bounded.

Theorem (4-2): Let condition (a) be fulfilled and the function f regularly achieves the limit

$$\lim_{x\to 0}\frac{\left\|f\left(t,x,D_{t_0,t}^{\beta}x\right)\right\|}{\|x\|} = 0$$

Then the zero solution of system (1.1) is asymptotically stable.

Proof: suppose that $t \in [t_0,\infty)$ then by lemma (2-17) we obtain:

$$\|x(t)\| \leq \|x_0\|\|E_\alpha(G(t-t_0)^\alpha)\| + \|x_1\|\|(t-t_0)E_{\alpha,2}(G(t-t_0)^\alpha)\| + \int_{t_0}^{t}\left\|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(G(t-\tau)^\alpha)f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau$$

Then we get by Lemma (2-6):

$$\|x(t)\| \leq \|x_0\|\frac{b_1}{1+\|G\|(t-t_0)^\alpha} + \|x_1\|\frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^\alpha} + \int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{1+\|G\|(t-\tau)^\alpha}\left\|f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau$$

Then we obtain:

$$\|x(t)\| \leq \|x_0\|\frac{b_1}{1+\|G\|(t-t_0)^\alpha} + \|x_1\|\frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^\alpha} + \frac{\|G\|(\alpha-1)}{2}\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{1+\|G\|(t-\tau)^\alpha}\|x(\tau)\|d\tau$$

We find using the Gronwall-Bellman inequality that:

$$\|x(t)\| \leq \|x_0\|\frac{b_1}{1+\|G\|(t-t_0)^\alpha} + \|x_1\|\frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^\alpha} + \frac{\|G\|(\alpha-1)}{2}\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{1+\|G\|(t-\tau)^\alpha}\left(\|x_0\|\frac{b_1}{1+\|G\|(t-t_0)^\alpha} + \|x_1\|\frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^\alpha}\right)\left(e^{\frac{\|G\|(\alpha-1)}{2}\int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{1+\|G\|(t-s)^\alpha}ds}\right)d\tau$$

Then by the definition of the limit, we find for the value of $\epsilon = \frac{\|G\|(\alpha-1)}{2}$, there exists at least $\delta > 0$, such that $\|x(t)\| < \delta$ then:

$$\left\| f\left(t, x(t), D^{\beta}_{t_0,t} x(t)\right) \right\| < \frac{\|G\|(\alpha-1)}{2} \|x(t)\|$$

Now let

$$I = \frac{\|G\|(\alpha-1)}{2} \int_{\tau}^{t} \frac{(t-s)^{\alpha-1}}{1+\|G\|(t-s)^{\alpha}} ds$$

And if $u = (t-s)^{\alpha}$ then $du = -\alpha(t-s)^{\alpha-1} ds$ so that:

$$I = \frac{(\alpha-1)}{2\alpha} \ln(1+\|G\|(t-\tau)^{\alpha})$$

Taking the exponential function of the above relationship, we find that:

$$e^{I} = (1+\|G\|(t-\tau)^{\alpha})^{\frac{(\alpha-1)}{2\alpha}}$$

So, we have:

$$\begin{aligned} \|x(t)\| \leq \|x_0\| \frac{b_1}{1+\|G\|(t-t_0)^{\alpha}} + \|x_1\| \frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}} \\ + \frac{\|G\|(\alpha-1)}{2} \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{1+\|G\|(t-\tau)^{\alpha}} \Bigg(\|x_0\| \frac{b_1}{1+\|G\|(t-t_0)^{\alpha}} \\ + \|x_1\| \frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}} \Bigg) \left((1+\|G\|(t-\tau)^{\alpha})^{\frac{(\alpha-1)}{2\alpha}} \right) d\tau \end{aligned}$$

