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RESEARCH ARTICLE

Quasi-Mixing C_0-Semigroups and Their Relations With Recurrent C_0-Semigroups

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ABSTRACT

This article presents and investigates the concept of the quasi-mixing C_0-semigroups. It is proved that the quasi-mixing of an invertible C_0-semigroup is equivalent to the quasi-mixing of its inverse. The quasi-mixing vectors are defined and they are used to prove a sufficient condition for the quasi-mixing. It is established that the quasi-mixing of a C_0-semigroup is equivalent to the quasi-mixing of any of its discretizations. The quasi-mixing of a C_0-semigroup $(T_t)_{t\geq 0}$, as it is demonstrated by this study, implies the quasi-mixing for each operator T_t, $t \geq 0$. It is proved that the quasi-mixing of the direct sum of two C_0-semigroups is equivalent to the quasi-mixing of both semigroups. Furthermore, some relations between the quasi-mixing and the recurrence are explored in this article. It is shown that the operators of a quasi-mixing C_0-semigroup are recurrent. Also, it is demonstrated that the recurrence of every discretization of a C_0-semigroup is equivalent to the quasi-mixing of the C_0-semigroup.

Keywords: Direct sum, Mixing, Quasi-mixing semigroups, Recurrent semigroups, Semigroups

Introduction

For decades, researchers have investigated dynamical systems and related topics on operators, and structures such as C_0-semigroups.[1–3] A bounded linear operator T on a Banach space X is called hypercyclic if there exists $x \in X$ such that $\{x, Tx, T^2x, \ldots, T^nx, \ldots\}$ is dense in X.[4] A C_0-semigroup on a Banach space X is a family of operators on X like $(T_t)_{t\geq 0}$ subject to three conditions: first, $T_0 = I$; second, $T_{t+s} = T_tT_s$ for any $s, t \geq 0$ and third, $\lim_{s\to t} T_sx = T_tx$ for any $t \geq 0$ and all $x \in X$.[5]

Presuming that for any open and non-empty sets $U, V \subseteq X$, $t > 0$ can be chosen so that $T_t(U) \cap V \neq \emptyset$, then $(T_t)_{t\geq 0}$ is called hypercyclic.[5] The hypercyclicity of a C_0-semigroup implies that each of its operators is also hypercyclic.[6] It is proved that hypercyclic C_0-semigroups can be constructed on any Banach space with infinite dimension.[4]

Assume that for $(T_t)_{t\geq 0}$ on X, there exists an increasing sequence (t_n) of positive real numbers and a dense subset Y of X so that $T_{t_n}y \to y$ for any $y \in Y$. Also, there is a dense subset Z of X and maps $S_{t_n} : Z \to X$ so that $S_{t_n}z \to 0$ and $T_{t_n}S_{t_n}z \to z$ for any $z \in Z$. In this case, it is said that $(T_t)_{t\geq 0}$ satisfies in Hypercyclicity Criterion.[5] Also,[7] states interesting statements about the hypercyclicity criterion. A review of these investigations is available in.[8] Also, one can see some newer concepts, such as super-recurrent in,[9] and Subspace-supercyclic abelian linear semigroups in.[10]

A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called "mixing" if for any open and non-empty sets $U, V \subseteq X$, $s > 0$ can be chosen so that $T_t(U) \cap V \neq \emptyset$ for every $t \geq s$. The mixing C_0-semigroups are hypercyclic.[5] If $(T_t)_{t\geq 0}$ satisfies the Hypercyclicity Criterion concerning the whole positive real numbers, then it is mixing.[5] Moreover, mixing C_0-semigroups of left multiplication operators are investigated in.[11]

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Presuming that an operator T on a Banach space X has a condition that for any non-empty open set V of X, $n \in \mathbb{N}$ can be chosen so that $T^n(V) \cap V \neq \emptyset$. Then T is called "recurrent".[12] Recurrent operators on function spaces are studied in.[13] In addition,[14] investigates recurrent C_0-semigroups in light of recurrent operators. A C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X is called recurrent if for any non-empty open subset V of X, there exists $s > 0$ such that $T_s(V) \cap V \neq \emptyset$.[14] It is proved in Lemma 4 of[9] that if $(T_t)_{t\geq 0}$ satisfies the Hypercyclicity Criterion, the direct sum of $(T_t)_{t\geq 0}$ with itself would be recurrent. An element $x \in X$ is termed recurrent for $(T_t)_{t\geq 0}$ if an increasing sequence $(t_n) \subseteq \mathbb{R}^+$ can be chosen so that $T_{t_n}x \to x$. Having a dense set of recurrent vectors is an equivalent condition for the recurrence of C_0-semigroups.[14] One can see some newer concepts like super-recurrence for C_0-semigroups in.[9]

