Wasit Journal for Pure Science

Journal Homepage: https://wjps.uowasit.edu.iq/index.php/wjps/index
e-ISSN: 2790-5241 p-ISSN: 2790-5233

Bayesian Estimation for the Odd Chen–Gamma Distribution Under Asymmetric Loss Functions

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DOI: https://doi.org/10.31185/wjps.874
Received 20 March 2025; Accepted 26 August 2025; Available online 30 June 2026

ABSTRACT: In this study, a new two-parameter lifetime distribution called the Odd Chen–Gamma distribution is proposed. The probability density function, cumulative distribution function, reliability function, and several important statistical properties of this distribution are derived. Bayesian analysis is performed using two prior functions for the unknown shape parameter: the exponential (informative) prior and a hypothetical prior. The Bayesian estimator of the shape parameter is obtained under two asymmetric loss functions, namely the De-Groot and the Al-Bayyati loss function. A simulation study is conducted to evaluate the performance of the Bayesian estimators by examining their mean squared errors under different parameter values and various sample sizes. All numerical computations and simulations were performed using MATLAB 2024b.

Keywords: Odd Chen–Gamma distribution, Bayesian estimation, Exponential prior, hypothetical prior, Asymmetric loss functions.

1. INTRODUCTION

Here The Odd Chen–Gamma (OC-Gamma) distribution is a newly proposed lifetime model that combines the flexibility of the classical Gamma distribution with the shape-enhancing capability of the Odd Chen transformation. This transformation enables the model to capture diverse hazard rate behaviors, making it suitable for reliability and survival analysis [1-3]. Despite its attractive properties and the increasing interest in Odd Chen-generated families, the Bayesian estimation of the OC-Gamma distribution has not yet been addressed in the existing literature. This represents a significant research gap, particularly when compared to other distributions such as the Chen, Exponentiated Generalized Chen, and Odd Generalized Exponential Chen models, which have been extensively studied under Bayesian approaches [4-6]. Bayesian methods have become increasingly valuable in reliability modeling due to their flexibility in incorporating prior information and their ability to handle complex models. They have been successfully employed in various applications, including inference based on record values, stress-strength analysis, and system reliability modeling under different lifetime distributions [7-9]. The transformation itself has led to the development of many generalized distributions, including the Odd Chen–Exponential, Odd Chen–Fréchet, and Odd Burr families, which have found applications in environmental data analysis, engineering systems, and industrial processes [10-13].In this study, estimate the shape parameter α of the OC-Gamma distribution using Bayesian method, assuming that the scale parameter β is known and equal (2). Two prior distributions are considered: the exponential prior (informative) and a non-informative prior. Estimation is conducted under two asymmetric loss functions: the De-Groot loss function and the Al-Bayyati loss function. Furthermore, a simulation study is performed to evaluate the efficiency of the Bayesian estimators based on their mean squared errors under different scenarios of parameter values and sample sizes [14-16].

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2. THE ODD CHEN–GAMMA DISTRIBUTION

2.1 PROBABILITY DENSITY FUNCTION (PDF)

First, we start with the general Odd Chen transformation, which allows any baseline CDF G(x) to be transformed into a new distribution:

$$f(x;\alpha,\beta,\eta)=\frac{\alpha\beta G(x;\alpha,\beta,\eta)^{\beta-1}g(x;\alpha,\beta,\eta)}{[1-G(x;\alpha,\beta,\eta)]^{\beta+1}}\left[e^{\left[\frac{G(x;\alpha,\beta,\eta)}{1-G(x;\alpha,\beta,\eta)}\right]^{\beta}}\right]e^{-\alpha\left[e^{\left[\frac{G(x;\alpha,\beta,\eta)}{1-G(x;\alpha,\beta,\eta)}\right]^{\beta}}-1\right]}$$

Next, we specify the baseline distribution as the Gamma distribution, Its CDF and PDF are given by:

$$F(x;\alpha,\beta)=\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right),\quad x>0$$

and

$$f(x;\alpha,\beta)=\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)},\quad x>0$$

Then, by differentiating the transformed CDF with respect to x, we obtain the PDF of the Odd Chen–Gamma distribution:

$$f(x;\alpha,\beta)=\frac{\alpha\beta[\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})]^{\beta-1}\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}}{[1-\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})]^{\beta+1}}\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}{1-\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}\right]^{\beta}}\right]e^{-\alpha\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}{1-\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}\right]^{\beta}}-1\right]}\qquad(1)$$