$$\begin{aligned} \leq \|x_0\| \frac{b_1}{1+\|G\|(t-t_0)^{\alpha}} + \|x_1\| \frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}} \\ + \frac{\|G\|(\alpha-1)}{2} \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{(1+\|G\|(t-\tau)^{\alpha})^{\frac{\alpha+1}{2\alpha}}} \Bigg(\|x_0\| \frac{b_1}{1+\|G\|(t-t_0)^{\alpha}} \\ + \|x_1\| \frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}} \Bigg) d\tau \end{aligned}$$

$$\begin{aligned} \leq \|x_0\| \frac{b_1}{1+\|G\|(t-t_0)^{\alpha}} + \|x_1\| \frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}} \\ + \frac{b_1\|G\|(\alpha-1)\|x_0\|}{2} \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{(1+\|G\|(t-\tau)^{\alpha})(1+\|G\|(t-\tau)^{\alpha})^{\frac{\alpha+1}{2\alpha}}} d\tau \\ + \frac{b_2\|G\|(\alpha-1)\|x_1\|}{2} \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}(\tau-t_0)}{(1+\|G\|(t-\tau)^{\alpha})(1+\|G\|(t-\tau)^{\alpha})^{\frac{\alpha+1}{2\alpha}}} d\tau \end{aligned}$$

Since $1+\|G\|(t-\tau)^{\alpha} > \|G\|(t-\tau)^{\alpha}$ and $1+\|G\|(\tau-t_0)^{\alpha} > \|G\|(\tau-t_0)^{\alpha}$ then we get:

$$\begin{aligned} \|x(t)\| \leq \|x_0\| \frac{b_1}{1+\|G\|(t-t_0)^{\alpha}} + \|x_1\| \frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}} \\ + \frac{b_1\|G\|(\alpha-1)\|x_0\|}{2} \int_{t_0}^{t} (t-\tau)^{\frac{\alpha-1}{2}-1} (\tau-t_0)^{-\alpha} d\tau \\ + \frac{b_2\|G\|(\alpha-1)\|x_1\|}{2} \int_{t_0}^{t} (t-\tau)^{\frac{\alpha-1}{2}-1} (\tau-t_0)^{-\alpha+1} d\tau \end{aligned}$$

Now we can calculate the integral

$$I=\int_{t_0}^{t}(t-\tau)^{\frac{\alpha-1}{2}-1}(\tau-t_0)^{-\alpha+1}\,d\tau$$

Assuming that:

$\tau=\mathrm{t}-v(t-t_0)$

We find:

$$I=\int_{t_0}^{t}(t-\tau)^{\frac{\alpha-1}{2}-1}(\tau-t_0)^{-\alpha+1}\,d\tau=(t-t_0)^{\frac{\alpha-1}{2}-1-\alpha+1+1}\int_0^1 v^{\frac{\alpha-1}{2}-1}(1-v)^{-\alpha+1}dv$$
$$=(t-t_0)^{\frac{-\alpha}{2}+\frac{1}{2}}B\left(\frac{\alpha-1}{2},2-\alpha\right)$$

Similarly, we conclude that:

$$\int_{t_0}^{t}(t-\tau)^{\frac{\alpha-1}{2}-1}(\tau-t_0)^{-\alpha}\,d\tau=(t-t_0)^{\frac{-\alpha}{2}-\frac{1}{2}}B\left(\frac{\alpha-1}{2},1-\alpha\right)$$

So, for that:

$$\|x(t)\|\le\|x_0\|\frac{b_1}{1+\|G\|(t-t_0)^{\alpha}}+\|x_1\|\frac{b_2(t-t_0)}{1+\|G\|(t-t_0)^{\alpha}}+\frac{b_1\|G\|^{\frac{-\alpha-1}{2\alpha}}(\alpha-1)\|x_0\|}{2}B\left(\frac{\alpha-1}{2},1-\alpha\right)$$
$$+\frac{b_2\|G\|^{\frac{-\alpha-1}{2\alpha}}(\alpha-1)\|x_1\|}{2}B\left(\frac{\alpha-1}{2},2-\alpha\right)$$

As a result, $\|x(t)\|\to 0$ when $t\to\infty$, so that, the zero solution of system (1.1) is asymptotically stable.

