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RESEARCH ARTICLE

Evaluation of Home Loan Candidates Using Energy and Laplacian Energy in Hesitancy Fuzzy Zagreb Matrices

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ABSTRACT

A hesitancy fuzzy graph is a type of generalized fuzzy graphs that reflects reluctance or uncertainty in interactions by assigning a variety of potential membership values rather than a single value to each vertex and edge. It offers an adaptable framework for simulating intricate systems with ambiguous, reluctant, or imperfect input. Topological measurements that capture and characterize a graph's structural features are called Zagreb indices, and they are obtained from vertex degrees. Although Zagreb indices and graph energies have been studied for intuitionistic fuzzy graphs, fuzzy graphs, and classical graphs, no work has been done to apply these ideas to hesitancy fuzzy graphs. The framework of hesitancy fuzzy Zagreb matrices is extended in this study by investigating their energy and demonstrating the concept of Laplacian energy. We propose upper and lower bounds for the energy of the hesitancy fuzzy Zagreb matrix and its Laplacian energy in the hypothetical context of choosing a house loan applicant, which is frequently impacted by variables like the CIBIL score. Additionally, the energy and Laplacian energy of the hesitancy fuzzy Zagreb matrix are compared.

Keywords: Hesitancy fuzzy graphs, Hesitancy fuzzy first Zagreb index, Hesitancy fuzzy second Zagreb index, Hesitancy fuzzy Zagreb indices, Hesitancy fuzzy Zagreb matrices

Introduction

By inventing fuzzy set (FS) theory in 1965, Zadeh expanded on the conventional idea of set theory. According to this theory, a set is defined by elements that have different levels of membership as opposed to absolute belonging. By using a membership function, each element is given a membership value between 0 and 1.[1] As a development of classical FSs, Atanassov created intuitionistic fuzzy sets (IFSs) in 1983. By combining degrees of membership and non-membership, this method offers a more flexible way to depict ambiguity.[2] The literature has presented a number of fuzzy set extensions, such as Atanassov's IFSs, fuzzy multisets, and type-2 FSs. This study concentrates on hesitant fuzzy sets (HFSs), which are different from fuzzy multisets in their interpretation but are strictly equivalent. A description of fundamental operations for HFSs and an analysis of their relationship with IFSs, demonstrating that an IFS may be thought of as an envelope of HFSs. As an extension of FSs, HFSs were created to better handle uncertainty since real-world issues frequently entail complexity and imprecise information.[3] When Leonhard Euler solved the well-known "Königsberg bridges" issue in 1735, graph theory was born. These days, graphs are used widely in many real-world situations and are essential in computer science for modeling networks, data structures, computational models, and processes.[4–6] A. Rosenfeld introduced the idea of fuzzy Graphs (FGs) in 1975, expanding FS theory. FGs assign different membership values to vertices and edges, in contrast to conventional graphs, where they are either present or missing. This paradigm enables the representation of networks and links with varying degrees of assurance or

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connection strength.[7–9] FGs and IFSs are extended to properly represent uncertainty in an intuitionistic fuzzy graph (IFG). This approach assigns a membership degree and a non-membership degree to every vertex and edge.[10,11] In contrast to conventional FGs, these values can represent complicated circumstances with more flexibility since they are not limited to adding up to one. An improvement on FSs intended to better reflect uncertainty is represented by reluctant FSs. A novel kind of graph based on HFs is presented in this paper, together with key ideas, symbols, observations, and proofs pertaining to hesitant fuzzy graphs (HFG).[12–14] Gutman proposed in 1978 that the entire sum of the absolute eigenvalues of a graph's adjacency matrix might be used to characterize its energy. By using the spectral qualities of an FG, FG energy captures its structural features while taking uncertainty into account through the membership values of vertices and edges.[15,16] By calculating the eigenvalues of the intuitionistic fuzzy adjacency matrix while accounting for both membership and non-membership values, the energy of an IFG shows its structural characteristics.[17] By showing how far a graph's Laplacian eigenvalues deviate from its average degree, Laplacian energy (LE) draws attention to the irregularities and structural features of the graph.[18,19] LE calculates the difference between the eigenvalues of the fuzzy Laplacian and the mean fuzzy degree, which represents structural fluctuations in an IFG, in order to account for uncertainty through membership values. LE in an IFG measures the divergence of its Laplacian eigenvalues from the mean degree, reflecting structural abnormalities, by considering both membership and non-membership information.[20–22] By using vertex degrees to create graph invariants, Zagreb indices (ZIs) offer numerical measurements that characterize the structure of the graph and the relationships between its vertices. By converting chemical structures into numerical values, topological indices (TIs) eliminate the need for expensive laboratory testing. Among them, Zagreb connection indices have drawn interest because of their usefulness and robust mathematical characteristics.[23–25] In this paper, get tight limits and precise formulations for several Zagreb connection indices, especially in chain graphs and graphs of diameter two.[26–29] The outcomes demonstrate how well they work for isomer discrimination and structure-property analysis. In order to assist multi-criteria decision-making (MCDM) issues and analyze linked fuzzy networks, research on topological fuzzy indices (TFIs) is useful.[30–33] In an MCDM paradigm, the first Zagreb index (FZI) of an FG has also been used to assess the top performers in an organization.[34–37] Developing a collection of operators for mixing hesitant fuzzy data is the goal of this work. In order to develop different operations and aggregation operators for hesitant fuzzy elements, it first looks at the link between HFSs and IFSs. The paper continues with a discussion of these operators' possible uses in decision-making difficulties after delving deeper into their relationships.[38] For intuitionistic fuzzy directed graphs (IFDGs), the LE has been defined by generalizing the notion of LE in FGs, and corresponding boundary conditions have been established.[39]

Research gap and motivation of the study: While ZIs and graph energy concepts have been thoroughly studied for IFG, FGs, and classical graphs, their application to HFG is still unexplored. Specifically, no previous research has defined or analyzed hesitancy fuzzy Zagreb matrices (HFZMs), nor has it looked at the corresponding energy and LE measures in this context. A major gap in the current literature is the lack of corresponding Zagreb-based energy measures for HFGs, which offer a richer and more flexible representation of uncertainty by permitting multiple possible membership values. The need to simulate real-world decision-making issues marked by uncertainty, hesitation, and insufficient information is what spurred this study. Such hesitation is frequently not adequately captured by IFG models or traditional FG models. A more realistic framework for depicting uncertain interactions is provided by HFGs, especially in decision-support systems. This work attempts to provide useful structural measures that improve decision analysis and comparative evaluation in complex uncertain environments by introducing the concepts of energy and LE of HFZMs.

Novelty and Contribution: This work establishes the first and second ZIs for HFGs and HFZM of HFGs for the first time. It also defines the energy and LE of HFZM and derives the upper and lower bounds for the energy of HFZM (E(HFZM)) and Laplacian energy of HFZM (LE(HFZM)). Furthermore, choosing a qualified applicant for a home loan is a decision-making (DM) problem to which the suggested measures are applied. Through the provision of a useful application framework and a fresh theoretical underpinning, these contributions advance the field of HFG research.

Some of the symbols utilized in this manuscript are shown in Table 1.

Intuitionistic fuzzy set[2]

Each element in an IFS is given a degree of membership and a degree of non-membership, with the remaining portion implicitly denoting hesitation, in order to characterize uncertainty. When two complementary degrees are sufficient to characterize uncertainty, this framework works well.

Table 1. Symbols and their descriptions.

Parameters	Description
$\tilde{\mu}(v_i)$	Truthfulness of V_i
$\tilde{\gamma}(v_i)$	Falsehood of V_i
$\tilde{\beta}(v_i)$	Abstention of V_i
HFG	Hesitancy fuzzy graph
HFZIs	Hesitancy fuzzy Zagreb indices
$HFM_1(G)$	Hesitancy fuzzy first Zagreb index
$HFM_2(G)$	Hesitancy fuzzy second Zagreb index
HFZM	Hesitancy fuzzy Zagreb matrix
E(HFZM)	Energy of hesitancy fuzzy Zagreb matrix
LE(HFZM)	Laplacian energy of hesitancy fuzzy Zagreb matrix

Hesitancy fuzzy set[3]

An HFS allows an element to have multiple possible membership values, thereby explicitly simulating hesitation. This approach more accurately conveys the decision-maker's indecision or assessment variability rather than reducing uncertainty to a single membership degree. Consequently, HFSs provide greater flexibility when representing situations where it is difficult to determine exact membership information.

Hesitancy fuzzy graph[12]

The form of a hesitancy fuzzy graph is G = (V, E), where $V = \{v_1, v_2, \ldots, v_n\}$. Three functions are associated with each vertex.

$\tilde{\mu} : V \to [0, 1]$, representing truthfulness.
$\tilde{\gamma} : V \to [0, 1]$, representing falsehood.
$\tilde{\beta} : V \to [0, 1]$, representing abstention. These functions meet the specified condition $\tilde{\mu}(v_i) + \tilde{\gamma}(v_i) + \tilde{\beta}(v_i) = 1$.

A set $E \subseteq V \times V$ denotes a collection of edges, with each edge $(V_i, V_j) \in$ E distinguished by specific characteristics: $\tilde{\mu} : V \times V \to [0, 1]$, indicating truthfulness.

$\tilde{\gamma} : V \times V \to [0, 1]$, indicating the falsehood.
$\tilde{\beta} : V \times V \to [0, 1]$, indicating abstention. The required functions need to adhere to the following specifications:

$$\tilde{\mu}\left(v_i, v_j\right) \leq \min\left[\tilde{\mu}\left(v_i\right), \tilde{\mu}\left(v_j\right)\right]$$

$$\tilde{\gamma}\left(v_i, v_j\right) \leq \max\left[\tilde{\gamma}\left(v_i\right), \tilde{\gamma}\left(v_j\right)\right]$$

$$\tilde{\beta}\left(v_i, v_j\right) \leq \min\left[\tilde{\beta}\left(v_i\right), \tilde{\beta}\left(v_j\right)\right]$$

and $0 \leq \tilde{\mu}(v_i, v_j) + \tilde{\gamma}(v_i, v_j) + \tilde{\beta}(v_i, v_j) \leq 1$ (see Fig. 1).

Example 1:

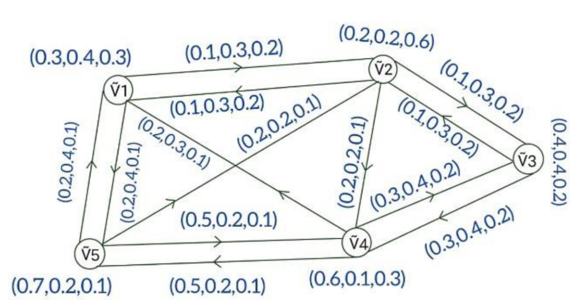

Fig. 1. Hesitancy fuzzy graph.

