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Ali Musaddak Delphi
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq, ali.m@csw.uobaghdad.edu.iq

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RESEARCH ARTICLE

Coefficient Bounds for a New Subclass of Meromorphic Multivalent Functions with Higher Derivatives Obtained from Bessel Function

Ali Musaddak Delphi

Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq

ABSTRACT

The goal of this paper is to introduce a new subclass of meromorphic multivalent functions with higher derivatives inside a punctured open unit disk U^*, by generalizing the Bessel function. The method used to derive the new findings on this topic is stated in section two of this paper. The product theorem between two functions and neighborhood properties was obtained using the recently discovered coefficient bounds for the new subclass. In addition, this paper attempted to address an important research gap in the subject of geometric function theory, which is the computational tools for coefficient bounds through the use of differential and integral operators and their generalizations, which led to knowing the behavior of the functions and comparing the results to know the best operator. Therefore, in this paper, after obtaining the coefficient bounds for the new subclass, the values of the coefficient bounds were calculated numerically, and an attempt was made to improve them using an integral operator. The numerical comparison of the two behaviors, which is mentioned in Table 1, revealed that the integral operator used had a positive effect by providing the best bounds of the coefficient in comparison to the Bessel function, whereas the Bessel function had a negative effect on the function in this topic. Finally, based on the results in Table 1, it was concluded that the image of U^* in the $uv-$plane is the set of all points that lie outside the open disk centered at (6,0) with radius 1, as shown in Fig. 1.

Keywords: Coefficient bounds, Hadamard product, Meromorphic functions, Multivalent functions, Neighborhood properties

Introduction

The study of complex analysis has a long history, dating back to the seventeenth century. It may be used not just for other analysis-related domains but also for other areas of mathematics and science in general. Geometric function theory is one of the most important areas of complex analysis that studies the relationship between the analytic properties of functions, such as the coefficients of their power series, and the geometric properties of their domain and images. One of its main objectives is to find coefficient bounds for new subclasses of analytic or meromorphic functions, whether univalent or multivalent, and to use these bounds to deduce geometric properties such as convexity and starlikeness. According to De Branges,[1] the conjecture that guided theory development, which states that the coefficients of any univalent analytic function on the open unit disk satisfy $|d_n| \leq n$ was conclusively proven in 1985. Researchers have been interested in this subject and have attempted to expand Bieberbach's conjecture to include further classes of geometric function theory with the help of differential or integral operators. As a result, new papers using these operators with related ideas, but unclear objectives are now available; this has led to a number of research gaps in this area. This paper attempts to fill one of these research gaps by calculating the coefficient bound numerically for a new subclass of meromorphic multivalent functions associated with Bessel function and improving these bounds using an integral operator.

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E-mail address: ali.m@csw.uobaghdad.edu.iq (A. M. Delphi).

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Now, assume $\mathcal{M}^*$ be the class of functions provided by

$$\mathcal{J}(\varpi) = \varpi^{-p} + \sum_{n=p}^{\infty} d_n \varpi^n, \tag{1}$$

which are multivalent and meromorphic in the open unit disk $U^* = \{\varpi \in \mathbb{C} : 0 < |\varpi| < 1\}$. If $\mathcal{L}$ is defined by

$$\mathcal{L}(\varpi) = \varpi^{-p} + \sum_{n=p}^{\infty} b_n \varpi^n, \tag{2}$$

belong to $\mathcal{M}^*$, then the convolution (Hadamard product) of $\mathcal{J}$ and $\mathcal{L}$ in U^* is determined by

$$(\mathcal{J} * \mathcal{L})(\varpi) = \varpi^{-p} + \sum_{n=p}^{\infty} d_n b_n \varpi^n. \tag{3}$$

For $\mathcal{J} \in \mathcal{M}^*$ is starlike of order $\varrho (0 \leq \varrho < p,\ p \in \{1, 2, 3, \ldots\})$ if $-Re\{\frac{\varpi \mathcal{J}'}{\mathcal{J}}\} > \varrho$ and is convex of order ϱ if $-Re\{1 + \frac{\varpi \mathcal{J}''}{\mathcal{J}'}\} > \varrho$, respectively represents by $\mathcal{J} \in \mathcal{S}^*_{\mathcal{M}}(\varrho)$ and $\mathcal{J} \in \mathcal{K}_{\mathcal{M}^*}(\varrho)$ for $0 < |\varpi| < 1$.