To prove that the function in (1) is a probability density function as:

$$\int_0^{\infty}f(x;\alpha,\beta)dx=1$$

$$\int_0^{\infty}\frac{\alpha\beta[\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})]^{\beta-1}\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}}{[1-\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})]^{\beta+1}}\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}{1-\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}\right]^{\beta}}\right]e^{-\alpha\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}{1-\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})}\right]^{\beta}}-1\right]}dx$$

We simplify the integral by using variable substitution.

$$u=\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right),\quad \frac{d}{dx}\gamma\left(\alpha,\frac{x}{\beta}\right)=\frac{1}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)}$$

$$du=\frac{1}{\Gamma(\alpha)\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)},\quad dx=\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$$

$$[\frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{x}{\beta})]^{\beta-1}=u^{\beta-1}$$

Thus, the integral transforms from $x\in[0,\infty)$ to $x\in[0,1]$ that

$[u(\infty)=1,u(0)=0]$.

$$I=\int_0^1\frac{\alpha\beta[u]^{\beta-1}\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}}{[1-u]^{\beta+1}}\left[e^{\left[\frac{u}{1-u}\right]^{\beta}}\right]e^{-\alpha\left[e^{\left[\frac{u}{1-u}\right]^{\beta}}-1\right]}dx$$

Substituting the second variable:

$v = \frac{u}{1-u}$ Note that… $u = \frac{v}{1+v} \Rightarrow 1-u = \frac{1}{1+v}$

$du = \frac{dv}{(1+v)^2}$

$$u^{\beta-1} = (\frac{v}{1+v})^{\beta-1} \Rightarrow (1-u)^{-(\beta+1)} = (1+v)^{\beta+1}$$

$$u^{\beta-1}(1-u)^{-(\beta+1)}du = (\frac{v}{1+v})^{\beta-1}(1+v)^{\beta+1}\frac{dv}{(1+v)^2}$$

$$= \frac{v^{\beta-1}}{(1+v)^{\beta-1}}(1+v)^{\beta+1}\frac{dv}{(1+v)^2}$$

$$= v^{\beta-1}(1+v)^{-\beta+1}(1+v)^{\beta+1}\frac{dv}{(1+v)^2}$$

$$= v^{\beta-1}(1+v)^2\frac{dv}{(1+v)^2}$$

$$= v^{\beta-1}dv \Rightarrow e^{\left[\frac{u}{1-u}\right]^{\beta}} = e^{v^{\beta}}$$

$$e^{-\alpha\left[e^{\left[\frac{u}{1-u}\right]^{\beta}}-1\right]} = e^{-\alpha\left[e^{v^{\beta}}-1\right]}$$

$$I = \alpha\beta e^{\alpha}\int_0^{\infty} v^{\beta-1}e^{v^{\beta}}e^{-\alpha\left[e^{v^{\beta}}-1\right]}dv$$

$$I = \alpha\beta e^{\alpha}\int_0^{\infty} v^{\beta-1}e^{v^{\beta}}e^{\alpha}e^{-\alpha e^{v^{\beta}}}dv$$

Substituting the third variable:

$$w = v^{\beta} \Rightarrow dw = \beta v^{\beta-1}dv$$

$$\frac{dw}{\beta} = v^{\beta-1}dv \quad , \quad w \in [0,\infty)$$

$$\Rightarrow I = \alpha\beta e^{\alpha}\int_0^{\infty}\frac{1}{\beta}e^{v^{\beta}}e^{w-\alpha e^{w}}dw$$

$$I = \alpha e^{\alpha}\int_0^{\infty} e^{v^{\beta}}e^{w-\alpha e^{w}}dw$$

Substituting variable:

$$z = \alpha e^{w} \Rightarrow dz = \alpha e^{w}dw = zdw$$

$$dw = \frac{dz}{z}$$

if $w = 0 \rightarrow z = \alpha$ and when is $w \rightarrow \infty$ and $z \rightarrow \infty$ and since $e^{w} = \frac{z}{\alpha}$,then $w = \ln\frac{z}{\alpha}$,that the integral becomes:

$$I = \alpha e^{\alpha}\int_{z=\alpha}^{\infty} e^{\ln\frac{z}{\alpha}-z}\frac{dz}{z}$$

So, substitute $e^{\ln\frac{z}{\alpha}}$ with $\frac{z}{\alpha}$

$$I = \alpha e^{\alpha}\int_{\alpha}^{\infty}\frac{z}{\alpha}\frac{e^{-z}}{z}dz$$

$$= \alpha e^{\alpha}\frac{1}{\alpha}\int_{\alpha}^{\infty}e^{-z}dz$$

$$= e^{\alpha}\int_{\alpha}^{\infty}e^{-z}dz$$

$$I = e^{\alpha}e^{-\alpha} = e^{0} = 1$$

Hence, it has been proven that $\int_0^{\infty} f(x;\alpha,\beta)dx = 1.$

After performing the integration, we find that:

$\int_0^{\infty} f(x;\alpha,\beta)dx = 1$ Since $f(x) > 0$, this confirms that $f(x)$ is a valid probability density function.