Methodology

Hesitancy Fuzzy Zagreb Indices (HFZIs)

Many well-established TIs in a crisp graph are frequently ineffective in resolving practical issues. The extension of such indices to FGs is necessary to get around this. Of them, the ZIs is one of the most popular degree-based indices, with important uses in chemistry and mathematics. A key component of FG theory is vertex strength, and fuzzy ZIs may be used to quantify edge connections, offering a way to assess the FG's overall strengthening. By combining each vertex's degrees of truthfulness, falsehood, and abstention with its membership values, HFZIs assist in determining the links between nearby vertices.

Definition 1 (Hesitancy fuzzy first Zagreb index)**:** Let G = (V, E) be an FG with hesitation. Next, the first ZI

$(HFM_1(G) = (HFM_1(G))_t, (HFM_1(G))_f, (HFM_1(G))_a)$of HFG is defined by $HFM_1(G)$

$$= \sum_{v_iv_j \in E(G)} [\tilde{\mu}(v_i)\, d_{\tilde{\mu}}(v_i) + \tilde{\mu}(v_j)\, d_{\tilde{\mu}}(v_j), \tilde{\gamma}(v_i)\, d_{\tilde{\gamma}}(v_i) + \tilde{\gamma}(v_j)\, d_{\tilde{\gamma}}(v_j), \tilde{\beta}(v_i)\, d_{\tilde{\beta}}(v_i) + \tilde{\beta}(v_j)\, d_{\tilde{\beta}}(v_j)]$$

Definition 2 (Hesitancy fuzzy second Zagreb index)**:** Let
G = (V, E) be an FG with hesitation. Next, the second ZI $(HFM_2(G)$

$= (HFM_2(G))_t, (HFM_2(G))_f, (HFM_2(G))_a)$ of HFG is defined by $HFM_2(G)$

$$= \sum_{v_iv_j \in E(G)} [\tilde{\mu}(v_i)\, d_{\tilde{\mu}}(v_i)\, \tilde{\mu}(v_j)\, d_{\tilde{\mu}}(v_j), \tilde{\gamma}(v_i)\, d_{\tilde{\gamma}}(v_i)\, \tilde{\gamma}(v_j)\, d_{\tilde{\gamma}}(v_j), \tilde{\beta}(v_i)\, d_{\tilde{\beta}}(v_i)\, \tilde{\beta}(v_j)\, d_{\tilde{\beta}}(v_j)]$$

Definition 3: A HFG G = (V, E) is examined. The HFZM is therefore described as

$$HFZM = \left([HFZM_{\tilde{\mu}}]_{ij}, [HFZM_{\tilde{\gamma}}]_{ij}, \left[HFZM_{\tilde{\beta}}\right]_{ij}\right),$$

in which

$$[HFZM_{\tilde{\mu}}]_{ij} = \begin{cases} \tilde{\mu}(v_i)\, d_{\tilde{\mu}}(v_i) + \tilde{\mu}(v_j)\, d_{\tilde{\mu}}(v_j), & v_iv_j \in E; \\ 0, & v_iv_j \notin E; \\ 0, & v_i = v_j. \end{cases}$$

$$[HFZM_{\tilde{\gamma}}]_{ij} = \begin{cases} \tilde{\gamma}(v_i)\, d_{\tilde{\gamma}}(v_i) + \tilde{\gamma}(v_j)\, d_{\tilde{\gamma}}(v_j), & v_iv_j \in E; \\ 0, & v_iv_j \notin E; \\ 0, & v_i = v_j. \end{cases}$$

$$\left[HFZM_{\tilde{\beta}}\right]_{ij} = \begin{cases} \tilde{\beta}(v_i)\, d_{\tilde{\beta}}(v_i) + \tilde{\beta}(v_j)\, d_{\tilde{\beta}}(v_j), & v_iv_j \in E; \\ 0, & v_iv_j \notin E; \\ 0, & v_i = v_j. \end{cases}$$

Self-loops at vertices are represented by diagonal elements in the HFZM. There is no interaction between a vertex and itself because the HFGs under consideration are simple graphs without self-loops. Furthermore, degree interactions between different vertices are used to define ZIs. In order to maintain consistency with the underlying graph structure and the theoretical underpinnings of ZIs, the diagonal entries of the HFZM are set to zero.

Definition 4: Let G = (V, E, $\tilde{\mu}$, $\tilde{\gamma}$, $\tilde{\beta}$) be an HFZM and let $\widetilde{\eta_i}, \widetilde{\xi_i}, \widetilde{\zeta_i}$ be a set of eigenvalues of a matrix of adjacency A(HFZM) of HFZM then the energy of HFZM is defined as

$$E(HFZM) = \left(E(A_{\tilde{\mu}}(HFZM)), E(A_{\tilde{\gamma}}(HFZM)), E\left(A_{\tilde{\beta}}(HFZM)\right)\right)$$
$$= \left(\sum_{i=1}^{n} |\widetilde{\eta_i}|, \sum_{i=1}^{n} |\widetilde{\epsilon_i}|, \sum_{i=1}^{n} |\widetilde{\zeta_i}|\right),$$

is the energy of truthfulness element, falsehood element, and abstention element.

The HFZM is not always symmetric due to the possibility of asymmetric hesitant membership values assigned to edges. Eigenvalues may therefore be real or complex. On the other hand, when the HFZM is symmetric, all eigenvalues are real.

Main results

Here, the LE of the HFZM is provided together with a number of upper and lower bounds. Examples that provide perspective on the calculated bounds are then utilized to show that the results are valid.

Definition 5: Assuming that G = (V, E, $\tilde{\mu}$, $\tilde{\gamma}$, $\tilde{\beta}$) is an HFZM, the Laplacian matrix of an HFZM can be expressed as

$$L\,(HFZM) = \left(L_{\tilde{\mu}}\,(HFZM)\,, L_{\tilde{\gamma}}\,(HFZM)\,, L_{\tilde{\beta}}\,(HFZM)\right)$$

Where, $L_{\tilde{\mu}}(HFZM) = D_{\tilde{\mu}}(HFZM) - A_{\tilde{\mu}}(HFZM)$,

$$L_{\tilde{\gamma}}\,(HFZM) = D_{\tilde{\gamma}}\,(HFZM) - A_{\tilde{\gamma}}\,(HFZM)\,,$$

$$L_{\tilde{\beta}}\,(HFZM) = D_{\tilde{\beta}}\,(HFZM) - A_{\tilde{\beta}}\,(HFZM)$$

Here $D_{\tilde{\mu}}(HFZM)$ be the truthfulness degree matrix, $D_{\tilde{\gamma}}(HFZM)$ be the falsehood degree matrix, $D_{\tilde{\beta}}(HFZM)$ be the abstention degree matrix. $A_{\tilde{\mu}}(HFZM)$ is the truthfulness element matrix, $A_{\tilde{\gamma}}(HFZM)$ is the falsehood element matrix, $A_{\tilde{\beta}}(HFZM)$ is the abstention element matrix.

Theorem 1: *If $L_{\tilde{\mu}}(HFZM)'s$ may be real or complex eigenvalues are $\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \widetilde{\eta_n}$, $L_{\tilde{\gamma}}(HFZM)'s$ may be real or complex eigenvalues are $\tilde{\epsilon}_1, \tilde{\epsilon}_2, \ldots, \tilde{\epsilon}_n$ and $L_{\tilde{\beta}}(HFZM)'s$ may be real or complex eigenvalues are $\tilde{\zeta}_1, \tilde{\zeta}_2, \ldots, \tilde{\zeta}_n$, then i) $\sum_{i=1}^{n} \tilde{\eta}_i = 2\sum_{1\le i<j\le n}[\widetilde{\mu_{ij}} + \widetilde{\mu_{ji}}]$, ii) $\sum_{i=1}^{n} \widetilde{\eta_i^2} = 2\sum_{1\le i<j\le n} \widetilde{\mu_{ij}}\widetilde{\mu_{ji}} + \sum_{i=1}^{n} (d_{\tilde{\mu}}(v_i))^2$, iii) $\sum_{i=1}^{n} \tilde{\epsilon}_i$ $=2\sum_{1\le i<j\le n}[\widetilde{\gamma_{ij}} + \widetilde{\gamma_{ji}}]$, iv) $\sum_{i=1}^{n} \widetilde{\epsilon_i^2} = 2\sum_{1\le i<j\le n} \widetilde{\gamma_{ij}}\widetilde{\gamma_{ji}} + \sum_{i=1}^{n} \left(d_{\tilde{\gamma}}\,(v_i)\right)^2$ v) $\sum_{i=1}^{n} \tilde{\zeta}_i = 2\sum_{1\le i<j\le n}[\widetilde{\beta_{ij}} + \widetilde{\beta_{ji}}]$, vi) $\sum_{i=1}^{n} \widetilde{\zeta_i^2} = 2\sum_{1\le i<j\le n} \widetilde{\beta_{ij}}\widetilde{\beta_{ji}} + \sum_{i=1}^{n} (d_{\tilde{\beta}}(v_i))^2$.*

Proof:

i) The trace of $L_{\tilde{\mu}}(HFZM)$ is identical to the sum of its eigenvalues, as is well known i.e., $\sum_{i=1}^{n} \tilde{\eta}_i = \text{trace}\,(L_{\tilde{\mu}}(HFZM)) = \sum_{i=1}^{n} d_{\tilde{\mu}}(v_i) = 2\sum_{1\le i<j\le n}[\widetilde{\mu_{ij}} + \widetilde{\mu_{ji}}]$
ii) Additionally, the trace of $(L_{\tilde{\mu}}(HFZM))^2$ is the sum of the squares of the eigenvalues of $L_{\tilde{\mu}}(HFZM)$

$$\text{i.e., } \sum_{i=1}^{n} \widetilde{\eta_i^2} = \text{trace}\left(L_{\tilde{\mu}}\,(HFZM)\right)^2$$

$$= \left(d_{\tilde{\mu}}\,(v_1)\right)^2 + \widetilde{\mu_{12}}\widetilde{\mu_{21}} + \widetilde{\mu_{13}}\widetilde{\mu_{31}} + \cdots + \widetilde{\mu_{1n}}\widetilde{\mu_{n1}} + \widetilde{\mu_{21}}\widetilde{\mu_{12}} + \left(d_{\tilde{\mu}}\,(v_2)\right)^2 + \cdots$$

$$+ \widetilde{\mu_{2n}}\widetilde{\mu_{n2}} + \cdots + \widetilde{\mu_{n1}}\widetilde{\mu_{1n}} + \widetilde{\mu_{n2}}\widetilde{\mu_{2n}} + \cdots + \left(d_{\tilde{\mu}}\,(v_n)\right)^2$$

$$\sum_{i=1}^{n} \widetilde{\eta_i^2} = 2\sum_{1\le i<j\le n} \widetilde{\mu_{ij}}\widetilde{\mu_{ji}} + \sum_{i=1}^{n} \left(d_{\tilde{\mu}}\,(v_i)\right)^2$$

The same is true for iii), iv), v), vi).