The first kind of the normalized generalized Bessel function of order υ, where $\upsilon \geq 0$, is provided by

$$\psi_v(\varpi) = \sum_{n=p}^{\infty} \frac{(-1)^{n-p}}{\Gamma(n-p+1)\Gamma(n+1-p+v)} \left(\frac{\varpi}{2}\right)^{2(n-p)+v}$$

Let's define

$$\mathcal{F}_p^v(\varpi) = \frac{2^v \Gamma(1+v)}{\varpi^{\frac{v}{2}+p}} \psi_v\left(\varpi^{1/2}\right) = \frac{1}{\varpi^p} + \sum_{n=p}^{\infty} \frac{(-1)^{n+1-p}\Gamma(1+v)}{4^{n+1-p}\Gamma(n+2-p)\Gamma(n+2-p+v)} \varpi^n, \tag{4}$$

observe that when $p = 1$, the operator $\mathcal{F}_p^v$ is simplified to the operator proposed by Popade et al.[2]

The following is the definition of the operator $\mathcal{F}_p^v$ using the Hadmard product (or convolution):

$$\mathcal{F}_p^v \mathcal{J}(\varpi) = \mathcal{F}_p^v(\varpi) * \mathcal{J}(\varpi) = \frac{1}{\varpi^p} + \sum_{n=p}^{\infty} \pi_n(v) d_n \varpi^n, \tag{5}$$

where $\pi_n(v) = \frac{(-1)^{n+1-p}\Gamma(1+v)}{4^{n+1-p}\Gamma(n+2-p)\Gamma(n+2-p+v)}$.

$\mathrm{T}^*(p, q, \tau, v, \epsilon)$ is a newly defined subclass of $\mathcal{J} \in \mathcal{M}^*$ that satisfies the following conditions:

$$-Re\left\{\frac{\varpi^q\left(\mathcal{F}_p^v \mathcal{J}(\varpi)\right)^q + \varpi^{q+1}\left(\mathcal{F}_p^v \mathcal{J}(\varpi)\right)^{q+1}}{\mathcal{F}_p^v \mathcal{J}(\varpi)}\right\} \geq \tau \left|\frac{\varpi^q\left(\mathcal{F}_p^v \mathcal{J}(\varpi)\right)^q + \varpi^{q+1}\left(\mathcal{F}_p^v \mathcal{J}(\varpi)\right)^{q+1}}{\mathcal{F}_p^v \mathcal{J}(\varpi)} + 1\right| + \epsilon, \tag{6}$$

where $\tau > 0,\ p \geq 1, q \geq 0, q < p,\ 0 < \epsilon < 1,\ \varpi \in U^*$.

Two general classifications can be distinguished in the literature on meromorphic functions. The first one, particularly in studies,[3–6] focus on univalent functions, whereas the second, as seen in studies,[7–10] has investigated multivalent functions with the goal of generalizing properties and finding new behaviors. This division helps organize the field and reflects the ongoing development of research in this area.

Materials and methods

Methodology

This work uses the two lemma listed below to establish new results about the characteristic of the geometric function of $\mathcal{J}(\varpi) \in \mathcal{M}^*$ in $\mathrm{T}^*(p, q, \tau, v, \epsilon)$

Lemma 1:[11] *Let* $\mathcal{Y} = -(\mathfrak{p} + i\mathfrak{q})$ *and* σ *is real number then* $Re(\mathcal{Y}) \geq \sigma$ *if and only if* $|\mathcal{Y} + (1-\sigma)| - |\mathcal{Y} - (1-\sigma)| \geq 0$.

Lemma 2:[11] *Let* $\mathcal{Y} = \mathfrak{p} + i\mathfrak{q}$ *and* σ, γ *are real numbers then* $-Re(\mathcal{Y}) \geq \sigma|\mathcal{Y}+1| + \gamma$ *if and only if* $-Re\{\mathcal{Y}(1+\sigma e^{i\phi}) + \sigma e^{i\phi}\} > \gamma, -\pi \leq \phi \leq \pi$.

Coefficient bounds results

A necessary and sufficient condition for $\mathcal{J} \in \mathrm{T}^*(p, q, \tau, \nu, \epsilon)$ is given by the following theorem

Theorem 1: *Let* $\mathcal{J}$ *be defined by* Eq. (1) *for* $\tau > 0,\ p \geq 1, q \geq 0, q < p,\ 0 < \epsilon < 1,\ \varpi \in U^*$, *then* $\mathcal{J} \in \mathrm{T}^*(p, q, \tau, \nu, \epsilon)$ *if and only if*

$$\sum_{n=p}^{\infty}\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(\nu)\,d_n \leq (1-\epsilon)\,. \tag{7}$$

