Baghdad Science Journal

Volume 23 | Issue 6 Article 19

6-22-2026

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How to Cite this Article

Sharba, Zainab Abdulmunim; Almurieb, Hawraa Abbas; and Alhashemi, Samah (2026) "Approximation in *S*-normed Orlicz Spaces," *Baghdad Science Journal*: Vol. 23: Iss. 6, Article 19.
DOI: https://doi.org/10.21123/2411-7986.5333

BAGHDAD SCIENCE JOURNAL 2026;23(6):2195–2205

RESEARCH ARTICLE

Approximation in $\mathcal{S}$-normed Orlicz Spaces

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ABSTRACT

Orlicz spaces generalize conventional Lebesgue spaces, providing a more flexible framework for analyzing functions. They are essential to functional analysis and related fields, particularly in approximation theory. This paper defines an Orlicz normed space equipped with an $\mathcal{S}$-norm, where $\mathcal{S}$ is an outer function that satisfies certain conditions. Introducing norms via the outer function $\mathcal{S}$ is a general method that encompasses many cases of norms in Orlicz spaces. The relationship between $\mathcal{S}$-norm and other norms on Orlicz spaces is established through several equivalences. Moreover, moduli of smoothness are of great interest in approximation theory, as they provide criteria for assessing the accuracy of approximations. Accordingly, the $\mathcal{S}$-norm was used to define the $\mathcal{S}$-modulus of smoothness of fractional order. Fundamental mathematical properties were also presented to support the proofs of the main approximation theorems. Furthermore, the modulus was employed to prove direct and inverse theorems by studying the degree of best approximation. This demonstrates that the upper and lower bounds of the degree of approximation approach zero as the $\mathcal{S}$-modulus of smoothness tends to zero.

Keywords: Approximation, Direct and inverse theorems, Modulus of smoothness, Orlicz space, $\mathcal{S}$-norm

Introduction

Interest in approximation problems in Orlicz spaces has increased because these spaces are more general than the classical L_p spaces, which consist of measurable functions $f(x)$ s.t $|f(x)|^p$, which are integrable functions. Direct and inverse approximation theorems have garnered significant attention in recent decades, particularly in Orlicz spaces and Lebesgue spaces with variable exponents. Several studies[1–4] have drawn analogous conclusions regarding Lebesgue functional spaces with variable exponents. Furthermore, many papers[5,6] have investigated continuous analogues of these issues. In,[7] the researchers presented a detailed analysis of recent developments in Lebesgue spaces concerning variable exponents. Additionally, Nekvinda's notable contributions[8] examined discrete weighted Lebesgue spaces (see[9] for details). In,[10] Stepanets studied the spaces S^p with finite norm of $2\pi-$perodic Lebesgue summable function $f \in L$.

$$\|g\|_{S^p} := \left\| \{\hat{g}(k)\}_{k\in\mathbb{Z}} \right\|_{l_p(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |\hat{g}(k)|^p \right)^{1/p}.$$

The authors in[11] introduced the modulus of smoothness in S^p space and studied the best approximation of functions, presenting direct and inverse theorems based on these moduli. In,[12] the authors studied the modulus

Received 5 October 2025; revised 11 March 2026; accepted 13 March 2026.
Available online 22 June 2026

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https://doi.org/10.21123/2411-7986.5333

of $p-$smoothness in terms of fractional order to obtain the fractional modulus of smoothness of functions with kth $p-$variation. In,[13] a modulus of smoothness was introduced for functions with locally absolutely continuous derivatives. The degree of approximation of bounded functions was estimated using the $k-$mixed modulus of smoothness.[14] Some findings[15] were expanded to Orlicz spaces l_M and l_p with variable summation exponents, identifying the most accurate approximations and Kolmogorov widths of particular diagonal operator image sets within these spaces. In,[16] the authors employed Jackson integrals to derive essential approximation theorems involving trigonometric polynomials. In,[17] a type of weighted Orlicz space $S_{p,u}$ was introduced, and a new version of the main approximation theorems concerning the degree of approximation via fractional order modulus of smoothness was established. The authors also demonstrated the equivalence of certain K-functionals and moduli of smoothness. Another equivalence was proved between a modified K-functional and the Jacobi weighted modulus of smoothness that approximates the Baskakov–Durrmeyer operator.[18] Wisla[19] defined Orlicz spaces concerning an outer function called the $\mathcal{S}$-norm. Later, the authors described the extreme points[20] and complex extreme points[21] of the unit ball of Orlicz spaces equipped with the $\mathcal{S}$-norm. In the same Orlicz spaces equipped with the $\mathcal{S}$-norm, many properties were studied, such as uniform monotonicity,[22] the Kadec-Klee property,[23] and fixed point properties.[24] On the other hand, approximation theorems for neural network operators were considered.[25] Exact upper bounds for the trigonometric width, the best n-term approximation of compact sets, and the relationship between the Stepanets spaces $\mathcal{S}^p_\Phi$ and the Orlicz spaces $\mathcal{S}_M$ are given in.[26] An outline of recent developments in approximation properties was given by the authors in.[27] Their work extended the discussion of the framework of Orlicz spaces, which allows for the handling of functions that are not necessarily continuous. This article introduces the approximation of functions in Orlicz spaces related to the $\mathcal{S}$-norm. The aim is to establish such theorems within this framework.

Materials and methods

This section presents the preliminaries, including definitions, fundamental properties, and concepts required for our work. First, we provide basic definitions of Orlicz spaces, including Young, outer, and pseudo-modular functions.

Definition 1: Let ϕ be a Young function on a measurable space T, then the convex modular with respect to ϕ is given by

$$I_\phi\left(f\right) = \int_T \phi\left(f\left(\nu\right)\right) d\nu. \tag{1}$$

which is also called pseudo-modular.