Example 2: *Numerical illustration of Theorem 1 using Fig. 1, For the HFG shown in Fig. 1, the eigenvalues of the $L_{\tilde{\mu}}(HFZM)$ are computed. We obtain, i) $\sum_{i=1}^{n} \tilde{\eta}_i = 32.48$, $2\sum_{1\le i<j\le n}[\widetilde{\mu_{ij}} + \widetilde{\mu_{ji}}] = 32.48$. ii) $\sum_{i=1}^{n} \widetilde{\eta_i^2} = 264.31$,*

$2\sum_{1\le i<j\le n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\sum_{i=1}^{n}(d_{\tilde{\mu}}(v_i))^2=264.31$. Thus, the result stated in Theorem 1, i) and ii) is numerically verified for the given example. Similar numerical verification can be carried out for parts iii), iv), v), vi) of Theorem 1.

Definition 6: When $G=(V,E,\tilde{\mu},\tilde{\gamma},\tilde{\beta})$ is an HFZM, the LE of an HFZM is expressed as

$$\text{LE}\,(\text{HFZM})=\left(LE_{\tilde{\mu}}\,(HFZM)\,,LE_{\tilde{\gamma}}\,(HFZM)\,,\;LE_{\tilde{\beta}}\,(HFZM)\right)$$

$$=\left(\sum_{i=1}^{n}|\widetilde{\xi_i}|\,,\sum_{i=1}^{n}|\widetilde{\delta_i}|\,,\sum_{i=1}^{n}|\widetilde{\lambda_i}|\right)$$

Where, $\sum_{i=1}^{n}|\widetilde{\xi_i}|=\sum_{i=1}^{n}\left|\widetilde{\eta_i}-2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right|$

$$\sum_{i=1}^{n}|\widetilde{\delta_i}|=\sum_{i=1}^{n}\left|\widetilde{\epsilon_i}-2\sum_{1\le i<j\le n}\frac{(\widetilde{\gamma_{ij}}+\widetilde{\gamma_{ji}})}{n}\right|$$

$$\sum_{i=1}^{n}|\widetilde{\lambda_i}|=\sum_{i=1}^{n}\left|\widetilde{\zeta_i}-2\sum_{1\le i<j\le n}\frac{(\widetilde{\beta_{ij}}+\widetilde{\beta_{ji}})}{n}\right|$$

Theorem 2: *If $L_{\tilde{\mu}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\xi_1},\widetilde{\xi_2},\ldots,\widetilde{\xi_n}$, $L_{\tilde{\gamma}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\delta_1},\widetilde{\delta_2},\ldots,\widetilde{\delta_n}$ and $L_{\tilde{\beta}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\lambda_1},\widetilde{\lambda_2},\ldots,\widetilde{\lambda_n}$, then i) $\sum_{i=1}^{n}\widetilde{\xi_i}=0$, ii) $\sum_{i=1}^{n}\widetilde{\xi_i^2}=2Q$, iii) $\sum_{i=1}^{n}\widetilde{\delta_i}=0$, iv) $\sum_{i=1}^{n}\widetilde{\delta_i^2}=2R$, v) $\sum_{i=1}^{n}\widetilde{\lambda_i}=0$, vi) $\sum_{i=1}^{n}\widetilde{\lambda_i^2}=2S$, where,*

$$Q=\sum_{1\le i<j\le n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}\,(v_i)\;-\;2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2$$

$$R=\sum_{1\le i<j\le n}\widetilde{\gamma_{ij}}\widetilde{\gamma_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\gamma}}\,(v_i)-2\sum_{1\le i<j\le n}\frac{(\widetilde{\gamma_{ij}}+\widetilde{\gamma_{ji}})}{n}\right)^2$$

$$S=\sum_{1\le i<j\le n}\widetilde{\beta_{ij}}\widetilde{\beta_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\beta}}\,(v_i)-2\sum_{1\le i<j\le n}\frac{(\widetilde{\beta_{ij}}+\widetilde{\beta_{ji}})}{n}\right)^2$$

Proof:

i) According to Definition 6,

$$\sum_{i=1}^{n}\widetilde{\xi_i}=\sum_{i=1}^{n}\left(\widetilde{\eta_i}-2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)$$

$$=\sum_{i=1}^{n}\eta_i-2n\left(\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)$$

$$=2\sum_{1\le i<j\le n}(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})-2\sum_{1\le i<j\le n}(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})$$

$$\sum_{i=1}^{n}\widetilde{\xi_i}=0$$

ii) According to Definition 6,

$$\sum_{i=1}^{n}\widetilde{\xi}_i^2=\sum_{i=1}^{n}\left(\widetilde{\eta}_i-2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2$$

$$=\sum_{i=1}^{n}\left(\widetilde{\eta}_i^2+\left(2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2-2\widetilde{\eta}_i\left(2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)\right)$$

$$=\sum_{i=1}^{n}\widetilde{\eta_i}^2+\sum_{i=1}^{n}\left(2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2-2\sum_{i=1}^{n}\widetilde{\eta}_i\left(2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)$$

$$=2\sum_{1\le i<j\le n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\sum_{i=1}^{n}(d_{\tilde{\mu}}(v_i))^2+\sum_{i=1}^{n}\left(2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2$$

$$-2\sum_{i=1}^{n}d_{\tilde{\mu}}(v_i)\left(2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)$$

$$=2\sum_{1\le i<j\le n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}(v_i)\;-\;2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2$$

$$\sum_{i=1}^{n}\widetilde{\xi}_i^2=2\mathrm{Q}$$

The same is true for iii), iv), v) and vi).

Example 3: *Numerical illustration of Theorem 2 using Fig. 1, For the HFG shown in Fig. 1, the eigenvalues of the $L_{\tilde{\mu}}(HFZM)$ are evaluated. We obtain, i) $\sum_{i=1}^{n}\widetilde{\xi}_i = 0$, ii) $\sum_{i=1}^{n}\widetilde{\xi}_i^2 = 53.32$, $2Q = 53.32$. Thus, the result stated in Theorem 1, i) and ii) is numerically verified for the given example. Similar numerical verification can be carried out for parts iii), iv), v), vi) of Theorem 2.*

Theorem 3: *Consider $G=(V,E,\tilde{\mu},\tilde{\gamma},\tilde{\beta})$ an HFZM with n vertices, if $L_{\tilde{\mu}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\xi_1},\widetilde{\xi_2},\ldots,\widetilde{\xi_n}$, $L_{\tilde{\gamma}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\delta_1},\widetilde{\delta_2},\ldots,\widetilde{\delta_n}$ and $L_{\tilde{\beta}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\lambda_1},\widetilde{\lambda_2},\ldots,\widetilde{\lambda_n}$, then*

i) $LE_{\tilde{\mu}}(HFZM)\le\sqrt{2n\left(\sum_{1\le i<j\le n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}(v_i)\;-2\sum_{1\le i<j\le n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\right)^2\right)}$

ii) $LE_{\tilde{\gamma}}(HFZM)\le\sqrt{2n\left(\sum_{1\le i<j\le n}\widetilde{\gamma_{ij}}\widetilde{\gamma_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\gamma}}(v_i)-2\sum_{1\le i<j\le n}\frac{(\widetilde{\gamma_{ij}}+\widetilde{\gamma_{ji}})}{n}\right)^2\right)}$

iii) $LE_{\tilde{\beta}}(HFZM)\le\sqrt{2n\left(\sum_{1\le i<j\le n}\widetilde{\beta_{ij}}\widetilde{\beta_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\beta}}(v_i)-2\sum_{1\le i<j\le n}\frac{(\widetilde{\beta_{ij}}+\widetilde{\beta_{ji}})}{n}\right)^2\right)}$

Proof: In the context of the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n}a_i\widetilde{b_i}\right)\le\left(\sum_{i=1}^{n}\widetilde{a_i}^2\right)\left(\sum_{i=1}^{n}\widetilde{b_i}^2\right)$$

Put $\widetilde{a_i}=1,\widetilde{b_i}=|\widetilde{\xi_i}|$ gives,

$$\left(\sum_{i=1}^{n}|\widetilde{\xi_i}|\right)^2\le\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\widetilde{\xi}_i^2\right)$$

Theorem 2 gives us,

$$\left(LE_{\tilde{\mu}}\left(HFZM\right)\right)^{2} \leq \mathrm{n}\left(2\left(\sum_{1\leq i<j\leq n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}\left(v_i\right)\ -2\sum_{1\leq i<j\leq n}\frac{\left(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}}\right)}{n}\right)^{2}\right)\right)$$

$$LE_{\tilde{\mu}}\left(HFZM\right) \leq \sqrt{2n\left(\sum_{1\leq i<j\leq n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}\left(v_i\right)\ -2\sum_{1\leq i<j\leq n}\frac{\left(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}}\right)}{n}\right)^{2}\right)}$$

It may also demonstrate the ii) and iii) inequalities.