Proof: By Eq. (5), is gotten

$$-Re\left\{\frac{\varpi^q\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^q+\varpi^{q+1}\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^{q+1}}{\mathcal{F}_p^\nu\mathcal{J}(\varpi)}\right\} \geq \tau\left|\frac{\varpi^q\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^q+\varpi^{q+1}\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^{q+1}}{\mathcal{F}_p^\nu\mathcal{J}(\varpi)}+1\right|+\epsilon$$

Lemma 2 then provides us with

$$-Re\left\{\frac{\varpi^q\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^q+\varpi^{q+1}\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^{q+1}}{\mathcal{F}_p^\nu\mathcal{J}(\varpi)}(1+\tau e^{i\vartheta})+\tau e^{i\vartheta}\right\} \geq \epsilon$$

$-\pi < \vartheta \leq \pi$, alternatively,

$$-Re\left\{\frac{\left(\varpi^q\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^q+\varpi^{q+1}\left(\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)^{q+1}\right)(1+\tau e^{i\vartheta})+\tau e^{i\vartheta}\left(\Omega\mathcal{F}_p^\nu\mathcal{J}(\varpi)\right)}{\mathcal{F}_p^\nu\mathcal{J}(\varpi)}\right\} \geq \epsilon. \tag{8}$$

Let $F(\varpi) = -((\varpi^q(\mathcal{F}_p^\nu\mathcal{J}(\varpi))^q + \varpi^{q+1}(\mathcal{F}_p^\nu\mathcal{J}(\varpi))^{q+1}))\ (1+\tau e^{i\vartheta}) - \tau e^{i\vartheta}(\mathcal{F}_p^\nu\mathcal{J}(\varpi))$, this lead to $F(\varpi) = (|\Omega|\varpi^{-p} + \sum_{n=p}^{\infty}[\frac{n!}{(n-q)!} + \frac{n!}{(n-(q+1))!}]\pi_n(\nu)d_n\varpi^n)\ (1+\tau e^{i\vartheta}) + \tau e^{i\vartheta}(\varpi^{-p} + \sum_{n=p}^{\infty}\pi_n(\nu)d_n\varpi^n)$, where $|\Omega| = |((p+q)-1)(-1)^{q+1}\prod_{k=1}^{q}(p+k-1)|$ and $E(\varpi) = \mathcal{F}_p^\nu\mathcal{J}(\varpi)$
According to Lemma 1, Eq. (2) equals

$$|F(\varpi)+(1-\epsilon)E(\varpi)| \geq |F(\varpi)-(1+\epsilon)E(\varpi)|$$

But

$$\begin{aligned}|F(\varpi)+(1-\epsilon)E(\varpi)| &= \left|\left(|\Omega|\varpi^{-p}+\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right]\pi_n(\nu)\,d_n\varpi^n\right)(1+\tau e^{i\vartheta})\right.\\ &\quad\left.+\tau e^{i\vartheta}\left(\varpi^{-p}+\sum_{n=p}^{\infty}\pi_n(\nu)\,d_n\varpi^n\right)+(1-\epsilon)\left[\varpi^{-p}+\sum_{n=p}^{\infty}\pi_n(\nu)\,d_n\varpi^n\right]\right|\\ &\geq \left(|\Omega|-\epsilon+1+\tau e^{i\vartheta}+\tau e^{i\vartheta}\right)|\varpi|^{-p}-\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}+(1-\epsilon)\right]\pi_n(\nu)\,d_n|\varpi|^n\\ &\quad-\tau\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}+1\right]\pi_n(\nu)\,d_n|\varpi|^n\end{aligned}$$

Also

$$|F(\varpi)-(1-\epsilon)E(\varpi)|=\left|\left(|\Omega|\varpi^{-p}+\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right]\pi_n(\nu)d_n\varpi^n\right)(1+\tau e^{i\vartheta})\right.$$

$$\left.+\tau e^{i\vartheta}\left(\varpi^{-p}+\sum_{n=p}^{\infty}\pi_n(\nu)d_n\varpi^n\right)-(1-\epsilon)\left[\varpi^{-p}+\sum_{n=p}^{\infty}\pi_n(\nu)d_n\varpi^n\right]\right|$$

$$\leq\left(|\Omega|+\tau e^{i\vartheta}-1+\epsilon+\tau e^{i\vartheta}\right)|\varpi|^{-p}+\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}-(1+\epsilon)\right]\pi_n(\nu)\,d_n|\varpi|^n$$

$$+\tau\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}+1\right]\pi_n(\nu)\,d_n|\varpi|^n$$