The following definition is the primary definition of Orlicz space from.[19]

Definition 2: The Orlicz space L_ϕ is a linear space of measurable functions defined as follows:

$$L_\phi = \left\{y \in L_0 : \exists\mu > 0, I_\phi\left(\mu y\right) < \infty\right\}. \tag{2}$$

Note that a Young function ϕ from L_0 produces L_ϕ.

Next, we define the outer function.

Definition 3: Let $\mathcal{S} : [0, \infty) \to [1, \infty)$ be an outer convex linear function that satisfies the condition

$$\max\{\nu, 1\} \le \mathcal{S}\left(\nu\right) \le \nu + 1, \nu \ge 0. \tag{3}$$

Now, we are ready to carry out the $\mathcal{S}$-norm on the Orlicz space $L_{\phi,\mathcal{S}}$.

Definition 4: The $\mathcal{S}$-norm on the Orlicz space $L_{\phi,\mathcal{S}}$ is given by

$$L_{\phi,\mathcal{S}} = \left\{y \in L_\phi : \exists\mu > 0, \|y\|_{\phi,\mathcal{S}} < \infty\right\}, \tag{4}$$

where

$$\|y\|_{\phi,\mathcal{S}} = \inf_{\alpha>0} \frac{\mu_k}{\alpha}\mathcal{S}\left(I_\phi\left(\alpha y\right)\right). \tag{5}$$

Additionally, ϕ and $\mathcal{S}$ are a Young function and an outer function, respectively, and μ_k is presented in the following definition.

Definition 5: Let $k, K \in \mathbb{Z}, P = \{p_k\}_{k=-\infty}^{\infty}, q = \{q_k\}_{k=-\infty}^{\infty}$ be sequences that satisfy $\frac{1}{p_k} + \frac{1}{q_k} = 1$, and $\mu = \{\mu_k\}_{k=-\infty}^{\infty} : 1 < p_k \leq K$, where $\mho = \mho(p, \mu)$ of all sequences

$$\mho = \left\{ \lambda = \{\lambda_k\}_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} \mu_k |\lambda_k|^{q_k} \leq 1 \right\}. \tag{6}$$

For approximation purposes, Fourier sums and coefficients are defined as follows:

Definition 6: Let $\mathbb{T}_n$ be the space of trigonometric polynomials of order not greater than n, and define the function g's Fourier sum, and $\hat{g}$ is the Fourier coefficient of g.

$$S_{n-1}(g) = \sum_{|k|\leq n-1} \hat{g}(k)\, e^{iky}. \tag{7}$$

For simplicity, denote $\hat{g}_{\ell h}(k) = \hat{g}(k)e^{-ik\ell h}$. The Fourier coefficients of g and g^{ψ} satisfy the following equality:

$$\hat{g}(k) = \psi_k \hat{g}^{\psi}(k), \ k \in \mathbb{Z}\setminus\{0\}. \tag{8}$$

In the next definition, the Orlicz norm is defined with respect to the $\mathcal{S}$-norm.

Definition 7: For any function $g \in L_{\phi,\mathcal{S}}$, ϕ and $\mathcal{S}$ are a Young function and an outer function, respectively, define the Orlicz norm as follows:

$$\|g\|^{*}_{\phi,\mathcal{S}} = \sup_{\lambda\in\Lambda} \sum_{k\in\mathbb{Z}} \mu_k \lambda_k \left|\hat{g}(k)\right|. \tag{9}$$

Next, the following definitions conclude the degree of approximation and its relation to the modulus of smoothness.

Definition 8: The best approximation of a function g in the space of its Fourier sum is given by

$$\begin{aligned} E_n(g)_{\phi,\mathcal{S}} &= \|g - S_{n-1}(g)\|_{\phi,\mathcal{S}} \\ &= \inf_{\alpha} \frac{\mu_k}{\alpha} \mathcal{S}\left(\sum_k \phi\left(\alpha g(k)\right)\right). \end{aligned} \tag{10}$$

Definition 9: The $\mathcal{S}$-difference of $g \in L_{\phi,\mathcal{S}}$ is defined in terms of fractional order $\alpha > 0$.

$$\|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}} = \inf_{\alpha} \frac{1}{\alpha} \mathcal{S}\left(\sum_k \mu_k \left|\hat{g}(k) \sum_{\ell} (-1)^{\ell} \binom{\alpha}{\ell} e^{-ik\ell h}\right|^{k}\right), \tag{11}$$

where

$$\Delta_h^{\alpha} g(y) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \binom{\alpha}{\ell} g(y - \ell h). \tag{12}$$

Definition 10: The $\mathcal{S}-$modulus of smoothness of fractional order is given by

$$\omega_{\alpha}(g, \delta)_{\phi,\mathcal{S}} = \sup_{|h|\leq h} \|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}}. \tag{13}$$

$\mathcal{S}$-Orlicz spaces

This section is dedicated to proving that our defined $\mathcal{S}-$Orlicz norm in (5) is in fact a norm, so that the $\mathcal{S}-$Orlicz space in (4) is a normed space.

Theorem 1: $(L_{\phi,\mathcal{S}}, \|\cdot\|_{\phi,\mathcal{S}})$ is a normed space, where ϕ is a Young function, and $\mathcal{S}$ is an outer function.

Proof: Let $f \in L_{\phi,\mathcal{S}}$; then it is clear that $\|f\|_{\phi,\mathcal{S}} \geq 0$. Additionally, $\|0\|_{\phi,\mathcal{S}} = \inf_{\alpha>0} \frac{\mu_k}{\alpha}\mathcal{S}(I_\phi(0)) = 0$.