Example 4: *Numerical illustration of Theorem 3 using Fig. 1, For the HFG shown in Fig. 1, the eigenvalues of the $L_{\tilde{\mu}}(HFZM)$ are evaluated. The computed results satisfy i) $LE_{\tilde{\mu}}(HFZM) =$* 14.2985, $\sqrt{2n(\sum_{1\leq i<j\leq n}\widetilde{\mu_{ij}}\widetilde{\mu_{ji}}+\frac{1}{2}\sum_{i=1}^{n}(d_{\tilde{\mu}}(v_i)\ -2\sum_{1\leq i<j\leq n}\frac{(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}})}{n}\ )^{2})}$ *= 16.3275. Similar numerical validation may be performed for the remaining parts ii) and iii) of Theorem 3.*

Theorem 4: *Let $G = (V, E, \tilde{\mu}, \tilde{\gamma}, \tilde{\beta})$ be a comprised n vertices, if $L_{\tilde{\mu}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\xi_1}, \widetilde{\xi_2}, \ldots, \widetilde{\xi_n}$, $L_{\tilde{\gamma}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\delta_1}, \widetilde{\delta_2}, \ldots, \widetilde{\delta_n}$ and $L_{\tilde{\beta}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_n}$, then*

i) $LE_{\tilde{\mu}}(HFZM) \leq \sqrt{n(\sum_{i=1}^{n}|\widetilde{\xi_i}|^2)}$
ii) $LE_{\tilde{\gamma}}(HFZM) \leq \sqrt{n(\sum_{i=1}^{n}|\widetilde{\delta_i}|^2)}$
iii) $LE_{\tilde{\beta}}(HFZM) \leq \sqrt{\mathrm{n}(\sum_{\mathrm{i}=1}^{\mathrm{n}}|\widetilde{\lambda_{\mathrm{i}}}|^2)}$

Proof: The Cauchy-Schwarz inequality again puts $\widetilde{a_i} = 1, \widetilde{b_i} = |\widetilde{\xi_i}|$

$$\left(\sum_{i=1}^{n}|\widetilde{\xi_i}|\right)^{2} \leq \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\widetilde{\xi_i}^{2}\right)$$

$$\sum_{i=1}^{n}|\widetilde{\xi_i}| \leq \sqrt{\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\widetilde{\xi_i}|^{2}\right)}$$

$$\sum_{i=1}^{n}|\widetilde{\xi_i}| \leq \sqrt{n\left(\sum_{i=1}^{n}|\widetilde{\xi_i}|^{2}\right)}$$

$$LE_{\tilde{\mu}}\left(HFZM\right) \leq \sqrt{n\left(\sum_{i=1}^{n}|\widetilde{\xi_i}|^{2}\right)}$$

Likewise, it can demonstrate ii) and iii).

Example 5: *Numerical illustration of Theorem 4 using Fig. 1, For the HFG shown in Fig. 1, the $L_{\tilde{\mu}}(HFZM)$ is constructed and its eigenvalues are determined. The numerical evaluation yields i) $LE_{\tilde{\mu}}(HFZM) =$* 14.2985, $\sqrt{n(\sum_{i=1}^{n}|\widetilde{\xi_i}|^2)}$ *= 16.3274. Analogous numerical verification can be carried out for parts ii) and iii) of Theorem 4.*

Theorem 5: *Consider the HFZM $G = (V, E, \tilde{\mu}, \tilde{\gamma}, \tilde{\beta})$ to have n vertices, if $L_{\tilde{\mu}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\xi_1}, \widetilde{\xi_2}, \ldots, \widetilde{\xi_n}$, $L_{\tilde{\gamma}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\delta_1}, \widetilde{\delta_2}, \ldots, \widetilde{\delta_n}$ and $L_{\tilde{\beta}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_n}$, then*

i) $LE_{\tilde{\mu}}(HFZM) \geq \sqrt{2Q + n(n-1)(\prod_{i=1}^{n} |\widetilde{\xi_i}|)^{2/n}}$

ii) $LE_{\tilde{\gamma}}(HFZM) \geq \sqrt{2R + n(n-1)(\prod_{i=1}^{n} |\widetilde{\delta_i}|)^{2/n}}$

iii) $LE_{\tilde{\beta}}(HFZM) \geq \sqrt{2S + n(n-1)(\prod_{i=1}^{n} |\widetilde{\lambda_i}|)^{2/n}}$

Proof: It is known that,

$$\left(LE_{\tilde{\mu}}\left(HFZM\right)\right)^2 = \left(\sum_{i=1}^{n} |\widetilde{\xi_i}|\right)^2$$

$$= \sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + 2\sum_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}|$$

$$= 2\left[\sum_{1\leq i<j\leq n} \widetilde{\mu_{ij}}\widetilde{\mu_{ji}} + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}\left(v_i\right) - 2\sum_{1\leq i<j\leq n}\frac{\left(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}}\right)}{n}\right)^2\right] + \frac{2n\left(n-1\right)}{2} AM_{1\leq i<j\leq n|\widetilde{\xi_i}||\widetilde{\xi_j}|}$$

because,

$$AM_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}| \geq GM_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}|$$

gives,

$$LE_{\tilde{\mu}}(HFZM) \geq \sqrt{2\left[\sum_{1\leq i<j\leq n} \tilde{\mu}_{ij}\tilde{\mu}_{ji} + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}(v_i) - \frac{2}{n}\sum_{1\leq i<j\leq n}(\tilde{\mu}_{ij}+\tilde{\mu}_{ji})\right)^2\right] + n(n-1)GM_{1\leq i<j\leq n}|\widetilde{\xi_i}||\widetilde{\xi_j}|}$$

however,

$$GM_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}| = \left(\prod_{i=1}^{n} |\widetilde{\xi_i}|\right)^{2/n}$$

Thus, it follows that

$$LE_{\tilde{\mu}}\left(HFZM\right) \geq \sqrt{2\left[\sum_{1\leq i<j\leq n} \widetilde{\mu_{ij}}\widetilde{\mu_{ji}} + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{\mu}}\left(v_i\right) - 2\sum_{1\leq i<j\leq n}\frac{\left(\widetilde{\mu_{ij}}+\widetilde{\mu_{ji}}\right)}{n}\right)^2\right] + n\left(n-1\right)\left(\prod_{i=1}^{n}|\widetilde{\xi_i}|\right)^{2/n}}$$

$$LE_{\tilde{\mu}}\left(HFZM\right) \geq \sqrt{2Q + n\left(n-1\right)\left(\prod_{i=1}^{n}|\widetilde{\xi_i}|\right)^{2/n}}$$

Inequalities ii) and iii) can be derived using the same method.

Example 6: *Numerical illustration of Theorem 5 using Fig. 1, For the HFG shown in Fig. 1, the $L_{\tilde{\mu}}(HFZM)$ is obtained and its spectrum is analyzed. The computed values satisfy i) $LE_{\tilde{\mu}}(HFZM) = 14.2985$, $\sqrt{2Q + n(n-1)(\prod_{i=1}^{n} |\widetilde{\xi_i}|)^{2/n}} = 13.2586$. Similar numerical validation may be extended to parts ii) and iii) of Theorem 5.*

Theorem 6: *Consider an HFZM with n vertices, $G = (V, E, \tilde{\mu}, \tilde{\gamma}, \tilde{\beta})$. If $L_{\tilde{\mu}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\xi_1}, \widetilde{\xi_2}, \ldots, \widetilde{\xi_n}$, $L_{\tilde{\gamma}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\delta_1}, \widetilde{\delta_2}, \ldots, \widetilde{\delta_n}$ and $L_{\tilde{\beta}}(HFZM)'s$ may be real or complex eigenvalues are $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_n}$, then*

i) $LE_{\tilde{\mu}}(HFZM) \geq \sqrt{\sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + n(n-1)(\prod_{i=1}^{n} |\widetilde{\xi_i}|)^{2/n}}$

ii) $LE_{\tilde{\gamma}}(HFZM) \geq \sqrt{\sum_{i=1}^{n} |\widetilde{\delta_i}|^2 + n(n-1)(\prod_{i=1}^{n} |\widetilde{\delta_i}|)^{2/n}}$

iii) $LE_{\tilde{\beta}}(HFZM) \geq \sqrt{\sum_{i=1}^{n} |\widetilde{\lambda_i}|^2 + n(n-1)(\prod_{i=1}^{n} |\widetilde{\lambda_i}|)^{2/n}}$

Proof: It is known that,

$$\left(LE_{\tilde{\mu}}\left(HFZM\right)\right)^2 = \left(\sum_{i=1}^{n} |\widetilde{\xi_i}|\right)^2$$

$$= \sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + 2 \sum_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}|$$

$$= \sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + \frac{2n\left(n-1\right)}{2} AM\left\{|\widetilde{\xi_i}|\,|\widetilde{\xi_j}|\right\}$$

$$LE_{\tilde{\mu}}\left(HFZM\right) \geq \sqrt{\sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + n\left(n-1\right) GM_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}|}$$

However,

$$GM_{1\leq i<j\leq n} |\widetilde{\xi_i}|\,|\widetilde{\xi_j}| = \left(\prod_{i=1}^{n} |\widetilde{\xi_i}|\right)^{2/n}$$

$$LE_{\tilde{\mu}}\left(HFZM\right) \geq \sqrt{\sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + n\left(n-1\right)\left(\prod_{i=1}^{n} |\widetilde{\xi_i}|\right)^{2/n}}$$

The results corresponding to inequalities ii) and iii) can likewise be demonstrated.

Example 7: *Numerical illustration of Theorem 6 using Fig. 1, For the HFG shown in Fig. 1, the $L_{\tilde{\mu}}(HFZM)$ is formulated and its eigenvalues are evaluated. The numerical results obtained satisfy i)* $LE_{\tilde{\mu}}(HFZM) = 14.2985$, $\sqrt{\sum_{i=1}^{n} |\widetilde{\xi_i}|^2 + n(n-1)(\prod_{i=1}^{n} |\widetilde{\xi_i}|)^{2/n}} = 13.2586$. *Similar numerical validation may be extended to parts ii) and iii) of Theorem 6.*

Application: The selection of a candidate for a home loan is often determined based on their CIBIL score

In this section, the example of choosing a home loan candidate, often based on their CIBIL score, is used to illustrate the value of HFGs.

Home loan selection is fundamentally a problem of balancing outreach with portfolio quality; therefore, every application carries a measurable probability of default alongside the prospect of interest income. In practice, 'Eligibility' is not a single test but the synthesis of credit behavior, income stability, property quality, and regulatory compliance. A salaried professional and a self-employed trader may both qualify for the same loan quantum, yet the risk and monitoring requirements differ.

Variety of Home loans:

- Home loan for the purchase of a new house or construction of a new house
- Home loan for the purchase of a Resale house
- Home loan balance transfer
- Home extension loan
- Home improvement loan
- Composite loan for plot purchase and construction
- Loan for commercial property purchase

The process of selecting a suitable candidate for a home loan involves evaluating both the applicant's financial stability and the potential risks associated with lending.

Key aspects include a good credit score, the CIBIL score, a three-digit number typically ranging from 300-900, is one of the most important indicators used by financial institutions to evaluate the creditworthiness of an applicant for a home loan. It reflects the individual's credit history, repayment patterns, and overall financial discipline, as reported by banks and other lenders. A higher score indicates lower credit risk, thereby increasing the likelihood of loan approval on favorable terms.

In the selection process, the CIBIL score serves as an initial screening tool. Applicants with excellent scores are generally fast-tracked for approval with minimal additional scrutiny, whereas those with lower scores undergo more rigorous evaluation. The decision-making framework often categorizes candidates into different risk tiers based on score ranges, with each tier corresponding to specific eligibility conditions, interest rates, and documentation requirements.