So,

$$|F(\varpi)+(1-\epsilon)E(\varpi)|-|F(\varpi)-(1-\epsilon)E(\varpi)|$$

$$\geq(1-\epsilon)|\varpi|^{-p}-\sum_{n=p}^{\infty}\left[(1+\tau)\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}+(\tau-\epsilon)\right]\pi_n(\nu)\,d_n|\varpi|^n\ \geq 0,$$

according to Eq. (7), which is equivalent to

$$\sum_{n=p}^{\infty}\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(\nu)\,d_n\ \leq(1-\epsilon)$$

Let $\mathcal{J}\in \mathrm{T}^{*}(p,q,\tau,\nu,\epsilon)$. On the other hand, Lemma 2 satisfies Eq. (8), when the values of ϖ on the positive real axis are selected, the Eq. (8) becomes

$$-Re\left\{\frac{\begin{array}{c}\left(|\Omega|\left(1+\tau e^{i\phi}\right)+\left(\tau e^{i\phi}-\epsilon\right)\right)\varpi^{-p}+\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right]\pi_n(\nu)d_n\varpi^n\left(1+\tau e^{i\phi}\right)\right)\\ +\left(\tau e^{i\phi}-\epsilon\right)\sum_{n=p}^{\infty}\pi_n(\nu)d_n\varpi^n\end{array}}{\varpi^{-p}+\sum_{n=p}^{\infty}\pi_n(\nu)d_n\varpi^n}\right\}>0$$

Since $Re(-e^{i\vartheta})\geq-|e^{i\vartheta}|=-1$, above, the inequality turns into

$$-(1-\epsilon)r^{-p}+\sum_{n=p}^{\infty}\left[(1+\tau)\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}+(\tau-\epsilon)\right]\pi_n(\nu)\,d_nr^n\leq 0$$

Eq. (7) is the result of applying the mean value theorem. ■

Corollary 1: *Let $\mathcal{J}(\varpi)\in \mathrm{T}^{*}(p,q,\tau,\nu,\epsilon)$, then*

$$\sum_{n=p}^{\infty}d_n\leq\frac{(1-\epsilon)}{\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)},\tag{9}$$

$$\sum_{n=p}^{\infty}nd_n\ \leq\frac{p(1-\epsilon)}{\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)}.\tag{10}$$

Proof: Theorem 1, from Eq. (7) yields

$$\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(v)\sum_{n=p}^{\infty}d_n$$
$$\leq\sum_{n=p}^{\infty}\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(v)\,d_n\ \leq(1-\epsilon)$$

which satisfied the Eq. (9) of Corollary 1, furthermore by the Eq. (7) it had

$$\left(\left[(1+\tau)\left(\frac{(p-1)!}{(p-q)!}+\frac{(p-1)!}{(p-(q+1))!}\right)+\frac{(\tau-\epsilon)}{p}\right]\pi_p(v)\right)\sum_{n=p}^{\infty}nd_n$$
$$\leq\sum_{n=p}^{\infty}n\left(\left[(1+\tau)\left(\frac{(n-1)!}{(n-q)!}+\frac{(n-1)!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(v)\right)d_n$$

When the coefficient Eq. (9) is taken into account, it fits into this formula:

$$\left(\left[(1+\tau)\left(\frac{(p-1)!}{(p-q)!}+\frac{(p-1)!}{(p-(q+1))!}\right)+\frac{(\tau-\epsilon)}{p}\right]\pi_p(v)\right)\sum_{n=p}^{\infty}nd_n\ \leq(1-\epsilon),\tag{11}$$

following modifications to Eq. (11), this can finally lead to Eq. (10) of Corollary 1. ■

The sharpness for ($n=p,\ p+1,p+2,\ldots$), is thus satisfied from the Eq. (9), as may be seen below.

$$\mathcal{J}(\varpi)=\varpi^{-p}+\frac{(1-\epsilon)}{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(v)}\varpi^{n}.\tag{12}$$

Theorem 2: *Let* $\mathrm{T}^*(p,q,\tau,v,\epsilon)$ *contains* $\mathcal{J}$ *defined by* Eq. (1) *and* $\mathcal{L}$ *defined by* Eq. (2) *then* $\mathcal{K}(\varpi)=\varpi^{-p}+\sum_{n=p}^{\infty}d_nb_n\varpi^n\in\mathrm{T}^*(p,q,\tau,v,\delta)$ *and* δ *is given by*

$$\delta=\frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]^2\pi_n(v)-\left[\tau(1-\epsilon)^2+(1-\epsilon)^2(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)\right]}{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]^2\pi_n(v)-(1-\epsilon)^2}.\tag{13}$$