Recurrent and mixing C_0-semigroups are defined. Now it will be interesting to note that if $(T_t)_{t\geq 0}$ is a semigroup with the property that for every open set U, we have an infinitely long interval of non-negative real numbers t such that $T_t(U) \cap U \neq \emptyset$, what properties does this new semigroup have? We call this property quasi-mixing.

In this paper, the concept of the quasi-mixing C_0-semigroups is introduced which is a concept between mixing C_0-semigroups and recurrent C_0-semigroups. The second section shows that mixing C_0-semigroups are quasi-mixing, and quasi-mixing C_0-semigroups are hypercyclic.

It notes some properties of the quasi-mixing C_0-semigroups such as inverse and preservation under quasiconjugacy. Moreover, this section introduces the quasi-mixing vectors. The third section proves some relations between the quasi-mixing and recurrence. It proves that the quasi-mixing of a C_0-semigroup indicates the recurrence of any of its operators. In addition, the recurrence of every one of the discretizations of a C_0-semigroup $(T_t)_{t\geq 0}$ would be an equivalent condition to the quasi-mixing of $(T_t)_{t\geq 0}$. Fourth section, investigates the direct sum of the quasi-mixing C_0-semigroups. It demonstrates that the quasi-mixing of the direct sum of two C_0-semigroups is equivalent to the quasi-mixing of both semigroups. Moreover, it proves some conclusions about the recurrence of the direct sum of C_0-semigroups from the quasi-mixing.

Some preliminaries and properties

As a concept between mixing and recurrent, a quasi-mixing C_0-semigroup is defined as follows.

Definition 1: For a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X, suppose that for every non-empty open set $U \subseteq X$, $t_0 \in \mathbb{R}$ can be chosen so that $T_t(U) \cap U \neq \emptyset$ for any $t \geq t_0$. Then $(T_t)_{t\geq 0}$ is called a quasi-mixing C_0-semigroup.

In this regard, mixing C_0-semigroups are quasi-mixing, and quasi-mixing C_0-semigroups are hypercyclic. Also, quasi-mixing C_0-semigroups are a subset of recurrent C_0-semigroups, as shown in the following analogy.

Lemma 1: *A quasi-mixing C_0-semigroup is recurrent.*

Proof. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and suppose $U \subseteq X$ is a non-empty open set. By Definition 1, there exists $t_0 \in \mathbb{R}$ such that $T_t(U) \cap U \neq \emptyset$ for any $t \geq t_0$. Especially, there is some s such that $T_s(U) \cap U \neq \emptyset$. Hence, $(T_t)_{t\geq 0}$ is recurrent.*

In this regard, the following relation can be shown: mixing C_0-semigroups$\subseteq$ quasi-mixing C_0-semigroups $\subseteq$ recurrent C_0-semigroups.

Example 1. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X such that $T_t = I$ for every $t \geq 0$. Assume $U \subseteq X$ is a non-empty open set. Then for every $t \geq 0$,

$$T_t\,(U) \cap U = I\,(U) \cap U = U \cap U = U.$$

Hence, $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup. Clearly, $(T_t)_{t\geq 0}$ is recurrent. Consider $V \subseteq X$ is a non-empty open set such that $U \cap V = \emptyset$. Then, for any $t \geq 0$,

$$T_t\,(U) \cap V = I\,(U) \cap V = U \cap V = \emptyset.$$

Hence, $(T_t)_{t\geq 0}$ is not a mixing C_0-semigroup.

Two C_0-semigroups $(T_t)_{t\geq 0}$ on Y and $(S_t)_{t\geq 0}$ on Z, where Y and Z Banach spaces, are considered quasi-conjugate if there is an operator $\Psi : Z \to Y$ with dense range exists so that $T_t o\Psi = \Psi o S_t$ for any $t > 0$.[5] In the following theorem, the stability of the quasi-mixing under quasiconjugacy is investigated.