Since a candidate's CIBIL score is frequently used to select them for a home loan, five candidates are examined $\tilde{P}_i(i = 1, 2, 3, 4, 5)$ and confer with a panel of four decision experts $\tilde{E}_j(j = 1, 2, 3, 4)$. Then the Candidate's CIBIL score is $\widetilde{P_1} = 500–599$, $\widetilde{P_2} = 800–900$, $\widetilde{P_3} = 600–699$, $\widetilde{P_4} = 550–650$, $\widetilde{P_5} = 700–799$.

Based on their experiences in the two different environments, each expert makes a different conclusion. Neutral, appropriate, improper, and based on their opinions are the three criteria.

The algorithm that describes our recommended approach is shown below:

Input: A unique set of applicants $\tilde{P} = \{\widetilde{P_1}, \widetilde{P_2}, \widetilde{P_3}, \widetilde{P_4}, \widetilde{P_5}\}$, a team of experts in bank decision makers $\tilde{E} = \{\widetilde{e_1}, \widetilde{e_2}, \widetilde{e_3}, \widetilde{e_4}\}$ in order to accomplish the target, and the formation of hesitancy fuzzy preference relations (HFPRs) $H_{\tilde{a}}, \tilde{a} = (1, 2, 3, 4)$ links for each expert.

Output: The most suitable candidate for a home loan is identified.

1. By examining four HFGs that correspond to four hesitancy preference relations, it is possible to match the designated HFGs with the HFZM, as defined in Definition 3.
2. Determine the E(HFZM) and LE(HFZM) for each $(\mathrm{HFZM})_i$, where i = 1, 2, 3, 4.
3. Find the weight vector for E(HFZM) and LE(HFZM) using these:

$$w_i = \left(\left(w_{\tilde{\mu}}\right)_i, \left(w_{\tilde{\gamma}}\right)_i, \left(w_{\tilde{\beta}}\right)_i\right)$$

i) $$w_i = \left(\frac{E(A_{\tilde{\mu}}(HFZM)_i)}{\sum_{i=1}^{n} E(A_{\tilde{\mu}}(HFZM)_i)}, \frac{E(A_{\tilde{\gamma}}(HFZM)_i)}{\sum_{i=1}^{n} E(A_{\tilde{\gamma}}(HFZM)_i)}, \frac{E(A_{\tilde{\beta}}(HFZM)_i)}{\sum_{i=1}^{n} E(A_{\tilde{\beta}}(HFZM)_i)}\right)$$

ii) $$w_i = \left(\frac{LE(A_{\tilde{\mu}}(HFZM)_i)}{\sum_{i=1}^{n} LE(A_{\tilde{\mu}}(HFZM)_i)}, \frac{LE(A_{\tilde{\gamma}}(HFZM)_i)}{\sum_{i=1}^{n} LE(A_{\tilde{\gamma}}(HFZM)_i)}, \frac{LE(A_{\tilde{\beta}}(HFZM)_i)}{\sum_{i=1}^{n} LE(A_{\tilde{\beta}}(HFZM)_i)}\right)$$

4. The hesitancy fuzzy Zagreb preference (HFZPR) matrices generated in step 2 must now be aggregated using one of the suitable aggregation operations on the multi-agent data. The HFZM, a well-known basic ordered averaging operator,[38] is employed.
5. Construct a model of the collective HFZPR's impact.
6. Plot the partial impact model associated with the collective HFZPR using constraint $\tilde{\mu}_{ij} \geq 0.5$. Locate every vertex (alternative) in the HFZPR model along with its degrees.
7. Each option is rated according to the vertex truthfulness degree.
8. Depending on the vertex's highest degree of truthfulness, the most qualified applicant is put forward.

The flowchart in Fig. 2 provides an overview of the suggested algorithm.

Decision makers express their evaluations using HFPRs, which serve as input for constructing the HFG and implementing the proposed method.

Tables 2 to 5 provide the HFPRs $H_{\tilde{a}}$, where $\tilde{a} = (1, 2, 3, 4)$.

By using the membership values of the edges, such as the degree of truthfulness, degree of falseness, and degree of abstention, to represent each expert's preference among four distinct alternatives, $(HFGs)_j$, j = (1, 2, 3, 4) based on (HFPRs)$H_{\tilde{a}}$, $\tilde{a} = (1, 2, 3, 4)$ are constructed. Figs. 3 to 6 illustrate these $(\mathrm{HFGs})_j$, j = (1, 2, 3, 4).

All HFG's $(HFZMs)_i$, i = (1, 2, 3, 4) have now been created. These matrices are based on how much the impact of a connection will be provided by the vertices in the neighborhood. Below is a list of all HFG's $(HFZM)_i$,

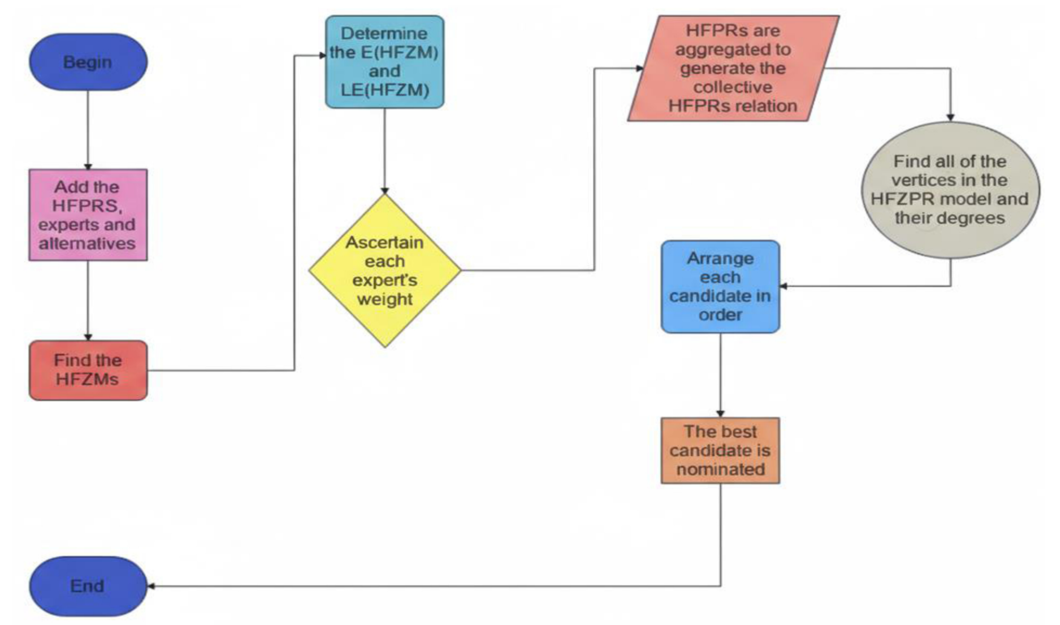

Fig. 2. An algorithm flowchart.

Table 2. HFPRs of the initial expert in decision-making.

$\widetilde{H}_1$	$\widetilde{P}_1$	$\widetilde{P}_2$	$\widetilde{P}_3$	$\widetilde{P}_4$	$\widetilde{P}_5$
$\widetilde{P}_1$	(0.56, 0.4, 0.04)	(0.43, 0.3, 0.03)	(0, 0, 0)	(0, 0, 0)	(0.56, 0.3, 0.03)
$\widetilde{P}_2$	(0.43, 0.3, 0.03)	(0.82, 0.05, 0.13)	(0.61, 0.2, 0.05)	(0,0,0)	(0, 0, 0)
$\widetilde{P}_3$	(0, 0, 0)	(0.61, 0.2, 0.05)	(0.63, 0.3, 0.07)	(0, 0, 0)	(0, 0, 0)
$\widetilde{P}_4$	(0, 0, 0)	(0.47, 0.28, 0.05)	(0.51, 0.31, 0.06)	(0.57, 0.35, 0.08)	(0.49,0.31,0.07)
$\widetilde{P}_5$	(0, 0, 0)	(0.71, 0.12, 0.07)	(0.61, 0.2, 0.05)	(0.49, 0.31, 0.07)	(0.79,0.12,0.09)

Table 3. HFPRs of the second expert in decision-making.

$\widetilde{H}_2$	$\widetilde{P}_1$	$\widetilde{P}_2$	$\widetilde{P}_3$	$\widetilde{P}_4$	$\widetilde{P}_5$
$\widetilde{P}_1$	(0.51, 0.45, 0.04)	(0.33, 0.41, 0.03)	(0, 0, 0)	(0, 0, 0)	(0.49,0.37,0.01)
$\widetilde{P}_2$	(0.33, 0.41, 0.03)	(0.89, 0.05, 0.06)	(0.63, 0.28, 0.02)	(0,0,0)	(0, 0, 0)
$\widetilde{P}_3$	(0, 0, 0)	(0.63, 0.28, 0.02)	(0.66, 0.32, 0.02)	(0, 0, 0)	(0, 0, 0)
$\widetilde{P}_4$	(0, 0, 0)	(0.53, 0.35, 0.05)	(0.50, 0.35, 0.01)	(0.56, 0.37, 0.07)	(0.53,0.34,0.04)
$\widetilde{P}_5$	(0, 0, 0)	(0.71, 0.20, 0.04)	(0.61, 0.29, 0.02)	(0.53, 0.34, 0.04)	(0.73,0.22,0.05)

Table 4. HFPRs of the third expert in decision-making.

$\widetilde{H}_3$	$\widetilde{P}_1$	$\widetilde{P}_2$	$\widetilde{P}_3$	$\widetilde{P}_4$	$\widetilde{P}_5$
$\widetilde{P}_1$	(0.52, 0.41, 0.07)	(0.47, 0.39, 0.02)	(0, 0, 0)	(0, 0, 0)	(0.47,0.35,0.04)
$\widetilde{P}_2$	(0.47, 0.39, 0.02)	(0.86, 0.03, 0.11)	(0.56, 0.29, 0.01)	(0,0,0)	(0, 0, 0)
$\widetilde{P}_3$	(0, 0, 0)	(0.56, 0.29, 0.01)	(0.64, 0.31, 0.05)	(0, 0, 0)	(0, 0, 0)
$\widetilde{P}_4$	(0, 0, 0)	(0.49, 0.28, 0.01)	(0.53, 0.33, 0.04)	(0.57, 0.38, 0.05)	(0.52,0.35,0.02)
$\widetilde{P}_5$	(0, 0, 0)	(0.72, 0.11, 0.1)	(0.61, 0.27, 0.01)	(0.52, 0.35, 0.02)	(0.76,0.12,0.12)

Table 5. HFPRs of the fourth expert in decision-making.