Proof: The largest δ must be found such that

$$\sum_{n=p}^{\infty}\frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\delta)\right]\pi_n(v)}{(1-\delta)}d_nb_n\ \leq1.\tag{14}$$

Since $\mathcal{J}$ & $\mathcal{L}\in\mathrm{T}^*(p,q,\tau,v,\epsilon)$, then

$$\sum_{n=p}^{\infty}\frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(v)}{(1-\epsilon)}d_n\ \leq1,\tag{15}$$

and

$$\sum_{n=p}^{\infty}\frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(v)}{(1-\epsilon)}b_n\ \leq1,\tag{16}$$

by cauchy-schwarz inequality, it gives

$$\sum_{n=p}^{\infty}\frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(v)}{(1-\epsilon)}\sqrt{d_nb_n}\ \leq1,\tag{17}$$

only that is wanted to be shown

$$\frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\delta)\right]\pi_n(\nu)}{(1-\delta)}d_nb_n \leq \frac{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(\nu)}{(1-\epsilon)}\sqrt{d_nb_n}, \quad (18)$$

this equivalently to

$$\sqrt{d_nb_n} \leq \frac{(1-\delta)\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]}{(1-\epsilon)\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\delta)\right]}. \quad (19)$$

From the Eq. (17), it gives

$$\sqrt{d_nb_n} \leq \frac{(1-\epsilon)}{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(\nu)}, \quad (20)$$

thus, it is enough to show that

$$\frac{(1-\epsilon)}{\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(\nu)} \leq \frac{(1-\delta)\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]}{(1-\epsilon)\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\delta)\right]},$$

which simplifies to Eq. (13) and the proof is complete. ■

Neighborhood properties

Based on the definition in Al-khafaji et al.,[12] which is extended for a subclass of meromorphically multivalent functions, the inclusion relation involving (n, δ)-neighborhood is founded in this section. For a function $\mathcal{J} \in \mathcal{M}^*$ and $0 \leq \delta < 1$, the state is as follows:

$$N_{n,\delta}(\mathcal{J}) = \left\{\mathcal{L} \in \mathcal{M}^* : \mathcal{L}(\varpi) = \varpi^{-p} + \sum_{n=p}^{\infty} b_n\varpi^n \& \sum_{n=p}^{\infty} n\left|d_n - b_n\right| \leq \delta\right\}, \quad (21)$$

For the identity function $e(\varpi) = \varpi^{-p}$, it is had

$$N_{n,\delta}(e) = \left\{\mathcal{L} \in \mathcal{M}^* : \mathcal{L}(\varpi) = \varpi^{-p} + \sum_{n=p}^{\infty} b_n\varpi^n \& \sum_{n=p}^{\infty} n\left|b_n\right| \leq \delta\right\}. \quad (22)$$

Definition 1: A function $\mathcal{J} \in \mathcal{M}^*$ is said to be in $\mathrm{T}^*(p, q, \tau, \nu, \epsilon)$ if there exists a function $\mathcal{L} \in \mathrm{T}^*(p, q, \tau, \nu, \epsilon)$ such that $|\frac{\mathcal{J}(\varpi)}{\mathcal{L}(\varpi)} - 1| < 1 - \delta, \ (0 \leq \delta < 1)$.

Theorem 3: *If* $\mathcal{L} \in \mathrm{T}^*(p, q, \tau, \nu, \epsilon)$ *and*

$$\Upsilon = 1 - \frac{\delta\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)}{\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)-(1-\epsilon)}, \quad (23)$$

then $N_{n,\delta}(\mathcal{L}) \subset \mathrm{T}^*_{\Upsilon}(p, q, \tau, \epsilon, \nu)$.

Proof: Let $\mathcal{J} \in N_{n,\delta}(\mathcal{L})$, then it is gotten from 21 that

$$\sum_{n=p}^{\infty} n\left|d_n - b_n\right| \leq \delta,$$

implying the inequality of the coefficient

$$\sum_{n=p}^{\infty}\left|d_n-b_n\right|\leq\delta,\ n\in\mathbb{N}$$

Given that $\mathcal{L}\in \mathrm{T}^{*}(p,q,\tau,\nu,\epsilon)$, Corollary 1 has it from the Eq. (9)

$$\sum_{n=p}^{\infty}b_n\leq\frac{(1-\epsilon)}{\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)}$$

Such that

$$\left|\frac{\mathcal{J}(\varpi)}{\mathcal{L}(\varpi)}-1\right|<\frac{\sum_{n=p}^{\infty}|d_n-b_n|}{1-\sum_{n=p}^{\infty}b_n}\leq\frac{\delta\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)}{\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)-(1-\epsilon)}=1-\Upsilon$$

Therefore, according to Definition 1, $\mathcal{J}\in \mathrm{T}_{\Upsilon}^{*}(p,q,\tau,\nu,\epsilon)$ for Υ provided by Eq. (23). ■

Theorem 4: *If*

$$\delta=\frac{p(1-\epsilon)}{\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)},\tag{24}$$

then $\mathrm{T}^{*}(p,q,\tau,\nu,\epsilon)\subset N_{n,\delta}(e)$.