2.2 CUMULATIVE DISTRIBUTION FUNCTION (CDF)

The cumulative distribution function CDF of the OC–Gamma distribution is given as:

$$F(x;\alpha,\beta) = 1 - e^{-\alpha\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)}{1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)}\right]^{\beta}}-1\right]} \tag{2}$$

The function F(x) denotes the cumulative distribution function (CDF), and it is defined as the integral of the probability density function f(x) given in equation (1), as follows:

$$F(x;\alpha,\beta) = \int_0^{x} f(x;\alpha,\beta)dx$$

where $f(x)$ is the corresponding probability density function.

2.3 RELIABILITY FUNCTION

The reliability function R(x) gives the probability that a random variable X exceeds a certain value x:

$$R(x;\alpha,\beta) = 1 - F(x;\alpha,\beta)$$

$$R(x;\alpha,\beta) = e^{-\alpha\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)}{1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)}\right]^{\beta}}-1\right]} \tag{3}$$

2.4 HAZARD FUNCTION

The hazard function, which quantifies the instantaneous risk or failure rate at any given x, is defined as the ratio of the probability density function $f(x;\alpha,\beta)$, given in Equation (1), to the reliability function $R(x;\alpha,\beta)$, given in Equation (3):

$$h(x;\alpha,\beta) = \frac{f(x;\alpha,\beta)}{R(x;\alpha,\beta)}$$

By substituting the expressions from Equations (1) and (3), the hazard function for the Odd Chen–Gamma distribution can be derived.

$$h(x;\alpha,\beta) = \frac{\alpha\beta\left[\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)\right]^{\beta-1}\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)\right]^{\beta+1}}\left[e^{\left[\frac{\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)}{1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)}\right]^{\beta}}\right] \tag{4}$$

2.5 QUINTILE FUNCTION

For the OC–Gamma distribution, the quintile function (inverse CDF) returns the value x such that:

$$F(x;\alpha,\beta) = u, \quad 0 < u < 1.$$

In other words,

$$Q(u) = F^{-1}(u) \Rightarrow F(Q(u);\alpha,\beta) = u$$

From our derivation, the closed form quintile function is

$$x = Q(u) = \beta\gamma^{-1}\left(\alpha, \Gamma(\alpha)\frac{\left[1+\ln\left(\frac{-\ln(1-u)}{\alpha}\right)\right]^{\frac{1}{\beta}}}{1+\left[1+\ln\left(\frac{-\ln(1-u)}{\alpha}\right)\right]^{\frac{1}{\beta}}}\right), \quad 0 < u < 1. \tag{5}$$

3. SOME STATISTICAL PROPERTIES OF THE ODD CHEN-GAMMA DISTRIBUTION

THEOREM-1-

Let $X \sim OC - Gamma(\alpha, \beta)$ be a continuous random variable following the Odd Chen–Gamma distribution with shape parameter $\alpha > 0$and scale parameter $\beta > 0$. Then the moment generating function (MGF) of X, denoted by $M_X(t) = E[e^{tx}]$, is given by:

$$M_X(t) = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]} v e^{(j+1)v^{\beta}} \, dv$$

Proof

The moment generating function define as:

$$M_X(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x; \alpha, \beta) \, dx,$$

where $f(x; \alpha, \beta)$ is the probability density function (PDF) of the OC–Gamma distribution:

$$f(x) = \frac{\alpha\beta G(x)^{\beta-1} g(x)}{[1 - G(x)]^{\beta+1}} \left[e^{\left[\frac{G(x)}{1-G(x)}\right]^{\beta}} \right] e^{-\alpha\left[e^{\left[\frac{G(x)}{1-G(x)}\right]^{\beta}} - 1\right]},$$

where $G(x)$and $g(x)$are the CDF and PDF of the Gamma distribution, respectively.