Theorem 1: *Quasi-mixing of C_0-semigroups is preserved under quasiconjugacy.*

Proof. *Presume $(S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on a Banach space Z and $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space Y and $\Psi : Z \to Y$ is an operator so that $T_t o\Psi = \Psi o S_t$ for any $t > 0$ and Ψ has a dense range. Suppose $U \subseteq Y$ is non-empty and open. Hence, $\Psi^{-1}(U)$ is a non-empty and open subset of Z. By definition of the quasi-mixing, $t_0 > 0$ can be chosen so that for any $t \geq t_0$,*

$$S_t\left(\Psi^{-1}(U)\right) \cap \Psi^{-1}(U) \neq \emptyset. \tag{1}$$

Thus, by Eq. (1), for any $t \geq t_0$, there exists $x_t \in \Psi^{-1}(U)$ with the property that $S_t x_t \in \Psi^{-1}(U)$. Accordingly, for any $t \geq t_0$, $\Psi o S_t x_t \in U$. But, $T_t o \Psi = \Psi o S_t$. So, for any $t \geq t_0$, $T_t(\Psi x_t) \in U$. Hence, for any $t \geq t_0$, $\Psi x_t \in U$ and $T_t(\Psi x_t) \in U$. Therefore, $T_t(U) \cap U \neq \emptyset$ for any $t \geq t_0$. Consequently, $(T_t)_{t\geq 0}$ is quasi-mixing.

In this regard, one might wonder whether the quasi-mixing of an invertible C_0-semigroup implies the quasi-mixing of its inverse or vice versa. The next theorem presents the answer.

Theorem 2: *Consider a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X. Assume T_t is invertible for every $t > 0$. Then the quasi-mixing of $(T_t)_{t\geq 0}$ is equivalent to the quasi-mixing of $(T_t^{-1})_{t\geq 0}$.*

Proof. *Assume $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup. Suppose $U \subseteq X$ is a non-empty and open set. By definition of the quasi-mixing, there is $s_0 > 0$ such that for any $t \geq s_0$,*

$$T_t(U) \cap U \neq \emptyset.$$

Note that $T_t(U) \cap U \neq \emptyset$ implies $T_t^{-1}(U) \cap U \neq \emptyset$. In consequence, $(T_t^{-1})_{t\geq 0}$ is quasi-mixing. The converse follows by the same reasoning.

In accordance with the idea of the recurrent vector, a quasi-mixing vector is defined.

Definition 2: Consider a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X. An element $x \in X$ is called a quasi-mixing vector if $T_t x \to x$ as $t \to \infty$.

In the following theorem, it is shown that if the quasi-mixing vectors form a dense set, then the semigroup is quasi-mixing.

Theorem 3: *A C_0-semigroup $(T_t)_{t\geq 0}$ on X is quasi-mixing if its quasi-mixing vectors form a dense set of X.*

Proof. *Let $A = \{x \in X : T_t x \to x\}$. By hypothesis, A is a dense set in X. Assume U is a non-empty open set. Because A is dense in X, $U \cap A \neq \emptyset$. Let $a \in U \cap A$. Therefore, there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq U$. Also, $T_t a \to a$. Hence, there is $t_0 > 0$ so that*

$$||T_t a - a|| < \varepsilon \tag{2}$$

for any $t \geq t_0$. Hence, for any $t \geq t_0$, $T_t a \in B(a, \varepsilon)$ by Eq. (2). Thus, for any $t \geq t_0$, $T_t a \in U$. Therefore, $T_t(U) \cap U \neq \emptyset$ for every $t \geq t_0$. This means $(T_t)_{t\geq 0}$ is quasi-mixing.

Another sufficient condition for the quasi-mixing is as follows.

Theorem 4: *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Presume for any non-empty open set $V \subseteq X$ and any neighborhood W of zero in X, there exist positive real numbers t_1 and t_2 such that:*

(i) $T_t(W) \cap V \neq \emptyset$ for any $t \geq t_1$,
(ii)$T_t(V) \cap W \neq \emptyset$ for any $t \geq t_2$.

Then $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup.