Now, for any positive λ, that is

$$\|\lambda f\|_{\phi,\mathcal{S}} = \inf_{\alpha>0} \frac{\mu_k}{\alpha}\mathcal{S}\left(I_\phi\left(\alpha\lambda y\right)\right)$$

$$= |\lambda| \inf_{\alpha>0} \frac{\mu_k}{\alpha}\mathcal{S}\left(I_\phi\left(\alpha y\right)\right)$$

$$= |\lambda| \; \|f\|_{\phi,\mathcal{S}}.$$

Let $\epsilon > 0$; choose $\alpha, \beta > 0$ so that for any $x, y \in L_{\phi,\mathcal{S}}$,

$$\inf_{\alpha>0} \frac{\mu_k}{\alpha}\mathcal{S}\left(I_\phi\left(\alpha x\right)\right) \leq \|x\|_{\phi,\mathcal{S}} + \frac{\epsilon}{2}$$

and

$$\inf_{\beta>0} \frac{\mu_k}{\beta}\mathcal{S}\left(I_\phi\left(\beta y\right)\right) \leq \|y\|_{\phi,\mathcal{S}} + \frac{\epsilon}{2}$$

Thus,

$$\|x+y\|_{\phi,\mathcal{S}} \leq \inf_{\alpha,\beta>0} \frac{\mu_k(\alpha+\beta)}{\alpha\beta}\mathcal{S}\left(I_\phi\left(\frac{\alpha+\beta}{\alpha\beta}(x+y)\right)\right).$$

Since $\mathcal{S}$ and ϕ are convex, then

$$\|x+y\|_{\phi,\mathcal{S}} \leq \inf_{\alpha,\beta>0} \frac{\mu_k(\alpha+\beta)}{\alpha\beta}\mathcal{S}\left(I_\phi\left(\frac{\alpha+\beta}{\alpha\beta}x\right)\right) \inf_{\alpha,\beta>0} \frac{\mu_k(\alpha+\beta)}{\alpha\beta}\mathcal{S}\left(I_\phi\left(\frac{\alpha+\beta}{\alpha\beta}y\right)\right)$$

$$= \inf_{\alpha>0} \frac{\mu_k}{\alpha}\mathcal{S}\left(I_\phi\left(\alpha x\right)\right) + \inf_{\beta>0} \frac{\mu_k}{\beta}\mathcal{S}\left(I_\phi\left(\beta y\right)\right) + \epsilon$$

$$\leq \|x\|_{\phi,\mathcal{S}} + \|y\|_{\phi,\mathcal{S}}.$$

Properties of $\mathcal{S}$-modulus of smoothness

This section presents some fundamental properties of $\mathcal{S}$-fractional difference and $\mathcal{S}$-modulus of smoothness in the following lemmas.

Lemma 1:[17] Let $g \in L_{\phi,\mathcal{S}},\ h \in \mathbb{R}$. Let $k \in \mathbb{Z},\ \alpha > 0$; then from (12)

i. $[\Delta_h^\alpha g]^\wedge(k) = [\sum_{\ell=0}^{\infty}(-1)^\ell\binom{\alpha}{\ell}g_{\ell h}]^\wedge(k) = \hat{g}(k)\sum_{\ell=0}^{\infty}(-1)^\ell\binom{\alpha}{\ell}e^{-ik\ell h} = (1-e^{-ikh})^\alpha\hat{g}(k).$

ii. $|[\Delta_h^\alpha g]^\wedge(k)| = |1-e^{-ikh}|^\alpha|\hat{g}(k)| = 2^\alpha|\sin\frac{kh}{2}|^\alpha|\hat{g}(k)|.$

iii. Set $F_\nu(y) := F_\nu(g, y) = |\hat{g}(\nu)|e^{-i\nu y} + |\hat{g}(-\nu)|e^{-i\nu y},\ \nu = 1, 2, \ldots$

$$\sum_{\nu=1}^{n-1} \nu^{\beta} F_{\nu}(x) = \sum_{\nu=1}^{N} F_{\nu}(x) + \sum_{\nu=2}^{n-1} \left(\nu^{\beta} - (\nu-1)^{\beta}\right) \sum_{i=\nu}^{N} F_i(x) - (n-1)^{\beta} \sum_{\nu=n}^{N} F_{\nu}(x).$$

Lemma 2: Let $g \in L_{\phi,\mathcal{S}},\ \alpha, \beta > 0, y, h \in \mathbb{R}$, then

i. $\lim_{h\to 0} \|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}} = 0$.

ii. $\|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}} \le M(\alpha)\|g\|_{\phi,\mathcal{S}}$, where $M(\alpha) := \sum_{\ell=0}^{\infty} |\binom{\alpha}{\ell}| \le 2^{\{\alpha\}}$, $\{\alpha\} := \inf\{k \in \mathbb{N} : k \ge \alpha\}$.

iii. $[\Delta_h^{\alpha} g]^{\wedge}(k) = (1 - e^{-ikh})^{\alpha} \hat{g}(k), k \in \mathbb{Z}$.

iv. $(\Delta_h^{\alpha}(\Delta_h^{\beta} g))(y) = \Delta_h^{\alpha+\beta} g(y)$ $(a.e.)$.

v. $\|\Delta_h^{\alpha+\beta} g\|_{\phi,\mathcal{S}} \le 2^{\{\beta\}} \|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}}$.

vi. $\|\Delta_{mh}^{\alpha} g\|_{\phi,\mathcal{S}} \le m^{\alpha} \|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}}$, where $m, \alpha \in \mathbb{N}$.

vii. $\|\Delta_{\eta}^{\alpha} g\|_{\phi,\mathcal{S}} \le h^{-\alpha}(h+\eta)^{\alpha} \|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}}$, where $\eta, \alpha \in \mathbb{R}$.