Theorem (4-3): Let condition (a) be fulfilled. If there exists a positive constant W_0 such that:

$$\int_{t_0}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau\le\|x_1\|\le W_0\|x_0\|$$

Then, the zero solution of the system (1.1) is uniformly stable according to Lipschitz's concept.

Proof: by Lemma (2-17) we find:

$$\|x(t)\|\le\|x_0\|\|E_{\alpha}(G(t-t_0)^{\alpha})\|+\|x_1\|\|(t-t_0)E_{\alpha,2}(G(t-t_0)^{\alpha})\|$$
$$+\int_{t_0}^{t}\left\|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(G(t-\tau)^{\alpha})f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau$$

And using the same upper bounds of the function $E_{\alpha,\alpha-i}(G(t-\tau)^{\alpha}),i=0,1,2$ and $t\in[t_0,\infty)$ in Theorem (2-4) we get:

$$\|x(t)\| \le W_1\|x_0\| + W_2\|x_1\| + W_3\int_{t_0}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x(\tau),D_{t_0,t}^{\beta}x(\tau)\right)\right\|d\tau$$

And from the given condition we obtain:

$$\|x(t)\| \le W_1\|x_0\| + W_2\|x_1\| + W_3\|x_1\| = W_1\|x_0\| + (W_2+W_3)\|x_1\|$$

$$\le W_1\|x_0\| + W_0(W_2+W_3)\|x_1\|$$

$$\le \left(W_1 + W_0(W_2+W_3)\right)\|x_0\|$$

So,

$$\|x(t)\| \le W\|x_0\|, W := \left(W_1 + W_0(W_2+W_3)\right) > 0$$

So, for that, the zero solution of the system (1.1) is uniformly stable according to Lipschitz's concept.

Theorem (4-4): If the zero solution of the system (1.1) is uniformly stable according to Lipschitz's concept, then it is uniformly stable:

Proof: suppose that the zero solution of the system (1.1) is uniformly stable according to Lipschitz's concept, then there exists $W>0,\delta>0$ such that:

$$\|x(t)\| \le W\|x_0\| \text{ When } \|x_0\| \le \delta_0, t \ge t_0$$

Let $\epsilon>0$ be given and let $\delta = min\left(\delta_0,\frac{\epsilon}{W}\right)$ then for the value of $\|x_0\|\le\delta$ we have:

$\|x(t)\| \le W\|x_0\| \le W\delta < \epsilon$ for each value $t\ge t_0$ so the zero solution of the system is uniformly stable.

5. Practical examples:

In this section, we will provide some illustrative examples that show the effectiveness of the theorems we arrived at in section four.

Example (5-1) Suppose:

$$D_{0,t}^{1.3}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} + \begin{bmatrix}e^{-t}(x_1^2+x_2^2)\sin\left(D_{0,t}^{0.3}|x_1(t)x_2(t)|\right)\\e^{-t}(x_1^2+x_2^2)\cos\left(D_{0,t}^{0.3}|x_1(t)x_2(t)|\right)\end{bmatrix}$$

With the initial conditions:

$$x(0)=\begin{bmatrix}0\\0\end{bmatrix}, x'(0)=\begin{bmatrix}1\\1\end{bmatrix}$$

We have:

$$f\left(t,x(t),D_{0,t}^{\beta}x(t)\right) = \begin{bmatrix}e^{-t}(x_1^2+x_2^2)\sin\left(D_{0,t}^{0.3}|x_1(t)x_2(t)|\right)\\e^{-t}(x_1^2+x_2^2)\cos\left(D_{0,t}^{0.3}|x_1(t)x_2(t)|\right)\end{bmatrix}$$

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Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN2790-1254

$$G = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

By finding the eigenvalues of the $matrix\ G$, which they are:

$$\lambda_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \lambda_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