Proof: Let $\mathcal{J}\in \mathrm{T}^{*}(p,q,\tau,\nu,\epsilon)$ then by Eq. (9) in Corollary 1

$$\sum_{n=p}^{\infty}d_n\leq\frac{(1-\epsilon)}{\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)},$$

and by the Eq. (10) in Corollary 1

$$\sum_{n=p}^{\infty}nd_n\ \leq\frac{p(1-\epsilon)}{\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)}=\delta.$$

Thus by 22 $\mathcal{J}\in N_{n,\delta}(e)$ and the proof is complete. ■

Modified coefficient bounds for $\mathrm{T}^{*}(p,q,\tau,\nu,\epsilon)$ by integral operator

The integral operator with$(0<u\leq 1, 0<c<\infty)$, defined as

$$F(\varpi)=(c-p+1)\int_0^1 u^c\mathcal{J}(u\varpi)\,du,\tag{25}$$

will be acted in this section on Eq. (1), therefore giving

$$\begin{aligned}F(\varpi)&=(c-p+1)\int_0^1 u^c\left(\varpi^{-p}+\sum_{n=p}^{\infty}d_n\varpi^n\right)du=(c-p+1)\int_0^1\left(u^{c-p}\varpi^{-p}+\sum_{n=p}^{\infty}d_nu^{n+c}\varpi^n\right)du\\&=\varpi^{-p}+\sum_{n=p}^{\infty}h_n\varpi^n,\end{aligned}\tag{26}$$

where $h_n=(\frac{c-p+1}{c+n+1})d_n$.

Currently, if $\mathcal{J}(\varpi)$ belongs to the $T^*(p, q, \tau, \nu, \epsilon)$, then $F(\varpi)$ belongs to a new subclass $T^*(p, q, \tau, \nu, \epsilon, c)$, and the coefficient bounds for this new subclass use the formula

$$\sum_{n=p}^{\infty} h_n \leq \frac{(c-p+1)(1-\epsilon)}{(c+p+1)\left(\left[(1+\tau)\left(\frac{p!}{(p-q)!}+\frac{p!}{(p-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_p(\nu)\right)}, \tag{27}$$

And the Eq. (27) satisfied the sharpness for $(n = p,\ p+1, p+2, \ldots)$.

$$F(\varpi) = \varpi^{-p} + \frac{(c-p+1)(1-\epsilon)}{(c+n+1)\left(\left[(1+\tau)\left(\frac{n!}{(n-q)!}+\frac{n!}{(n-(q+1))!}\right)+(\tau-\epsilon)\right]\pi_n(\nu)\right)}\varpi^n. \tag{28}$$

Results and discussion

Computational results

In Table 1, a comparison will be made between the coefficient bounds in Eq. (12) for subclass $T^*(p, q, \tau, \nu, \epsilon)$ with the coefficient bounds of Eq. (28) for subclass $T^*(p, q, \tau, \nu, \epsilon, c)$.

Table 1. Compare the values of the two subclasses.

n	2	3	4	5	6
d_n	−0.3076923077	0.2352941176	−0.0846560847	0.0177777778	−0.0024521073
h_n	−0.0615384615	0.0392156863	−0.0120937264	0.0022222222	−0.0002724564

n	7	8	9	10	11	12
d_n	0.0002394314	−0.0000174324	0.0000009832	−0.0000000442	0.0000000016	0.0000000000
h_n	0.0000239431	−0.0000015848	0.0000000819	−0.0000000034	0.0000000001	0.0000000000

The values of (d_n) derived from the influence of the Bessel function on Eq. (1) in the subclass $T^*(p, q, \tau, \nu, \epsilon)$, are found to alternate based on the findings above, which were computed by entering $(p = 2,\ q = 0, \nu = 0, \tau = 1, c = 2\&c > p-1)$. This shows that the coefficients have changed significantly from positive to alternating, demonstrating how the Bessel function effects the behavior of Eq. (1) in $T^*(p, q, \tau, \nu, \epsilon)$. The integral operator was used to improve the results in order to remove these oscillation values and return the function to its normal state. The values that were obtained were (h_n), which is also alternating but better than the value of (d_n). A straightforward comparison of the two sets of data shows that the function in $T^*(p, q, \tau, \nu, \epsilon, c)$ behaves better than the function in $T^*(p, q, \tau, \nu, \epsilon)$. Accordingly, it can be said that the integral operator has a positive effect and provides the best coefficient bounds when compared to the Bessel function.