$\widetilde{H}_4$	$\widetilde{P}_1$	$\widetilde{P}_2$	$\widetilde{P}_3$	$\widetilde{P}_4$	$\widetilde{P}_5$
$\widetilde{P}_1$	(0.57, 0.36, 0.07)	(0.49, 0.3, 0.04)	(0, 0, 0)	(0, 0, 0)	(0.55,0.31,0.06)
$\widetilde{P}_2$	(0.49, 0.30, 0.04)	(0.85, 0.03, 0.12)	(0.63, 0.19, 0.03)	(0,0,0)	(0, 0, 0)
$\widetilde{P}_3$	(0, 0, 0)	(0.63, 0.19, 0.03)	(0.68, 0.23, 0.09)	(0, 0, 0)	(0, 0, 0)
$\widetilde{P}_4$	(0, 0, 0)	(0.61, 0.25, 0.07)	(0.60, 0.21, 0.06)	(0.63, 0.28, 0.09)	(0.57,0.23,0.03)
$\widetilde{P}_5$	(0, 0, 0)	(0.73, 0.1, 0.11)	(0.61, 0.22, 0.07)	(0.57, 0.23, 0.03)	(0.77,0.1,0.13)

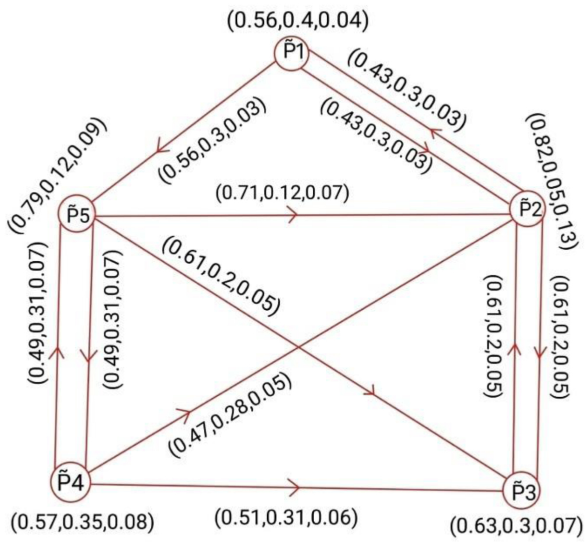

Fig. 3. $(\mathrm{HFG})_1$.

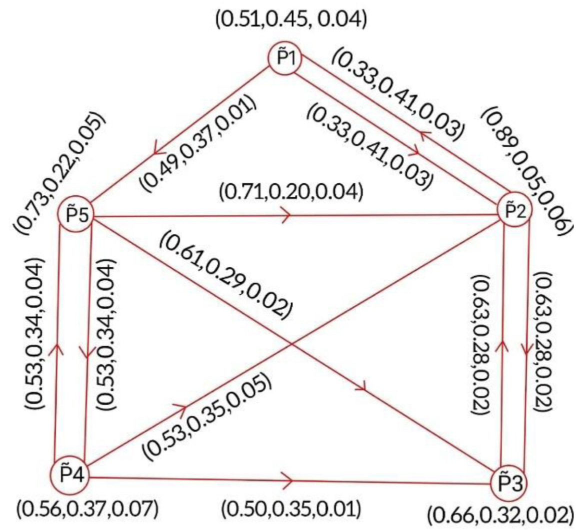

Fig. 4. $(\mathrm{HFG})_2$.

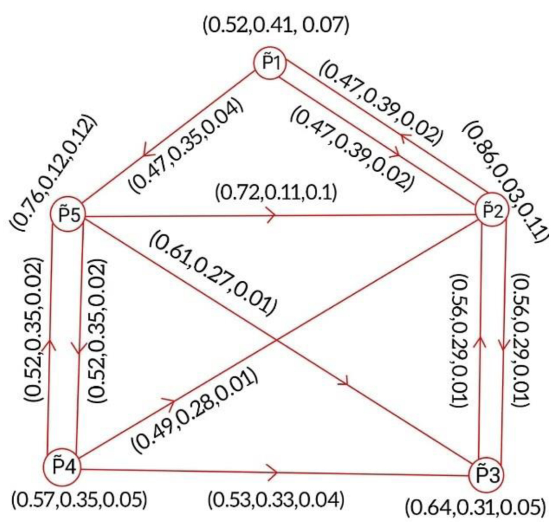

Fig. 5. $(\mathrm{HFG})_3$.

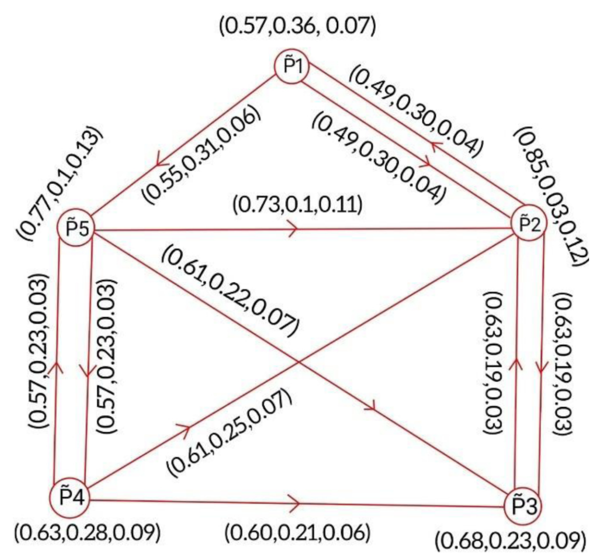

Fig. 6. $(\mathrm{HFG})_4$.

$i = (1, 2, 3, 4)$.

$(\mathrm{HFZM})_1 =$

$$\begin{bmatrix} (0,0,0) & (3.4684, 0.4300, 0.0400) & (0,0,0) & (0,0,0) & (3.0546, 0.5088, 0.0297) \\ (3.4684, 0.4300, 0.0400) & (0,0,0) & (4.1240, 0.3430, 0.0511) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.1240, 0.3430, 0.0511) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (3.7904, 0.4935, 0.0564) & (2.5680, 0.6965, 0.0347) & (0,0,0) & (3.3766, 0.5723, 0.0461) \\ (0,0,0) & (4.9326, 0.2188, 0.0625) & (3.7102, 0.4218, 0.0408) & (3.3766, 0.5723, 0.0461) & (0,0,0) \end{bmatrix}$$

$(\mathrm{HFZM})_2 =$

$$\begin{bmatrix} (0,0,0) & (3.3989, 0.6320, 0.0142) & (0,0,0) & (0,0,0) & (2.6816, 0.8743, 0.0103) \\ (3.3989, 0.6320, 0.0142) & (0,0,0) & (4.3766, 0.4805, 0.0128) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.3766, 0.4805, 0.0128) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (3.9828, 0.6071, 0.0212) & (2.7346, 0.8946, 0.0112) & (0,0,0) & (3.2655, 0.8494, 0.0173) \\ (0,0,0) & (4.9075, 0.4353, 0.0189) & (3.6593, 0.7228, 0.0089) & (3.2655, 0.8494, 0.0173) & (0,0,0) \end{bmatrix}$$

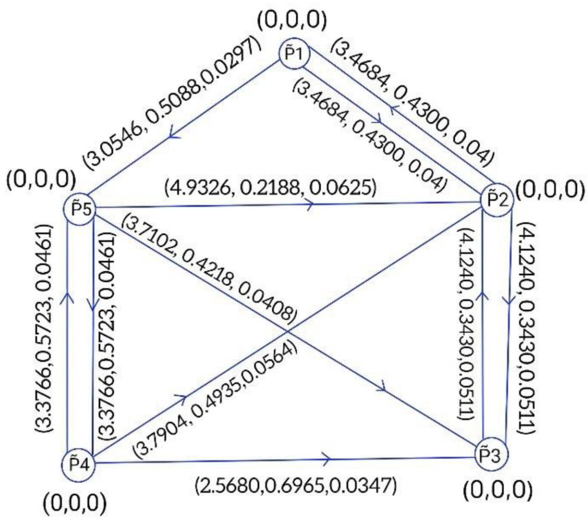

Fig. 7. $(\text{HFZM})_1$.

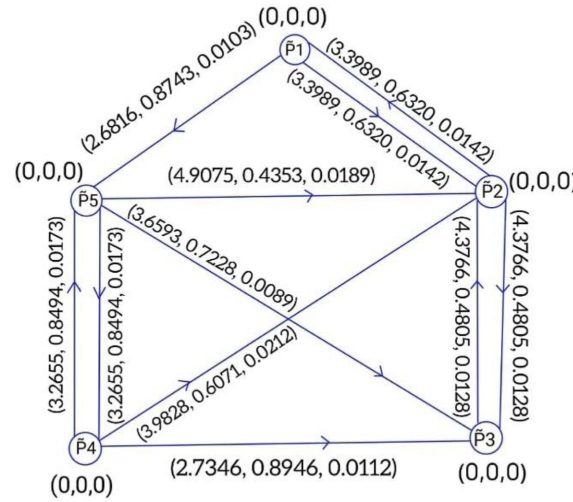

Fig. 8. $(\text{HFZM})_2$.

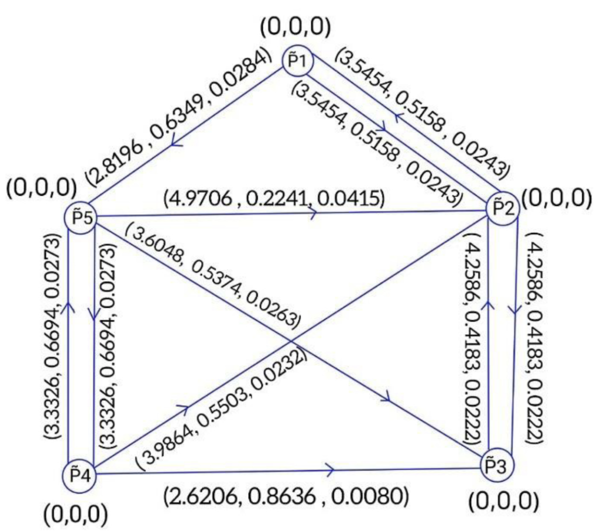

Fig. 9. $(\text{HFZM})_3$.

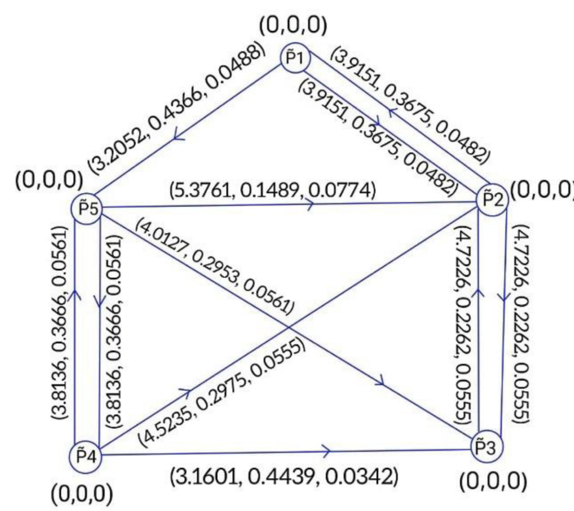

Fig. 10. $(\text{HFZM})_4$.