Let: $u = G(x) \Rightarrow x = G^{-1}(u), \quad du = g(x)\, dx$

As $x \to 0$, we get $u \to 0$; and as $x \to \infty$, $u \to 1$. Substituting into the MGF, we obtain:

$$M_X(t) = \alpha\beta \int_0^1 e^{tG^{-1}(u)} \frac{u}{(1-u)^{\beta+1}} e^{\left[\left(\frac{u}{1-u}\right)^{\beta}\right]} e^{-\alpha\left[e^{\left(\frac{u}{1-u}\right)^{\beta}} - 1\right]} du$$

Let

$$u = \frac{v}{1-v} \quad \Rightarrow \quad v = \frac{u}{1+u}, \quad du = \frac{dv}{(1+v)^2}$$

Under this substitution:

$$\frac{u}{(1-u)^{\beta+1}} du = v\, dv, \quad \left(\frac{u}{1-u}\right)^{\beta} = v^{\beta}$$

Thus, the MGF becomes:

$$M_X(t) = \alpha\beta \int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]} v e^{[v^{\beta}]} e^{-\alpha\left[e^{v^{\beta}} - 1\right]} dt$$

Recall the exponential identity:

$$e^{-\alpha\left[e^{v^{\beta}} - 1\right]} = e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} e^{(j+1)v^{\beta}}$$

Substitute this into the expression for the MGF:

$$M_X(t) = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]^{r}} v e^{(j+1)v^{\beta}} \, dv \tag{6}$$

where G^{-1}denotes its inverse function of $G(x; \alpha, \beta)$ which is the cumulative distribution function (CDF) of the Gamma distribution

This completes the proof.

COROLLARY 1: Raw Moments of the OC–Gamma Distribution

Let $X \sim OC - Gamma(\alpha, \beta)$ be a continuous random variable, and let $\mu'_r = E[X^r]$ denote the r^{th}raw moment about the origin. As a direct result of the series expansion of the moment generating function $M_X(t)$ equation (6), the raw moments are given by:

$$\mu'_r = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]^r} ve^{(j+1)v^{\beta}}\, dv \quad (7)$$

We define the auxiliary integral:

$$I_r(j) = \int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]^r} ve^{(j+1)v^{\beta}}\, dv \quad (8)$$

so that the expression for the raw moment becomes:

$$\mu'_r = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_r(j) \quad (9)$$

- For $r = 1$ (mean):

$$\mu'_1 = E[X^1] = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_1(j) \quad (10)$$

- For $r = 2$

$$\mu'_2 = E[X^2] = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_2(j) \quad (11)$$

- For $r = 3$

$$\mu'_3 = E[X^3] = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_3(j) \quad (12)$$

- For $r = 4$

$$\mu'_4 = E[X^4] = \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_4(j) \quad (13)$$

COROLLARY 2: Central Moments of the OC–Gamma Distribution.

Let $X \sim OC - Gamma(\alpha, \beta)$. Denote the r^{th} raw moment by $\mu'_r = E[X^r]$ and the r^{th} central moment by $\mu_r = E[(X - \mu)^r]$, where $\mu = \mu'_1$ is the mean. Then:

$$\mu'_r = E[X^r], \mu_r = E[(X - \mu)^r]$$

The first central moment is zero:

$$\mu_1 = \mu'_1 - \mu'_1 = 0$$

The second central moment (variance) is

$$\mu_2 = \mu'_2 - (\mu'_1)^2$$

$$= \alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_2(j) - \left(\alpha\beta e^{\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} I_1(j)\right)^2, \quad (14)$$

where

$$I_r(j) = \int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]^r} ve^{(j+1)v^{\beta}}\, dv$$

The third central moment is

$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$

$$= \alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_3(j) - 3\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_2(j)\right)\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right) + 2\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right)^3 \quad (15)$$

The fourth central moment is

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$$

$$= \alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_4(j) - 4\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_3(j)\right)\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right) + 6\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_2(j)\right)\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right)^2 - 3\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right)^4 \quad (16)$$

Additionally, the mean and variance are expressed as:

$$\mu = \mu'_1 = E[X] = \alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]^1} v e^{(j+1)v^{\beta}}\, dv \quad (17)$$

$$\sigma^2 = E[(X-\mu)^2] = \mu'_2 - \mu^2 = \alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\int_0^{\infty} e^{\left[tG^{-1}\left(\frac{v}{1+v}\right)\right]^2} v e^{(j+1)v^{\beta}}\, dv \quad (18)$$