Linear fractional transformation

Based on the results in Table 1, Eq. (26) can be written as follows:

$$F(\varpi) = \varpi^{-2} - 0.0615384615\varpi^2 + 0.0392156863\varpi^3 - 0.0120937264\varpi^4 + 0.0022222222\varpi^5 - 0.0002724564\varpi^6$$
$$+0.0000239431\varpi^7 - 0.0000015848\varpi^8 + 0.0000000819\varpi^9 - 0.0000000034\varpi^{10} + 0.0000000001\varpi^{11}. \tag{29}$$

If the points $\varpi_1 = \frac{1}{2},\ \varpi_2 = 0, \varpi_3 = -\frac{1}{3}$ are substituted into Eq. (29), the values will be obtained

$$w_1 = F\left(\frac{1}{2}\right) = 3.9889 \cong 4,\ w_2 = F(0) = \infty,\ w_3 = F\left(-\frac{1}{3}\right) = 8.9870 \cong 9,$$

Now, for $a, d, c, d \in \mathbb{C}$ & $ad - bc \in 0$, the linear fractional transformation

$$w = \frac{a\varpi + b}{c\varpi + d},$$

that maps the points $(\frac{1}{2}, 0. -\frac{1}{3})$ into the points $(4, \infty, 9)$ will be found using the following implicit form with $w_4 = w$ and $\varpi_4 = \varpi$:

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(\varpi_1 - \varpi_2)(\varpi_3 - \varpi_4)}{(\varpi_2 - \varpi_3)(\varpi_4 - \varpi_1)}. \tag{30}$$

After substituting into the Eq. (30), the following is obtained

$$\frac{(4-\infty)(9-w)}{(\infty-9)(w-4)} = \frac{(\frac{1}{2}-0)(-\frac{1}{3}-\varpi)}{(0+\frac{1}{3})(\varpi-\frac{1}{2})} \Rightarrow \frac{(9-w)}{(w-4)} \lim_{w'\to 0} \frac{(4-\frac{1}{w'})}{(\frac{1}{w'}-9)} = \frac{(\frac{1}{2}-0)(-\frac{1}{3}-\varpi)}{(0+\frac{1}{3})(\varpi-\frac{1}{2})} \Rightarrow \frac{(w-9)}{(w-4)} = \frac{(-1-3\varpi)}{(2\varpi-1)} \Rightarrow$$

$$w = \frac{6\varpi - 1}{\varpi}. \tag{31}$$

Since $\varpi \in U^*$, then the objective is to find the image of the punctured open unit disk in the $uv-$plane. So, after simple calculations, Eq. (31) converts to the following form

$$\varpi = \frac{-1}{w-6}. \tag{32}$$

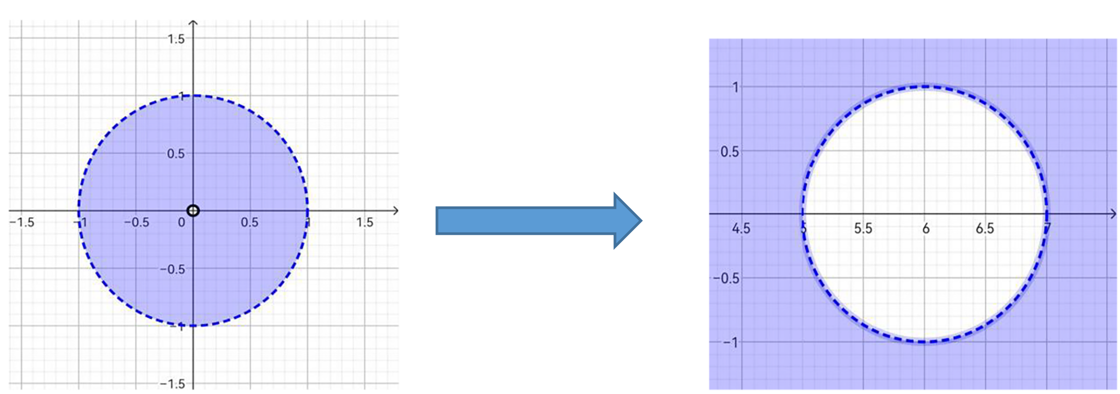
Since $0 < |\varpi| < 1$ and using Eq. (32), the derived formula is obtained