$(\text{HFZM})_3 =$

$$\begin{bmatrix} (0,0,0) & (3.5454, 0.5158, 0.0243) & (0,0,0) & (0,0,0) & (2.8916, 0.6349, 0.0284) \\ (3.5454, 0.5158, 0.0243) & (0,0,0) & (4.2586, 0.4183, 0.0222) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.2586, 0.4183, 0.0222) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (3.9864, 0.5503, 0.0232) & (2.6206, 0.8636, 0.0080) & (0,0,0) & (3.3326, 0.6694, 0.0273) \\ (0,0,0) & (4.9706, 0.2241, 0.0415) & (3.6048, 0.5374, 0.0263) & (3.3326, 0.6694, 0.0273) & (0,0,0) \end{bmatrix}$$

$(\text{HFZM})_4 =$

$$\begin{bmatrix} (0,0,0) & (3.9151, 0.3675, 0.0482) & (0,0,0) & (0,0,0) & (3.2052, 0.4366, 0.0488) \\ (3.9151, 0.3675, 0.0482) & (0,0,0) & (4.7226, 0.2262, 0.0555) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.7226, 0.2262, 0.0555) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.5235, 0.2975, 0.0555) & (3.1601, 0.4439, 0.0342) & (0,0,0) & (3.8136, 0.3666, 0.0561) \\ (0,0,0) & (5.3761, 0.1489, 0.0774) & (4.0127, 0.2953, 0.0561) & (3.8136, 0.3666, 0.0561) & (0,0,0) \end{bmatrix}$$

In Figs. 7 to 10 visually depict the $(\text{HFZM})_{i,}$ $(i = 1, 2, 3, 4)$. In these figures, the concept of edge connectivity is expanded upon. HFZM are represented by these graphs, which are stronger than the HFGs in Figs. 3 to 6.

The energy is calculated for each HFZM as

$$E(HFZM)_1 = (19.0047,\ 2.6601,\ 0.2316)$$

$$E(HFZM)_2 = (18.9049,\ 3.9040,\ 0.0754)$$

$$E(HFZM)_3 = (19.1049,\ 2.6670,\ 0.1229)$$

$$E(HFZM)_4 = (21.4981,\ 1.9233,\ 0.2759)$$

The LE is calculated for each HFZM as

$$LE(HFZM)_1 = (28.1828,\ 2.7955,\ 0.3614)$$

$$LE(HFZM)_2 = (28.1548,\ 4.0956,\ 0.1195)$$

$$LE(HFZM)_3 = (28.4607,\ 3.2995,\ 0.2222)$$

$$LE(HFZM)_4 = (31.3965,\ 1.8761,\ 0.4302)$$

For each expert, the weight is calculated as

$$w_i = \left((w_{\tilde{\mu}})_i, (w_{\tilde{\gamma}})_i, \left(w_{\tilde{\beta}}\right)_i\right)$$

$$w_i = \left(\frac{E\left(A_{\tilde{\mu}}(HFZM)_i\right)}{\sum_{i=1}^{n} E\left(A_{\tilde{\mu}}(HFZM)_i\right)}, \frac{E\left(A_{\tilde{\gamma}}(HFZM)_i\right)}{\sum_{i=1}^{n} E\left(A_{\tilde{\gamma}}(HFZM)_i\right)}, \frac{E\left(A_{\tilde{\beta}}(HFZM)_i\right)}{\sum_{i=1}^{n} E\left(A_{\tilde{\beta}}(HFZM)_i\right)}\right)$$

$$w_1 = (0.2422,\ 0.2385,\ 0.3281)$$

$$w_2 = (0.2409,\ 0.3500,\ 0.1068)$$

$$w_3 = (0.2435,\ 0.2391,\ 0.1741)$$

$$w_4 = (0.2740,\ 0.1724,\ 0.3909)$$

$$w_i = \left(\frac{LE\left(A_{\tilde{\mu}}(HFZM)_i\right)}{\sum_{i=1}^{n} LE\left(A_{\tilde{\mu}}(HFZM)_i\right)}, \frac{LE\left(A_{\tilde{\gamma}}(HFZM)_i\right)}{\sum_{i=1}^{n} LE\left(A_{\tilde{\gamma}}(HFZM)_i\right)}, \frac{LE\left(A_{\tilde{\beta}}(HFZM)_i\right)}{\sum_{i=1}^{n} LE\left(A_{\tilde{\beta}}(HFZM)_i\right)}\right)$$

$$w_1 = (0.2424,\ 0.2317,\ 0.3189)$$

$$w_2 = (0.2424,\ 0.3394,\ 0.1054)$$

$$w_3 = (0.2450,\ 0.2734,\ 0.1961)$$

$$w_4 = (0.2703,\ 0.1555,\ 0.3796)$$

The collective HFZPRs for energy are calculated by summing four HFZPRs as illustrated in Fig. 11.

$$HFZM = \sum_{i=1}^{4} w_i (HFZM)_i$$

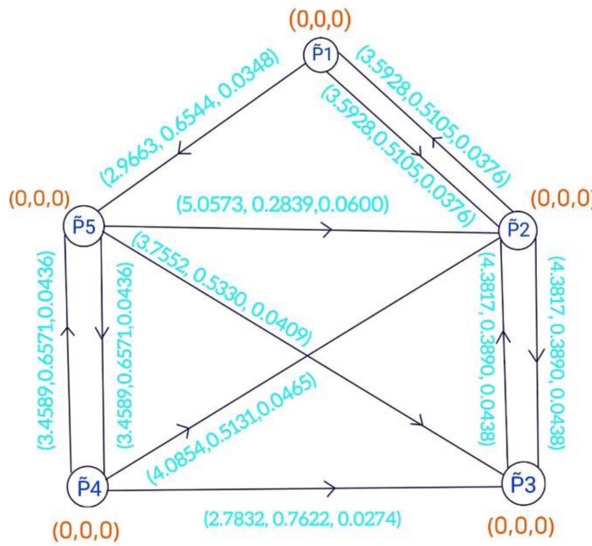

Fig. 11. Combined HFZPR.

(HFZM) =

$$\begin{bmatrix} (0,0,0) & (3.5928, 0.5105, 0.0376) & (0,0,0) & (0,0,0) & (2.9663, 0.6544, 0.0348) \\ (3.5928, 0.5105, 0.0376) & (0,0,0) & (4.3817, 0.3890, 0.0438) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.3817, 0.3890, 0.0438) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.0854, 0.5131, 0.0465) & (2.7832, 0.7622, 0.0274) & (0,0,0) & (3.4589, 0.6571, 0.0436) \\ (0,0,0) & (5.0573, 0.2839, 0.0600) & (3.7552, 0.5330, 0.0409) & (3.4589, 0.6571, 0.0436) & (0,0,0) \end{bmatrix}$$

The computation of degree $deg(\widetilde{P}_i)$(i = 1, 2, 3, 4, 5) as shown in Fig. 11,

$$deg\left(\widetilde{P}_1\right) = (10.1519,\ 1.6754,\ 0.1100)$$

$$deg\left(\widetilde{P}_2\right) = (25.0917,\ 2.5960,\ 0.2693)$$

$$deg\left(\widetilde{P}_3\right) = (15.3018,\ 2.0732,\ 0.1559)$$

$$deg\left(\widetilde{P}_4\right) = (13.7864,\ 2.5895,\ 0.1611)$$

$$deg\left(\widetilde{P}_5\right) = (18.6966,\ 2.7855,\ 0.2229)$$

with the maximum degree $\widetilde{P}_2$, the choices $\widetilde{P}_i(i = 1, 2, 3, 4, 5)$ positioned as

$$\widetilde{P}_2 > \widetilde{P}_5 > \widetilde{P}_3 > \widetilde{P}_4 > \widetilde{P}_1$$

The collective HFZPRs for LE are calculated by summing four HFZPRs as illustrated in Fig. 12.

$$HFZM = \sum_{i=1}^{4} w_i (HFZM)_i$$

(HFZM) =

$$\begin{bmatrix} (0,0,0) & (3.5915, 0.5122, 0.0374) & (0,0,0) & (0,0,0) & (2.9652, 0.6561, 0.0347) \\ (3.5915, 0.5122, 0.0374) & (0,0,0) & (4.3805, 0.3922, 0.0431) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.3805, 0.3922, 0.0431) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (4.0836, 0.5171, 0.0458) & (2.7816, 0.7701, 0.0269) & (0,0,0) & (3.4574, 0.6609, 0.0432) \\ (0,0,0) & (5.0563, 0.2829, 0.0594) & (3.7542, 0.5358, 0.0404) & (3.4574, 0.6609, 0.0432) & (0,0,0) \end{bmatrix}.$$

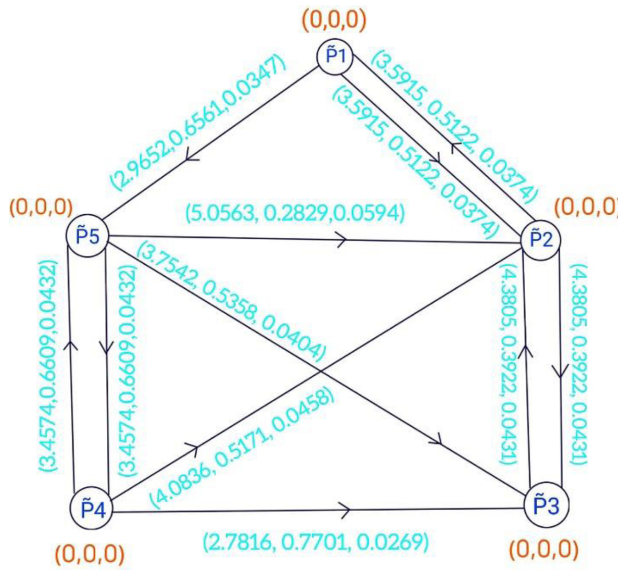

Fig. 12. Combined HFZPR.