Proof. Let $V \subseteq X$ be a non-empty open set. Now, Lemma 2.36 of[5] indicates that a non-empty open set V_1 and a neighborhood W_1 of zero in X can be chosen so that

$$V_1 + W_1 \subseteq V. \tag{3}$$

By part (i) of hypothesis, there is $t_1 \in \mathbb{R}$ so that for any $t \geq t_1$,

$$T_t(W_1) \cap V_1 \neq \emptyset. \tag{4}$$

By part (ii) of hypothesis, there is $t_2 \in \mathbb{R}$ so that for any $t \geq t_2$,

$$T_t(V_1) \cap W_1 \neq \emptyset. \tag{5}$$

Consider $t_0 = \max\{t_1, t_2\}$. Hence, by Eqs. (4) and (5), for any $t \geq t_0$,

$$T_t(V_1) \cap W_1 \neq \emptyset \text{ and } T_t(W_1) \cap V_1 \neq \emptyset. \tag{6}$$

Suppose $t \geq t_0$. By Eq. (6), correspondence to this t, there are $w_t \in W_1$ and $v_t \in V_1$ such that

$$T_t w_t \in V_1 \text{ and } T_t v_t \in W_1. \tag{7}$$

Thus, $v_t + w_t \in V$ and by Eqs. (3) and (7),

$$T_t(v_t + w_t) = T_t(v_t) + T_t(w_t) \in V.$$

Hence, for any $t \geq t_0$, $T_t(V) \cap V \neq \emptyset$. This indicates that $(T_t)_{t\geq 0}$ is quasi-mixing.

Relations between quasi-mixing and recurrent

Recurrent vectors are mentioned in the introduction. In accordance with Theorem 3 of,[14] a recurrent C_0-semigroup has a dense set of recurrent vectors.

The next lemma asserts that this is true for the quasi-mixing C_0-semigroups, too.

Lemma 2: *A quasi-mixing C_0-semigroup has a dense set of recurrent vectors.*

Proof. *In accordance with Lemma 1, a quasi-mixing C_0-semigroup is recurrent. In light of Theorem 3 of,[14] a recurrent C_0-semigroup has a dense set of recurrent vectors. Therefore, a quasi-mixing C_0-semigroup has a dense set of recurrent vectors.*

As Theorem 2.3 of[6] proves, hypercyclicity of $(T_t)_{t\geq 0}$ implies that T_t is hypercyclic for each $t > 0$. The following theorem proves that this is correct for the quasi-mixing C_0-semigroups, too.

Theorem 5: *Presume $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on X. Then for each $s \geq 0$, T_s is a quasi-mixing operator.*

Proof. *If $s = 0$, then $T_s = T_0$ and $T_0 = I$. Clearly, the identity operator is quasi-mixing. Now, suppose $s > 0$. Assume $U \subseteq X$ is a non-empty open set. Because $(T_t)_{t\geq 0}$ is quasi-mixing, $t_0 > 0$ can be chosen so that for any $t \geq t_0$, $T_t(U) \cap U \neq \emptyset$. By Archimides property, a natural number M can be chosen such that $Ms > t_0$. Hence, for any $n \geq M$, $ns > t_0$. Hence, for any $n \geq M$,*

$$T_{ns}(U) \cap U \neq \emptyset. \tag{8}$$

But $(T_t)_{t\geq 0}$ is a C_0-semigroup and $T_{ns} = T_s^n$. Therefore, for any $n \geq M$, $T_s^n(U) \cap U \neq \emptyset$ by Eq. (8). Hence, T_s is a quasi-mixing operator.

Recall that an operator T on a Banach space X is called recurrent if for any non-empty open subset U of X, $n \in \mathbb{N}$ exists such that $T^n(U) \cap U \neq \emptyset$.[5] Clearly, quasi-mixing operators are recurrent. As Theorem 2 of[14] proves, if $(T_t)_{t\geq 0}$ satisfies the Hypercyclicity Criterion, then T_t is recurrent for any $t > 0$. Now, satisfying in the Hypercyclicity Criterion can be replaced with the weaker condition quasi-mixing due to Theorem 5 in the following corollary since the quasi-mixing operators are recurrent.

Corollary 1. Quasi-mixing of a C_0-semigroup $(T_t)_{t\geq 0}$ on X implies that T_t is recurrent for every $t > 0$.