Proof: Now, to prove property (i)

$$\|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}} = \lim_{h\to 0^+} \left\| \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k \left| \hat{g}(k) \sum_{\ell} (-1)^{\ell} \binom{a}{\ell} e^{-ik\ell h} \right|^k \right) \right\|_{\phi,\mathcal{S}}$$

$$= \lim_{h\to 0^+} \left\| \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k \left| \hat{g}(k) \left((-1)^0 \binom{a}{0} e^{-ik\ell h} + (-1)^1 \binom{a}{1} e^{-ik\ell h} + \cdots \right) \right|^k \right) \right\|_{\phi,\mathcal{S}}$$

$$= \lim_{h\to 0^+} \left\| \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k \left| \hat{g}(k) \left(\frac{\alpha!}{(\alpha-0)!} e^{-ik\ell h} - \frac{\alpha!}{(\alpha-1)!} e^{-ik\ell h} + \cdots \right) \right|^k \right) \right\|_{\phi,\mathcal{S}}$$

$$= \lim_{\alpha\to\infty} \left\| \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k \left| \hat{g}(k) \sum_{\ell=0}^{\alpha} (-1)^{\ell} \binom{a}{\ell} e^{-ik\ell h} \right) \right|^k \right\|_{\phi,\mathcal{S}} = 0.$$

To prove (ii), from Lemma 1(i) and (5)

$$\|\Delta_h^{\alpha} g\|_{\phi,\mathcal{S}} = \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k \left| \hat{g}(k) \sum_{\ell} (-1)^{\ell} \binom{a}{\ell} e^{-ik\ell h} \right| \right)$$

$$\le \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k \left| \sum_{\ell} (-1)^{\ell} \binom{a}{\ell} e^{-ik\ell h} \right| |\hat{g}(k)| \right)$$

$$\le \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(2^{\{\alpha\}} \left(1 - e^{-ikh}\right)^{\alpha} |\hat{g}(k)| \right)$$

$$\le \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k 2^{\{\alpha\}} \mu_k |\hat{g}(k)| \right)$$

$$\le C(\alpha) \inf_{\alpha} \frac{1}{\alpha} \mathcal{S} \left(\sum_k \mu_k |\hat{g}(k)| \right)$$

$$\le C(\alpha) \|g\|_{\phi,\mathcal{S}}.$$

Property (iii) is a consequence of Lemma 1. Property (iv) follows directly from (ii). Property (v) follows directly from (ii) and (iv).

To prove (vi)

$$\left|\left[\Delta_{mh}^{\alpha}g\right]^{\wedge}(k)\right| = \left|\left[\sum_{k_1}^{m-1}\cdots\sum_{k_\alpha}^{m-1}\Delta_h^{\alpha}g\left(x-\left(k_1+\cdots+k_\alpha\right)h\right)\right]^{\wedge}(k)\right|$$

$$\leq \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k_1}^{m-1}\cdots\sum_{k_\alpha}^{m-1}\sum_{j=0}^{\alpha}(-1)^j\binom{\alpha}{j}g_{jh}\left(x-\left(k_1+\cdots+k_\alpha\right)h\right)e^{-ikx}dx\right|$$

$$\leq m^{\alpha}\left|\left[\Delta_h^{\alpha}g\right]^{\wedge}(k)\right|.$$

Property (vii), if $\delta \geq \eta$, it is clear. Now for $h < \eta$, fix $m \in \mathbb{N}$, s.t. $\frac{\eta}{h} \leq m \leq \frac{\eta}{h+1}$, then

$$\Delta_\eta^{\alpha}g \leq \Delta_{mh}^{\alpha}g \leq m^{\alpha}\Delta_h^{\alpha}g \leq \left(\frac{\eta}{h+1}\right)^{\alpha}\Delta_h^{\alpha}g. \qquad \square$$

The proof of the following lemma is derived directly from Lemma 1 and Lemma 2, as well as Eq. (13).

Lemma 3: Let $g, f \in L_{\phi,\mathcal{S}}$, $\alpha \geq \beta > 0$ and $\delta, \delta_1, \delta_2 > 0$. Then

i. $\omega_\alpha(g,\delta)_{\phi,\mathcal{S}}$ is a nonnegative increasing function on $(0,\infty)$: $\lim_{\delta\to 0^+}\omega_\alpha(g,\delta)_{\phi,\mathcal{S}} = 0$.
ii. $\omega_\alpha(g,\delta)_{\phi,\mathcal{S}} \leq 2^{\{\alpha\}}\|g\|_{\phi,\mathcal{S}}$.
iii. $\omega_\alpha(g,\delta)_{\phi,\mathcal{S}} \leq 2^{\{\alpha-\beta\}}\omega_\beta(g,\delta)_{\phi,\mathcal{S}}$.
iv. $\omega_\alpha(g,m\delta)_{\phi,\mathcal{S}} \leq m^{\alpha}\omega_\alpha(g,\delta)_{\phi,\mathcal{S}}$, $(\alpha,\ m \in \mathbb{N})$.
v. $\omega_\alpha(g,\eta)_{\phi,\mathcal{S}} \leq \delta^{-\alpha}(\delta+\eta)^{\alpha}\omega_\alpha(g,\delta)_{\phi,\mathcal{S}}$.
vi. $\omega_\alpha(g+f,\delta)_{\phi,\mathcal{S}} \leq \omega_\alpha(g,\delta)_{\phi,\mathcal{S}} + \omega_\alpha(f,\delta)_{\phi,\mathcal{S}}$.
vii. $\omega_1(g,\delta_1+\delta_2)_{\phi,\mathcal{S}} \leq \omega_1(g,\delta_1)_{\phi,\mathcal{S}} + \omega_1(g,\delta_2)_{\phi,\mathcal{S}}$.
viii. If $g^{(\beta)} \in L_{\phi,\mathcal{S}}$, then $\omega_\alpha(g,\delta)_{\phi,\mathcal{S}} \leq \delta^{\beta}\omega_{\alpha-\beta}(g^{(\beta)},\delta)_{\phi,\mathcal{S}}$.