The computation of degree $deg(\widetilde{P}_i)$(i = 1, 2, 3, 4, 5) as shown in Fig. 12,

$$deg\left(\widetilde{P}_1\right) = (10.1482,\ 1.6805,\ 0.1095)$$

$$deg\left(\widetilde{P}_2\right) = (25.0839,\ 2.6088,\ 0.2662)$$

$$deg\left(\widetilde{P}_3\right) = (15.2968,\ 2.0903,\ 0.1535)$$

$$deg\left(\widetilde{P}_4\right) = (13.7800,\ 2.6090,\ 0.1591)$$

$$deg\left(\widetilde{P}_5\right) = (18.6905,\ 2.7966,\ 0.2209)$$

with the maximum degree $\widetilde{P}_2$, the choices $\widetilde{P}_i(i = 1, 2, 3, 4, 5)$ positioned as

$$\widetilde{P}_2 > \widetilde{P}_5 > \widetilde{P}_3 > \widetilde{P}_4 > \widetilde{P}_1$$

The energy and LE collective HFZPR are shown in Figs. 11 and 12. For the representation of this collective HFZPR, we choose hesitancy fuzzy membership whose truthfulness degrees are $\widetilde{\mu_{ij}} \geq 0.5$, as illustrated in Figs. 11 and 12. For both the energy and LE, $\widetilde{P}_2$ is the highest degree. Therefore, the best option for selecting a candidate is $\widetilde{P}_2$.

The candidates are ranked based on their corresponding energy and LE values based on the suggested decision-making framework. According to the comparative analysis, candidate $\widetilde{P}_2$ achieves the highest LE and the maximum energy of all the candidates. $\widetilde{P}_2$ is ranked first, followed by the other candidates, in descending order of their individual energy and LE values. Consequently, $\widetilde{P}_2$ is selected as the final option using the recommended methodology since it is found to be the most qualified applicant.

Comparative analysis

i) Performance comparison of HFG and IFG based on identical numerical values

A comparative analysis is conducted using identical numerical values under both the IFG and HFG frameworks in order to assess the efficacy of the suggested HFG–based model. The hesitation information present in the hesitant fuzzy representation is disregarded in the IFG model, which only takes into account the membership and non-membership values.

The suggested algorithm is used to calculate the corresponding Zagreb matrices, energy values, and LE values for the IFG-based model using the same membership and non-membership values. It is found that the rankings derived from the IFG-based analysis and the suggested HFG-based method are comparable. This consistency shows that the hesitant fuzzy framework offers a more expressive representation by explicitly incorporating hesitation, without changing the final decision outcome in this instance. It also validates the suggested model.

$$\widetilde{P}_2 > \widetilde{P}_5 > \widetilde{P}_3 > \widetilde{P}_4 > \widetilde{P}_1$$

ii) Comparison of energy and Laplacian energy in HFZM and IFDG

It was necessary to compare the proposed methodology with previous approaches in order to assess its viability and effectiveness. As a result, the current model and the earlier model are contrasted. The present approach was initially introduced in,[39] where an IFDG's LE is lower than its membership value's energy. However, the focus of our proposed model is on the energy and LE of HFZMs. The findings demonstrate that in an HFZM, the LE of the truthfulness value is higher than its equivalent energy. In Figs. 13 and 14, the proposed and existing approaches are graphically contrasted.

Fig. 13 indicates that the E(IFDG) is greater than or equal to its LE(IFDG). Using the HFZM approach, on the other hand, Fig. 14 shows that the LE(HFZM) is greater than or equal to its corresponding E(HFZM).

Results and discussion

The home loan selection problem is examined using the suggested HFZM-based method. Figs. 11 and 12 provide a summary of the calculated eigenvalues, energy, and LE values for each candidate. A final ranking of candidates is produced based on these values. Candidate $\widetilde{\boldsymbol{P}}_2$ receives the highest rank, according to the results, followed by $\widetilde{\boldsymbol{P}}_5$, $\widetilde{\boldsymbol{P}}_3$, $\widetilde{\boldsymbol{P}}_4$, and $\widetilde{\boldsymbol{P}}_1$. The relative strength of hesitant fuzzy interactions that the suggested measures capture is reflected in this ranking.

Using the same numerical inputs, a comparison is made between the IFG framework and the suggested HFG-based model. The outcomes show that both strategies produce the same ranking order of options, supporting the suggested method's consistency and dependability. Ranking consistency is presented in Table 6.

Furthermore, the energy and LE measures for the IFDG and the HFZM are compared. In contrast to the HFZM, where the LE is greater than or equal to the corresponding energy, it is found that in the case of IFDG, the

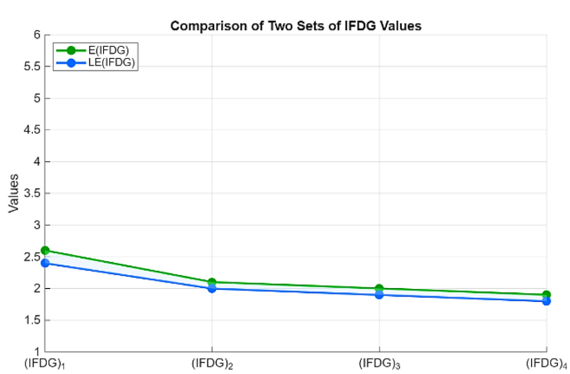

Fig. 13. IFDG models.

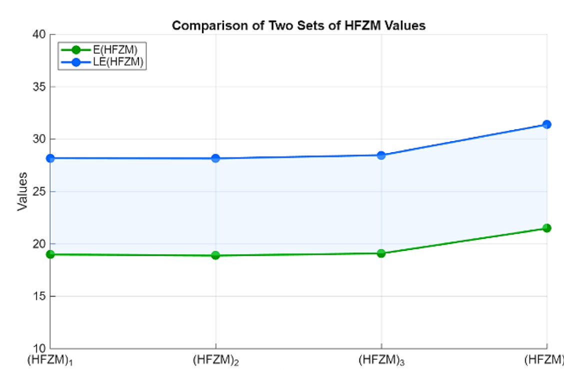

Fig. 14. HFZM models.

Table 6. Analysis of ranking consistency between IFG and HFG approaches.

Method	Ranking Order
IFG	$\widetilde{\boldsymbol{P}}_2 > \widetilde{\boldsymbol{P}}_5 > \widetilde{\boldsymbol{P}}_3 > \widetilde{\boldsymbol{P}}_4 > \widetilde{\boldsymbol{P}}_1$
HFG	$\widetilde{\boldsymbol{P}}_2 > \widetilde{\boldsymbol{P}}_5 > \widetilde{\boldsymbol{P}}_3 > \widetilde{\boldsymbol{P}}_4 > \widetilde{\boldsymbol{P}}_1$

energy is greater than or equal to its LE. These results support the efficacy of the suggested HFZM-based method while highlighting the different spectral properties of the two graph representations.

Conclusion

An HFG is a sophisticated graph structure that illustrates uncertainty by giving each vertex and edge a range of potential membership values as opposed to just one. ZIs play a crucial role in FG theory. The ZIs is a degree-based index that considers the strength of the vertices as well as the overall strength of the fuzzy network. The study investigates the HFZMs of HFGs and sets specified boundary parameters. Likewise, the descriptions of E(HFZM) and LE(HFZM) were presented. To evaluate home loan applicants based on their CIBIL scores, the proposed approach employs the energy and LE measures of the HFZM. A comparative analysis between the IFG and HFG frameworks reveals that both yield identical ranking orders of the applicants. Furthermore, a comparison of the energy and LE values of the HFZM and IFDG is also conducted to examine their consistency and effectiveness.

Advantages and Limitations: By efficiently capturing hesitant fuzzy interactions through spectral measures like eigenvalues, energy, and LE, the suggested HFZM-based method offers a clear and consistent ranking of home loan candidates. The approach's stability and dependability are demonstrated by the identical ranking results obtained under both IFG and HFG frameworks. Furthermore, the method's theoretical robustness is supported by the comparison with IFDG, which demonstrates the method's capacity to reveal distinct spectral characteristics.

Future Perspectives of the Study: The current study opens up a number of avenues for further investigation. It is possible to extend the suggested HFZM framework to generalized network structures and other classes of FGs. In order to improve DM accuracy, future research may also look into incorporating additional spectral measures and TIs. Additionally, the suggested method's applicability can be investigated in hybrid frameworks that combine HFGs with other uncertainty modeling paradigms, as well as in larger-scale and dynamic decision environments. Promising directions for future research include algorithmic optimization and computational efficiency analysis for big networks.

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Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the figures and the table in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Vellore Institute of Technology.

Authors' contributions statement

A. A. and G. D. were responsible for the design and implementation of the research. A. A. performed drafting the manuscript, conducted the analysis, acquired data, interpreted the results, and contributed to the writing of the manuscript. G. D. did the revision and interpretation. All authors read and agreed upon the published version of the manuscript.

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تقييم المتقدمين للحصول على القروض السكنية باستخدام الطاقة وطاقة لابلاس في مصفوفات زغرب الضبابية الترددية

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الخلاصة

يُعدّ الرسم البياني الضبابي الترددي (Hesitancy Fuzzy Graph) أحد أنواع الرسوم البيانية الضبابية المعممة التي تعكس التردد أو عدم اليقين في العلاقات، وذلك من خلال إسناد مجموعة من قيم الانتماء المحتملة لكل رأس وحافة بدلاً من قيمة واحدة فقط. ويوفر هذا النوع إطارًا مرنًا لنمذجة الأنظمة المعقدة التي تتسم بالغموض أو التردد أو عدم اكتمال المعلومات. تُعرف مؤشرات زغرب (Zagreb Indices) بأنها مقاييس طوبولوجية تُستخدم لالتقاط الخصائص البنائية للرسم البياني ووصفها، ويتم اشتقاقها من درجات الرؤوس. وعلى الرغم من دراسة مؤشرات زغرب وطاقات الرسوم البيانية في الرسوم البيانية التقليدية والضبابية والضبابية الحدسية، فإن تطبيق هذه المفاهيم على الرسوم البيانية الضبابية الترددية لم يحظَ بأي دراسة سابقة. في هذا البحث، تم توسيع إطار مصفوفات زغرب الضبابية الترددية من خلال دراسة طاقتها وتقديم مفهوم طاقة لابلاس (Laplacian Energy) الخاصة بها. كما تم اقتراح حدود عليا وحدود دنيا لطاقة مصفوفة زغرب الضبابية الترددية وطاقة لابلاس المرتبطة بها. وقد جرى توظيف هذه المفاهيم في سياق افتراضي يتعلق باختيار المتقدمين للحصول على قرض سكني، وهي عملية تتأثر عادةً بعوامل متعددة مثل درجة CIBIL الخاصة بالجدارة الائتمانية. إضافة إلى ذلك، يقدم البحث مقارنة بين طاقة مصفوفة زغرب الضبابية الترددية وطاقة لابلاس الخاصة بها.

الكلمات المفتاحية: الرسوم البيانية الضبابية الترددية، مؤشر زغرب الأول الضبابي الترددي، مؤشر زغرب الثاني الضبابي الترددي، مؤشرات زغرب الضبابية الترددية، مصفوفات زغرب الضبابية الترددية.