$$0 < \left|\frac{-1}{w-6}\right| < 1 \Rightarrow |w-6| > 1 \Rightarrow |(u+iv)-6| > 1 \Rightarrow |(u-6)+iv| > 1, \tag{33}$$

the Eq. (33) leads to the following result

$$(u-6)^2 + v^2 > 1. \tag{34}$$

The Eq. (34) represents the set of points that lie outside the open disk centered at $(6, 0)$ with radius 1 in the $uv-$plane(b) and this image of the punctured open unit disk in the $xy-$plane(a) under the Eq. (31), which is based on Eq. (29), as shown in the Fig. 1.



(a) **(b)**

Fig. 1. The effect of Eq. (31) on the punctured open unit disk.

Conclusion

In the current work, the coefficient bounds, product theorem, and neighborhood properties are computed using a new subclass of multivalent meromorphic function convolution with Bessel function on a punctured open unit disk. MICROSOFT EXCEL 2010 was used to determine the coefficient bounds values for this new subclass, and it was discovered that they alternated, this indicates Eq. (1) is significantly impacted by the Bessel

Function. The subclass $\tau^*(p, q, \tau, \nu, \epsilon, c)$ was constructed using the integral operator in Eq. (26) to improve the results and remove the oscillation values for the function in $\tau^*(p, q, \tau, \nu, \epsilon)$. Table 1 was used to compare the values of the two subclasses. It was found that this operator gave the function in the subclass $\tau^*(p, q, \tau, \nu, \epsilon, c)$ better values but it could not eliminate the values of the function from oscillation. Finally, based on the results in Table 1, it was concluded that the image of the punctured open unit disk U^* in the $uv-$plane is the set of all points that lie outside the open disk centered at $(6, 0)$ with radius 1, as shown in Fig. 1. In the future, further accurate programs can be utilized to compare and ascertain the impact of any operator on analytic and meromorphic functions. Since this area of study is regarded as a research gap and should be investigated in order to produce new findings.

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Authors' declaration

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Furthermore, any Figures and images that are not mine have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

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قيود المعاملات لصنف جزئي جديد من الدوال الميرومورفية المتعددة التكافؤ ذات المشتقات العليا المستمدة من دالة بيسل

علي مصدق دلفي

قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد,بغداد، العراق

الخلاصة

يهدف هذا البحث الى تقديم صنف جزئي جديد من الدوال الميرومورفية المتعددة التكافؤ ذات المشتقات العليا داخل قرص الوحدة المثقوب*U, من خلال تعميم دالة بيسل. الطريقة المستخدمة لايجاد نتائج جديدة في هذا الموضوع مذكورة في البند الثاني من هذا البحث. مبرهنة حاصل ضرب بين دالتين و خصائص الجوار تم الحصول عليها باستخدام النتائج الجديدة لقيود المعاملات الاخيرة لهذا الصنف الجديد. بالاضافة الى ذلك, يحاول هذا البحث معالجة فجوة بحثية مهمة في موضوع نظرية الدالة الهندسية, الا وهي الادوات الحسابية لقيود المعاملات المستمدة من خلال استخدام المؤثرات التفاضلية و التكاملية و تعميماتها والتي تؤدي الى معرفة سلوك الدوال و المقارنة بين النتائج لمعرفة المؤثر الافضل. لذلك في هذا البحث, و بعد الحصول على قيود المعاملات للصنف الجزئي الجديد , تم حساب قيود المعاملات عدديا و محاولة تحسينها باستخدام المؤثر التكاملي. اظهرت المقارنة العددية للسلوكين المذكورين في الجدول1 ان المؤثر التكاملي كان له تاثير ايجابي من خلال توفير افضل القيود للمعاملات بالمقارنة مع دالة بيسل, في حين كان لدالة بيسل تاثير سلبي على الدالة في هذا الموضوع . اخيرا, بناءا على النتائج في الجدول1, استنتج ان صورة U*, في المستوي uv-هي مجموعة جميع النقاط الواقعة خارج القرص المفتوح الذي مركزه (6,0) و نصف قطره 1 , كما هو موضح في الشكل 1.

الكلمات المفتاحية: قيود المعاملات، ضرب هادمرد، الدوال الميرومورفية، الدوال متعددة التكافؤ، خواص الجوار.