The following lemma introduces the relation between degrees of approximation.

Lemma 4: For any nonzero sequence $\psi = \{\psi_k\}_{k=-\infty}^{\infty}$ with $\lim_{|k|\to\infty}|\psi_k| = 0$ *and* $g^{\psi} \in L_{\phi,\mathcal{S}}$. Then

$$E_n(g)_{\phi,\mathcal{S}} \leq \varepsilon_n E_n(g^{\psi})_{\phi,\mathcal{S}},$$

where $\varepsilon_n = \max_{|k|\geq n}|\psi_k|$.

Proof: By (10), Lemma 2, and since $\mathcal{S}$ is monotone nondecreasing, then

$$E_n(g)_{\phi,\mathcal{S}} = \inf_{\alpha>0}\left\{\frac{\mu_k}{\alpha}\mathcal{S}\left(\sum_{|k|\geq n}\psi_k\phi\left(\alpha\hat{g}^{\psi}(k)\right)\right) \leq 1\right\}$$

$$\leq \inf_{\alpha>0}\left\{\frac{\mu_k}{\alpha}\mathcal{S}\left(\sum_{|k|\geq n}\varepsilon_n\phi\left(\alpha\hat{g}^{\psi}(k)\right)\right) \leq 1\right\}$$

$$\leq \varepsilon_n E_n(g^{\psi})_{\phi,\mathcal{S}}. \qquad \square$$

In particular, if $\varepsilon_n = \max_{|k|\geq n}|\psi_k| = |\psi_{k_0}|$, for some positive integer k_0, then

$$E_n(T_{k_0})_{\phi,\mathcal{S}} = \varepsilon_n E_n\left(T_{k_0}^{\psi}\right)_{\phi,\mathcal{S}},$$

for any polynomial $T_{k_0}(x) = ce^{i\varepsilon_n x}$.

In the following lemma, an equivalence between the Orlicz norm and $\mathcal{S}$-norm is held immediately from definitions of (4), (5), and (9).

Lemma 5: For any $g \in L_{\phi,\mathcal{S}}$, it follows that

$$\|g\|_{\phi,\mathcal{S}} \le \|g\|^*_{\phi,\mathcal{S}} \le 2\|g\|_{\phi,\mathcal{S}}.$$

For the technical steps of the proofs of the main results, we need the following lemma.

Lemma 6:[17] For any sequence of kernels $\{\kappa_n(x)\}$ satisfying $\int_{-\pi}^{\pi} \kappa_n(x)dx = 1$, and

$$\int_{-\pi}^{\pi} |x|^r |\kappa_n(x)|\, dx \le C(r)(n+1)^{-r}, \tag{14}$$

where $r = 0, 1, 2, \ldots$ then

$$\int_{-\pi}^{\pi} \left(|x| + n^{-1}\right)^r |\kappa_n(x)|\, dx \le C(r)\, n^{-r}, \tag{15}$$

where $r, n \in \mathbb{N}$.

Results and discussion

Direct approximation result

This section involves the most important part of the paper; attention is given to $E_n(g)_{\phi,\mathcal{S}}$ for functions from $L_{\phi,\mathcal{S}}$ by Fourier sums in terms of $\mathcal{S}-$modulus of smoothness. To prove it, some techniques from[17] are used in this work.

Theorem 2: Let $g \in L_{\phi,\mathcal{S}}$. Then for any $\beta > 0,\ n \in \mathbb{N}$

$$E_n(g)_{\phi,\mathcal{S}} \le C(\beta)\, \omega_\beta\left(g, n^{-1}\right)_{\phi,\mathcal{S}},$$

where $C = C(\beta)$ is independent of g and n.

Proof: Let $\{\kappa_n(t)\}_{n=1}^{\infty}$ be a sequence of kernels of trigonometric polynomials of degree less than n, that satisfies Lemma 6.

Let us start by looking at the case where $\beta \in \mathbb{N}$

$$\sigma_{n-1}(y) = (-1)^{\beta+1} \int_{-\pi}^{\pi} \kappa_{n-1}(t) \sum_{\ell=1}^{\beta} (-1)^{\ell} \binom{\beta}{\ell} g(y - \ell t)\, dt.$$

Therefore, σ_{n-1} is a trigonometric polynomial whose degree is at most $n-1$.

Now by (9), Lemma 5 and Lemma 6, and (6), it follows that

$$\begin{aligned}
E_n(g)_{\phi,\mathcal{S}} &\le \|g - \sigma_{n-1}\|_{\phi,\mathcal{S}} \\
&\le \|g - \sigma_{n-1}\|^*_{\phi,\mathcal{S}} \\
&= \left\|(-1)^{\beta} \int_{-\pi}^{\pi} \kappa_{n-1}(t) \sum_{\ell=0}^{\beta} (-1)^{\ell} \binom{\beta}{\ell} g(y - \ell t)\, dt\right\|^*_{\phi,\mathcal{S}} \\
&= \left\|(-1)^{\beta} \int_{-\pi}^{\pi} \kappa_{n-1}(t)\, \Delta_t^{\beta} g(y)\, dt\right\|^*_{\phi,\mathcal{S}} \\
&= \sup_{\lambda \in \mho} \sum_{k \in \mathbb{Z}} \mu_k \lambda_k \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \kappa_{n-1}(t)\, \Delta_t^{\beta} g(y)\, dt\right) e^{-iky} dy\right|.
\end{aligned}$$