The coefficient of skewness S.K is

$$S.K = \frac{\mu_3}{(\mu_2)^{3/2}}$$

$$= \frac{\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_3(j) - 3\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_2(j)\right)\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right) + 2\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right)^3}{\left(\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_2(j) - \left[\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}I_1(j)\right]^2\right)^{3/2}} \quad (19)$$

The coefficient of variation CV is

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\int_0^{\infty}\left[G^{-1}\left(\frac{v}{1+v}\right)\right]^2 v\, e^{(j+1)v^{\beta}}\, dv}}{\alpha\beta e^{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\int_0^{\infty}\left[G^{-1}\left(\frac{v}{1+v}\right)\right]^1 v e^{(j+1)v^{\beta}}\, dv} \quad (20)$$

4. BAYESIAN PARAMETER ESTIMATION

Consider a random $x_1, x_2, \dots, x_n$ drawn independently from the Odd Chen-Gamma distribution with probability density function $f(x|\alpha, \beta))$, where α is a shape parameter and β is a scale parameter. In this Bayesian analysis, the scale parameter β is assumed to be known equal 2, while the shape parameter α is to be estimated based on the observed data. The posterior distribution of α is given by[17]:

$$p(\alpha|x) = \frac{L(\underline{x}|\alpha)\, q(\alpha)}{\int_0^{\infty} L(\underline{x}|\alpha)\, q(\alpha)\, d\alpha} \quad (21)$$

Where:

- $L(\underline{x}|\alpha) = \prod_{j=1}^{n} f(x_j|\alpha)$ is the likelihood function.
- $q(\alpha)$ is the prior distribution for the shape parameter α.

4.1 LIKELIHOOD FUNCTION

The joint probability of the observed data, provided α, serves as the basis for the likelihood function:

$$L(\underline{x}|\alpha) = \prod_{j=1}^{n} f(x_j|\alpha)$$

$$L(\underline{x}|\alpha) = \alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^{n} x_j\right)^{\alpha-1} e^{-\alpha\sum_{j=1}^{n}\left(e^{u_j^2}-1\right)} e^{\sum_{j=1}^{n} u_j^2}\left(\prod_{j=1}^{n}\frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right) \quad (22)$$

Where

$$u = \frac{\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha, \frac{x}{2}\right)}{1 - \frac{1}{\Gamma(\alpha)}\gamma\left(\alpha, \frac{x}{2}\right)}$$

4.2 THE POSTERIOR DISTRIBUTION

4.2.1 The posterior distribution under exponential prior:
The exponential prior is defined as [16]:

$$q_1(\alpha) = \frac{1}{\lambda} e^{-\frac{\alpha}{\lambda}} \quad , \quad \alpha > 0 \quad , \quad \lambda > 0 \tag{23}$$

To find the posterior distribution under exponential prior we substitute the equation (21) and equation (23) we get:

$$p_1(\alpha|x) = \frac{\alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty \alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} \tag{24}$$

Then the Bayesian estimator under the two-loss function as:

a) De - Groot loss function:

This is from the asymmetric type given by: [15]

$$L(\hat{\alpha}, \alpha) = \left(\frac{\hat{\alpha} - \alpha}{\alpha}\right)^2$$

Then the Bayes estimator for the parameter (α) as:

$$\hat{\alpha}_{D_1} = \frac{E[\alpha^2|x]}{E[\alpha|x]} \tag{25}$$

Which is define as:

$$\hat{\alpha}_{D_1} = \frac{\int_0^\infty \alpha^2 \frac{\alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty \alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha}{\int_0^\infty \alpha \frac{\alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty \alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha} \tag{26}$$

b) Al-Bayyati loss function:

This is from asymmetric type given by: [14]

$$L(\hat{\alpha}, \alpha) = \alpha^c(\hat{\alpha}, \alpha)^2$$

Then the Bayes estimator for the parameter (α) as:

$$\hat{\alpha}_{B_1} = \frac{E[\alpha^{c+1}|x]}{E[\alpha^c|x]} \tag{27}$$

Which is define as

$$\hat{\alpha}_{B_1} = \frac{\int_0^\infty \alpha^{c+1} \frac{\alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty \alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha}{\int_0^\infty \alpha^{c} \frac{\alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty \alpha^n 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\frac{1}{\lambda}+\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha} \quad (28)$$