By a discretization of a C_0-semigroup, it means (T_{t_n}), where $(t_n)_{n=1}^{\infty} \subseteq \mathbb{R}^+$ is an increasing sequence tending to infinity.[5]

Theorem 6: *If $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on X, then every discretization of $(T_t)_{t\geq 0}$ is quasi-mixing.*

Proof. *Assume (T_{t_n}) is an arbitrary discretization of $(T_t)_{t\geq 0}$. Thus $(t_n)_{n=1}^{\infty} \subseteq \mathbb{R}^+$ is an increasing sequence tending to infinity. Consider $U \subseteq X$ is a non-empty and open set. The quasi-mixing of $(T_t)_{t\geq 0}$ gives us a positive real number t_0 such that for any $t \geq t_0$, $T_t(U) \cap U \neq \emptyset$. The sequence $(t_n)_{n=1}^{\infty}$ is an increasing sequence. Thus, there exists a natural number N such that $t_N > t_0$. So, $t_n > t_N > t_0$ for any $n \geq N$. Therefore, $T_{t_n}(U) \cap U \neq \emptyset$ for any $n \geq N$. Hence, (T_{t_n}) is a quasi-mixing C_0-semigroup.*

Surprisingly, the recurrence of every discretization of a C_0-semigroup implies that it is quasi-mixing as follows.

Theorem 7: *The following statements are equivalent for a C_0-semigroup $(T_t)_{t\geq 0}$ on X:*

(i) $(T_t)_{t\geq 0}$ is quasi-mixing.
(ii)Every discretization of $(T_t)_{t\geq 0}$ is quasi-mixing.
(iii)Every discretization of $(T_t)_{t\geq 0}$ is recurrent.

Proof. *(i) → (ii) is concluded from Theorem 6. Also, (ii) → (iii) is deduced from Lemma 1. For proving (iii) → (i), consider that $(T_t)_{t\geq 0}$ is not quasi-mixing. Consider $U \subseteq X$ is non-empty and open. Consequently, for any $t_0 > 0$, $t_1 > t_0$ can be chosen so that $T_{t_1}(U) \cap U = \emptyset$. Hence, an increasing sequence $(t_n)_{n=1}^{\infty}$ can be found with $\lim_{n\to+\infty} t_n = +\infty$ such that $T_{t_n}(U) \cap U = \emptyset$ for any n. Therefore, (T_{t_n}) is not recurrent, which is a contradiction.*

Properties of the direct sum

The direct sum of two Banach spaces X and Y is mentioned by $X \oplus Y$. It is a set of $x \oplus y$, where $x \in X$ and $y \in Y$. If $(T_t)_{t\geq 0}$ is a C_0-semigroup on X and $(S_t)_{t\geq 0}$ is a C_0-semigroup on Y, then for any $t > 0$, the direct sum $T_t \oplus S_t$ is defined with $(T_t \oplus S_t)(x \oplus y) = T_t x \oplus S_t y$ for any $x \oplus y \in X \oplus Y$.[5]

Similar to the proof of Proposition 2.40 of,[5] one can prove that $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are mixing if and only if $(T_t \oplus S_t)_{t\geq 0}$ is mixing. The next theorem proves this for the quasi-mixing operators.

Theorem 8: *If $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on X and $(S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on Y, then $(T_t \oplus S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on $X \oplus Y$.*

Proof. *Consider two non-empty open sets $U \subseteq X$ and $V \subseteq Y$. By the quasi-mixing of $(T_t)_{t\geq 0}$, $t_0 > 0$ can be chosen so that for any $t \geq t_0$,*

$$T_t(U) \cap U \neq \emptyset. \tag{9}$$

In accordance with the quasi-mixing of $(S_t)_{t\geq 0}$, there exists $s_0 > 0$ such that for any $t \geq s_0$,

$$S_t(V) \cap V \neq \emptyset. \tag{10}$$

Let $p = \max\{s_0, t_0\}$. Then by Eqs. (9) and (10), for any $t \geq p$,

$$(T_t \oplus S_t)(U \oplus V) \cap (U \oplus V) = (T_t(U) \cap U) \oplus (S_t(V) \cap V) \neq \emptyset.$$

Thus, $(T_t \oplus S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup. The converse of Theorem 8 is as follows.