By using Lemma 5 and the Fubini Theorem, it is concluded that

$$E_n(g)_{\phi,\mathcal{S}} \le \int_{-\pi}^{\pi} |\kappa_{n-1}(t)| \sup_{\lambda\in\mho} \sum_{k\in\mathbb{Z}} \mu_k\lambda_k \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_t^\beta g(y)\, e^{-iky} dy\right| dt$$
$$\le 2\int_{-\pi}^{\pi} |\kappa_{n-1}(t)| \left\|\Delta_t^\beta g(y)\right\|_{\phi,\mathcal{S}}^{*} dt$$
$$\le 2\int_{-\pi}^{\pi} |\kappa_{n-1}(t)| \left\|\Delta_t^\beta g(y)\right\|_{\phi,\mathcal{S}} dt$$
$$\le 2\int_{-\pi}^{\pi} |\kappa_{n-1}(t)|\, \omega_\beta(g,t)_{\phi,\mathcal{S}} dt.$$

Using property (vii) in Lemma 3, set $\eta = |t|,\ \delta = n^{-1}$ and Lemma 6, then

$$\int_{-\pi}^{\pi} |\kappa_{n-1}(t)|\, \omega_\beta(g,|t|)_{\phi,\mathcal{S}} dt \le n^\beta \omega_\beta(g,n^{-1})_{\phi,\mathcal{S}} \int_{-\pi}^{\pi} \left(|t|+n^{-1}\right)^\beta |\kappa_{n-1}(t)|\, dt$$
$$\le C(\beta)\, \omega_\beta(g,n^{-1})_{\phi,\mathcal{S}}.$$

Therefore, the theorem is proved for $\beta \in \mathbb{N}$.
Now, if $0 < \beta \notin \mathbb{N}$, then let $\alpha \in \mathbb{N}$ with $\alpha - 1 < \beta < \alpha$. By property (iii) of Lemma 3, it follows that

$$E_n(g)_{\phi,\mathcal{S}} \le C(\beta)\, \omega_\alpha(g,n^{-1})_{\phi,\mathcal{S}} \le C(\beta)\, \omega_\beta(g,n^{-1})_{\phi,\mathcal{S}}.$$

Inverse approximation result

Finally, using the $\mathcal{S}-$modulus of smoothness, the upper bound of the degree of approximation is estimated. Some of the proof techniques come from [17]

Theorem 3: Let $g \in L_{\phi,\mathcal{S}}$. Then for any $\beta > 0,\ n \in \mathbb{N}$, then

$$\omega_\beta\left(g,\frac{\pi}{n}\right)_{\phi,\mathcal{S}} \le \left(\frac{\pi}{n}\right)^\beta \sum_{\nu=1}^{n} \left(\nu^\beta - (\nu-1)^\beta\right) E_\nu(g)_{\phi,\mathcal{S}}. \tag{16}$$

Proof: In view of (i) of Lemma 1, since $g \in L_{\phi,\mathcal{S}}$, for $\varepsilon > 0$ there exist $M_\varepsilon \in \mathbb{N}$, $M_\varepsilon > n$, such that for any $N > M_\varepsilon$, let

$$E_N(g)_{\phi,\mathcal{S}} = \|g - S_{N-1}(g)\|_{\phi,\mathcal{S}} < 2^{-\beta}\varepsilon.$$

Set $g_0 := S_{M_\varepsilon}(g)$. By Lemma 1 (ii), these yield

$$\|\Delta_h^\beta g\|_{\phi,\mathcal{S}} \le \|\Delta_h^\beta g_0\|_{\phi,\mathcal{S}} + \|\Delta_h^\beta (g-g_0)\|_{\phi,\mathcal{S}}$$
$$\le \|\Delta_h^\beta g_0\|_{\phi,\mathcal{S}} + 2^\beta E_{M_\varepsilon+1}(g)_{\phi,\mathcal{S}}$$
$$< \|\Delta_h^\beta g_0\|_{\phi,\mathcal{S}} + \varepsilon. \tag{17}$$

Moreover, let $S_{n-1} := S_{n-1}(g_0)$ be as in (7). Then again, by Lemma 1 (ii), for $|h| \le \frac{\pi}{n}$, then

$$\|\Delta_h^\beta g_0\|_{\phi,\mathcal{S}} = \|\Delta_h^\beta (g_0 - S_{N-1}) + \Delta_h^\beta S_{N-1}\|_{\phi,\mathcal{S}}$$
$$\le \left\| 2^\beta (g_0 - S_{N-1}) + \sum_{|k|\le n-1} |kh|^\beta\, |\hat{g}(k)|\, e^{ik} \right\|_{\phi,\mathcal{S}}$$

$$\leq \left\| 2^{\beta}\left(g_{0}-S_{N-1}\right)+\left(\frac{\pi}{n}\right)^{\beta} \sum_{|k| \leq n-1}|k|^{\beta}|\hat{g}(k)| e^{ik} \right\|_{\phi, \mathcal{S}}$$

$$= \left\| 2^{\beta} \sum_{v=n}^{N_{0}} F_{v}+\left(\frac{\pi}{n}\right)^{\beta} \sum_{v=1}^{n-1} v^{\beta} F_{v} \right\|_{\phi, \mathcal{S}}, \tag{18}$$

where $F_v(y)$ is as in Lemma 1(iii)