4.2.2 The posterior distribution under Hypothetical prior:

The Hypothetical prior used here is[15]:

$$q_2(\alpha) = k\alpha^2 \quad ,\alpha > 0 \quad , \quad k > 0 \text{ positive number} \quad (29)$$

To find the posterior distribution under this prior we substitute the equation (21) and equation (29) we get:

$$p_2(\alpha|x) = \frac{k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} \quad (30)$$

Then the Bayesian estimator under the two-loss function as:

a) De - Groot loss function: by eq. (25):

$$\hat{\alpha}_{D_2} = \frac{\int_0^\infty \alpha^{2} \frac{k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha}{\int_0^\infty \alpha \frac{k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha} \quad (31)$$

b) Al-Bayyati loss function: by eq .(27):

$$\hat{\alpha}_{B_2} = \frac{\int_0^\infty \alpha^{c+1} \frac{k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha}{\int_0^\infty \alpha^{c} \frac{k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)}{\int_0^\infty k\,\alpha^{2+n} 2^{n-\alpha n}(\Gamma\alpha)^{-2n}\left(\prod_{j=1}^n x_j\right)^{\alpha-1} e^{-\alpha\left(\sum_{j=1}^n\left(e^{u_j^2}-1\right)\right)} e^{\sum_{j=1}^n u_j^2}\left(\prod_{j=1}^n \frac{\gamma\left(\alpha,\frac{x}{2}\right)}{\left[1-\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{2}\right)\right]^3}\right)d\alpha} d\alpha} \quad (32)$$

5. SIMULATION PROCEDURE

A simulation was conducted according to the Monte-Carlo method using MATLAB 2024b. first a random set,$\{u_i : i = 1 \ldots n\}$, of numbers were generated form a $U(0,1)$ distribution. The values u_i were then used to get the sample $x_i, i = 1 \ldots n$ by solving the equation (5).

The sample was then used in the estimators to acquire the estimates for α for each estimation method,
This process was repeated 1000 times, then the mean squared error for the values were then calculated from the formula:

$$MSE = \frac{\sum_{j=1}^{1000}(\hat{\alpha}_j - \alpha)^2}{1000}$$

MSE provides a single measure of an estimator's accuracy by combining both bias and variance.
The above value for each of the four estimators were then used to compare the performance between the two loss functions for each prior.
The sample size was taken to be $n = 10,50,100$ with $\alpha = 0.5, 1,$ and 2

Table 1. MSE Values for $\alpha = 0.5$

n	exponential prior		
	D_1	B_1	**Best**
10	0.0023	0.0346	D_1
50	0.0007	0.0208	D_1
100	0.0005	0.01819	D_1
n	**Hypothetical prior**		
	D_2	B_2	**Best**
10	0.0408	0.0457	D_2
50	0.0216	0.0223	D_2
100	0.0186	0.0189	D_2

Table 2. MSE Values for $\alpha = 1$

n	exponential prior		
	D_1	B_1	**Best**
10	0.0038	0.0665	D_1
50	0.0011	0.0628	D_1
100	0.0008	0.0248	D_1
n	**Hypothetical prior**		
	D_2	B_2	**Best**
10	0.0657	0.0652	B_2
50	0.0610	0.0597	B_2
100	0.0169	0.0115	B_2

Table 3. MSE Values for $\alpha = 2$

n	exponential prior		
	D_1	B_1	**Best**
10	0.0111	0.2152	D_1
50	0.0022	0.1904	D_1
100	0.0014	0.0008	B_1
n	**Hypothetical prior**		
	D_2	B_2	**Best**
10	0.2106	0.1964	B_2
50	0.1187	0.0132	B_2
100	0.0337	0.0005	B_2

6. CONCLUSIONS

This research focused on estimating the shape parameter α of the Odd Chen–Gamma distribution using the Bayesian approach, with a fixed scale parameter $\beta = 2$. Two prior distributions were considered: the informative exponential prior and a Hypothetical prior as non-informative. The Bayesian estimator obtained under two asymmetric loss functions, namely the De-Groot and the Al-Bayyati loss function.

The simulation results showed that the De-Groot estimator performed best at α = 0.5 under both prior distributions. At α = 1, De-Groot outperformed under the exponential prior, while the Al-Bayyati estimator was superior under the hypothetical prior. At α = 2, De Groot remained better under the exponential prior except at n = 100, where Al-Bayyati was better; under the hypothetical prior, Al-Bayyati consistently outperformed. These results highlight that estimator performance depends on both the value of α and the choice of prior distribution.

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