And

$$|\arg(spec(G))| > \frac{1.3\pi}{2}$$

Now by applying the special condition of theorem (2-4) where $\theta(\mathrm{t}) = e^{-t}$ then the integral will $\int_0^\infty (t-\tau)^{0.3}\, e^{-\tau} d\tau$ will be bounded. Then for that:

$$\left\| f\left(t, x(t), D_{0,t}^{\beta} x(t)\right) \right\| = \sqrt{\left(e^{-t}(x_1^2 + x_2^2)^{\frac{1}{2}} \sin\left(D_{0,t}^{0.3}|x_1(t)x_2(t)|\right)\right)^2 + \left(e^{-t}(x_1^2 + x_2^2)^{\frac{1}{2}} \cos\left(D_{0,t}^{0.3}|x_1(t)x_2(t)|\right)\right)^2}$$

$$\leq \sqrt{e^{-2t}(x_1^2 + x_2^2)} = e^{-t}\sqrt{x_1^2 + x_2^2} = \theta(t)\|x(t)\|$$

So, the zero solution is stable.

Example (2-5): consider the following system:

$$D_{0,t}^{1.2} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} x_2 D_{0,t}^{0.2}|x_1(t)| \\ 0 \\ x_2 D_{0,t}^{0.2}|x_2(t)| \end{bmatrix}$$

With the initial conditions:

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, x'(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we have:

$$f\left(t, x(t), D_{0,t}^{\beta} x(t)\right) = \begin{bmatrix} x_2 D_{0,t}^{0.2}|x_1(t)| \\ 0 \\ x_2 D_{0,t}^{0.2}|x_2(t)| \end{bmatrix}$$

$$G = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

And by finding the eigenvalues of the $matrix\ G$, which they are:

$$\lambda_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \lambda_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}, \lambda_3 = -1$$

Where

$|\arg(spec(G))| > \frac{1.2\pi}{2}$

Now by applying theorem (2-4) we obtain:

$$\lim_{x\to 0}\frac{\left\|f\left(t,x(t),D_{0,t}^{\beta}x(t)\right)\right\|}{\|x\|}=\lim_{x\to 0}\frac{\sqrt{x_2^2\left(\left(D_{0,t}^{0.2}|x_1(t)|\right)^2+\left(D_{0,t}^{0.2}|x_2(t)|\right)^2\right)}}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$\leq \lim_{x\to 0}\frac{\sqrt{x_2^2\left(\left(D_{0,t}^{0.2}|x_1(t)|\right)^2+\left(D_{0,t}^{0.2}|x_2(t)|\right)^2\right)}}{\sqrt{x_2^2}}=\lim_{x\to 0}\sqrt{\left(D_{0,t}^{0.2}|x_1(t)|\right)^2+\left(D_{0,t}^{0.2}|x_2(t)|\right)^2}=0$$

So, the zero solution is asymptotically stable.

Example(3-5): Suppose we have the system $D_{0,t}^{1.5}x(t)+3x(t)=x(t)D_{0,t}^{0.4}x(t)$ with the initial conditions $x(0)=0, x'(0)=1$. We have
$f\left(t,x(t),D_{0,t}^{\beta}x(t)\right)=x(t)D_{0,t}^{0.4}x(t)$ and $G=-3$. The eigenvalues for the matrix G are $\lambda=-3$ and $|\arg(-3)|=\pi>\frac{1.5\pi}{2}$ so,

$$\lim_{x\to 0}\frac{\left\|f\left(t,x(t),D_{0,t}^{\beta}x(t)\right)\right\|}{\|x\|}=\lim_{x\to 0}\frac{\left|f\left(t,x(t),D_{0,t}^{\beta}x(t)\right)\right|}{|x|}=\lim_{x\to 0}\frac{\left|x(t)D_{0,t}^{0.4}x(t)\right|}{|x|}=0$$

So, the zero solution is asymptotically stable.

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