$$\left\| 2^{\beta} \sum_{v=n}^{M_{\varepsilon}} F_{v}+\left(\frac{\pi}{n}\right)^{\beta} \sum_{v=1}^{n-1} v^{\beta} F_{v} \right\|_{\phi, \mathcal{S}} \leq \left(\frac{\pi}{n}\right)^{\beta} \left\| n^{\beta} \sum_{v=n}^{M_{\varepsilon}} F_{v}+\sum_{v=1}^{n-1}\left(v^{\beta}-(v-1)^{\beta}\right) \sum_{i=v}^{M_{\varepsilon}} F_{i}-(n-1)^{\beta} \sum_{v=n}^{M_{\varepsilon}} F_{v} \right\|_{\phi, \mathcal{S}}$$

$$\leq \left(\frac{\pi}{n}\right)^{\beta} \left\| \sum_{v=1}^{n}\left(v^{\beta}-(v-1)^{\beta}\right) \sum_{i=v}^{M_{\varepsilon}} F_{i} \right\|_{\phi, \mathcal{S}} \leq \left(\frac{\pi}{n}\right)^{\beta} \sum_{v=1}^{n}\left(v^{\beta}-(v-1)^{\beta}\right) E_{v}\left(g_{0}\right)_{\phi, \mathcal{S}}. \tag{19}$$

By (17), (18), (19), and by the definition of the function g_0, with $|h| \leq \frac{\pi}{n}$, the following inequality is obtained.

$$\|\Delta_{h}^{\beta} g\|_{\phi, \mathcal{S}} \leq \left(\frac{\pi}{n}\right)^{\beta} \sum_{v=1}^{n}\left(v^{\beta}-(v-1)^{\beta}\right) E_{v}(g)_{\phi, \mathcal{S}}+\varepsilon .$$

Finally, (13) completes the proof.

$$\omega_{\beta}\left(g, \frac{\pi}{n}\right)_{\phi, \mathcal{S}} \leq \left(\frac{\pi}{n}\right)^{\beta} \sum_{v=1}^{n}\left(v^{\beta}-(v-1)^{\beta}\right) E_{v}(g)_{\phi, \mathcal{S}} . \qquad \square$$

Conclusion

Approximation in Orlicz spaces plays a crucial role in functional analysis and approximation theory, involving the identification of functions within a given subspace that closely resemble a target function according to a specific criterion. This criterion may be the Luxemburg norm or the Orlicz norm. In this context, the approximation of functions is presented in Orlicz spaces in terms of the $\mathcal{S}$-norm, referred to as $\mathcal{S}$-Orlicz spaces. First, $\mathcal{S}$-norm, $\mathcal{S}$-difference, and $\mathcal{S}$-modulus of smoothness are defined. Then, their analytic properties—such as positivity, monotonicity, and limiting behavior approaching zero—are examined. Furthermore, several significant equivalences between these concepts, the $\mathcal{S}$-norm, and Fourier series coefficients are demonstrated. Finally, the focus is on presenting direct and inverse approximation theorems within $\mathcal{S}$-Orlicz spaces in terms of the $\mathcal{S}$-modulus of smoothness of fractional order, comparable to the familiar Jackson and Bernstein theorems.

Acknowledgment

I am grateful for the blessings God has bestowed upon me, and I appreciate the contributions of everyone involved in completing this paper.

Authors' declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Babylon.

Authors' contributions statement

Z. Sh. Conceived the research idea. H.A. designed definitions. S. A. improved the idea and suggested methodologies for the proofs. All authors shared the ideas, proofs and examples. They also wrote the manuscript together.

References

1. Volosivets S, Krotova Y. Direct and inverse approximation theorems connected with the q-Bessel Fourier transform in weighted L^2 space. Ramanujan J. 2025;66(3):29. https://doi.org/10.1007/s11139-024-01015-0.
2. Yadav R, Meher R, Mishra VN. Approximation properties of some modified Szász–Mirakjan–Kantorovich operators. Numer Anal Appl. 2022;15(2):170–185. https://doi.org/10.1134/s1995423922020082.
3. Ouadih S El. Direct and inverse approximation theorems for functions defined in Damek–Ricci spaces. Ukr Math J. 2024;Dec 26:1–17. https://doi.org/10.1007/s11253-024-02387-8.
4. Tyr O, Daher R, Ouadih SE, Fourchi OE. On the Jackson-type inequalities in approximation theory connected to the q-Dunkl operators in the weighted space $L^2_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$. Bol Soc Mat Mex. 2021;27(2):51. https://doi.org/10.1007/s40590-021-00358-8.
5. Guven A. Approximation of continuous functions by matrix means of hexagonal Fourier series. Results Math. 2018;73(1):18. https://doi.org/10.1007/s00025-018-0775-z.
6. Vinogradov O. Direct and inverse theorems of approximation theory in Banach function spaces. St Petersburg Mathe J. 2024;35(6):907–928. https://doi.org/10.1090/spmj/1836.
7. Diening L, Harjulehto P, Hästö P, Růžička M. Lebesgue and Sobolev spaces with variable exponents. Vol. 2017. Lecture Notes in Mathematics. Berlin/Heidelberg: Springer;2011 https://doi.org/10.1007/978-3-642-18363-8.
8. Nekvinda A. Embeddings between discrete weighted Lebesgue spaces with variable exponents. Math Inequalities Appl. 2007;10(1):165. https://doi.org/10.7153/mia-10-14.
9. Datt G, Bajaj DS, Fiorenza A. Weighted composition operators on variable exponent Lebesgue spaces. Adv Oper Theory. 2024;9(3):68. https://doi.org/10.1007/s43036-024-00366-1.
10. Stepanets AI. Approximation characteristics of spaces $\mathcal{S}^p_\varphi$. Ukrainian Math. J. 2001;53(3):446–475. https://doi.org/10.1023/a:1012348506457.
11. Stepanets A, Serdyuk A. Direct and inverse theorems in the theory of approximation of functions in the space S^p. Ukr Math J. 2002;54(1):126–148. https://doi.org/10.1023/a:1019701805228.
12. Sharba ZA, Almurieb HA. Why moduli of p−smoothness? J Interdiscip Math. 2021;24(7):1995–2004. https://doi.org/10.1080/09720502.2021.1964740.
13. Sharba ZA, Bhaya E S. Moduli of Smoothness for Weighted Lp, $0 < p < 1$ Approximation. J Phys.: Conf Ser. 2021;1804(1): IOP Publishing: 012081. https://doi.org/10.1088/1742-6596/1804/1/012081.
14. Kamel RA, Bhaya ES. K–mixed modulus of smoothness. Baghdad Sci J. 2025;22(1):224–234. https://doi.org/10.21123/bsj.2024.9726.
15. Chaichenko SO, Shydlich AL. Approximate characteristics of modular Orlicz spaces. J Math Sci. 2019 May 21;239(1):30–42. https://doi.org/10.1007/s10958-019-04285-y.
16. Almurieb HA, Sharba ZA, Kareem MA. Trigonometric Jackson integrals approximation by a k- generalize modulus of smoothness. Nonlinear Funct. Anal Appl. 2022;27(4):807–812. https://doi.org/10.22771/nfaa.2022.27.04.09.
17. Abdullayev F, Chaichenko S, KYZY MI, Shidlich A. Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent. Turk J Math. 2020;44(1):284–299. https://doi.org/10.3906/mat-1911-3.
18. Han L, Bai Y. Direct and inverse theorems of approximation by linear combinations of weighted Baskakov–Durrmeyer operators in Orlicz spaces. Anal. Theory Appl. 2024;40(3):221–239. https://doi.org/10.4208/ata.oa-2022-0013.
19. Wisła, M. Orlicz spaces equipped with s-norms. J Math Anal Appl. 2020;483(2):123659. https://doi.org/10.1016/j.jmaa.2019.123659.
20. Başar E, Öztop S, Uysal BH, Yaşar Ş. Extreme points in Orlicz spaces equipped with s-norms and closedness. Math Nachr. 2023;296(11):5042–5062. https://doi.org/10.1002/mana.202200374.
21. Başar E, Uysal BH, Yaşar Ş. Complex extreme points and complex rotundity in Orlicz spaces equipped with the *s*-norm. Istanbul J Math. 2023;1(1):40–47. https://doi.org/10.26650/ijmath.2023.00001.
22. Cui, Y, Wisła, M. Monotonicity properties and solvability of dominated best approximation problem in Orlicz spaces equipped with s-norms. Rev R Acad Cienc Exactas Fís. Nat Ser A Mat. 2021;115(4):198. https://doi.org/10.1007/s13398-021-01139-8.
23. Dong, J, Cui Y, Wisła, M. Kadec-Klee property in Orlicz function spaces equipped with S-norms. J Funct Spaces. 2022;1:5902635. https://doi.org/10.1155/2022/5902635.
24. Li X, Cui Y, Shang S. Weakly convergent sequence coefficient and fixed-point property in Orlicz sequence spaces equipped with s-norms: Positivity. 2025;29(4):51. https://doi.org/10.1007/s11117-025-01142-y.
25. Coroianu L, Costarelli D. Best approximation and inverse results for neural network operators. Results Math. 2024 Aug;79(5):193. https://doi.org/10.1007/s00025-024-02222-3.
26. Shydlich A. Approximation characteristics of Stepanets-Orlicz type spaces. In Altay Conference Proceedings in Mathematics. 2025;2(1):48–59. https://doi.org/10.64700/altay.24.
27. Piconi M, Vinti G. Semi-discrete sampling operators acting on function spaces. In Altay Conference Proceedings in Mathematics. 2025;1(1):95–111. https://doi.org/10.64700/altay.25.

التقريب في فضاءات $\mathcal{S}$ - أورلكز المعيارية

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الخلاصة

تُعمم فضاءات أورلكز فضاءات ليبيك التقليدية مما يوفر إطارًا أكثر مرونة لتحليل الدوال. وبالتالي فهي ضرورية للتحليل الدالي وغيره من الفروع المرتبطة به ولاسيما التقريب الدالي. في هذا البحث، عرفنا فضاء أورلكز المعياري بدلالة المعيار $\mathcal{S}$ ، حيث إن $\mathcal{S}$ تمثل دالة خارجية تحقق شروط معينة. حيث يُعد تقديم المعايير عبر الدالة الخارجية $\mathcal{S}$ طريقة عامة تشمل العديد من حالات المعايير في فضاءات أورلكز. إذ تظهر العلاقة بين معيار $\mathcal{S}$ والمعايير الأخرى في فضاءات أورلكز من خلال بعض التكافؤات. ومن ناحية أخرى، تُعتبر معاملات النعومة ذات أهمية كبيرة في نظرية التقريب الدالي، فهي تُوفر معايير للحكم على دقة التقريبات، لذا قمنا بتعريف معامل $\mathcal{S}$ للنعومة ذي الرتبة الكسرية بالمعيار$\mathcal{S}$. كذلك قدمنا بعض الخصائص الرياضية الأساسية لاستخدامها لاحقًا في اثبات نتائج التقريب الرئيسية. علاوة على ذلك، درسنا درجة أفضل تقريب باستخدام هذا المعامل لإثبات النظريات المباشرة والعكسية. والتي تظهر أن الحدود العليا والسفلى لدرجة التقريب تقترب من الصفر كما يفعل معامل $\mathcal{S}$ للنعومة.

الكلمات المفتاحية: التقريب، مقياس النعومة، النظرية المباشرة والعكسية، فضاء أورلكز، المعيار$\mathcal{S}$.