Theorem 9: *Presume $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on X and $(S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on Y. If $(T_t \oplus S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on $X \oplus Y$, then $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are quasi-mixing respectively on X and Y.*

Proof. *Consider two non-empty open sets $U \subseteq X$ and $V \subseteq Y$. So, $U \oplus V$ is an open non-empty set of $X \oplus Y$. Because the quasi-mixing of $(T_t \oplus S_t)_{t\geq 0}$, $t_0 > 0$ can be chosen such that for any $t \geq t_0$, $(T_t \oplus S_t)(U \oplus V) \cap (U \oplus V) \neq \emptyset$. Hence, for any $t \geq t_0$,*

$$(T_t(U) \oplus S_t(V)) \cap (U \oplus V) \neq \emptyset. \tag{11}$$

By Eq. (11), for any $t \geq t_0$,

$$(T_t(U) \cap U) \oplus (S_t(V) \cap V) \neq \emptyset.$$

Hence, $T_t(U) \cap U \neq \emptyset$ and $S_t(V) \cap V \neq \emptyset$ for any $t \geq t_0$. Therefore, $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are quasi-mixing.

Of note here is that from previous theorems, the consequent corollaries are concluded.

Corollary 2. For a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X and a C_0-semigroup $(S_t)_{t\geq 0}$ on a Banach space Y, $(T_t \oplus S_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup on $X \oplus Y$ if and only if both semigroups are quasi-mixing.

Corollary 3. For a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X, the quasi-mixing of $(T_t)_{t\geq 0}$ implies the quasi-mixing of $(T_t \oplus T_t)_{t\geq 0}$ on $X \oplus X$ and vice versa.

It is proved in Theorem 12 of[14] that $(T_t \oplus S_t)_{t\geq 0}$ is recurrent when $(S_t)_{t\geq 0}$ is recurrent and $(T_t)_{t\geq 0}$ is mixing. The following theorem proves that mixing can be replaced with the weaker condition, quasi-mixing.

Theorem 10: *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and $(S_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space Y. Then, if $(T_t)_{t\geq 0}$ is quasi-mixing and $(S_t)_{t\geq 0}$ is recurrent, then $(T_t \oplus S_t)_{t\geq 0}$ is recurrent.*

Proof. *Consider two non-empty open sets $U \subseteq X$ and $V \subseteq Y$. By hypothesis, $(T_t)_{t\geq 0}$ is quasi-mixing. Hence, there exists $t_0 > 0$ so that $T_t(U) \cap U \neq \emptyset$ for any $t \geq t_0$.*

On the other hand, $(S_t)_{t\geq 0}$ is recurrent. Thus, there exists $t_1 > 0$ such that $S_{t_1}(V) \cap V \neq \emptyset$.

In accordance with Lemma 1 of,[14] t_1 can be chosen so that $t_1 > t_0$. Hence,

$$T_{t_1}(U) \cap U \neq \emptyset \text{ and } S_{t_1}(V) \cap V \neq \emptyset.$$

Therefore,

$$(T_{t_1} \oplus S_{t_1})(U \oplus V) \cap (U \oplus V) = \left(T_{t_1}(U) \cap U\right) \oplus \left(S_{t_1}(V) \cap V\right) \neq \emptyset.$$

Consequently, $(T_t \oplus S_t)_{t\geq 0}$ is recurrent.

In accordance with Theorems 7 and 10, the following corollaries are concluded.

Corollary 4. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and $(S_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space Y. Presume $(S_t)_{t\geq 0}$ is a recurrent C_0-semigroup. If for any increasing sequence $(t_n)_{n=1}^{\infty} \subseteq \mathbb{R}^+$, tending to infinity, (T_{t_n}) is a quasi-mixing C_0-semigroup, then $(T_t \oplus S_t)_{t\geq 0}$ is recurrent.

It is proved in Lemma 4 of[14] that satisfying $(T_t)_{t\geq 0}$ in the Hypercyclicity Criterion indicates recurrence of $(T_t \oplus T_t)_{t\geq 0}$. Now, it is shown that if $(T_t)_{t\geq 0}$ satisfies the weaker condition, quasi-mixing, then $(T_t \oplus T_t)_{t\geq 0}$ is recurrent. The consequent assertion is a special case of Theorem 10.

Corollary 5. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then $(T_t \oplus T_t)_{t\geq 0}$ is recurrent if $(T_t)_{t\geq 0}$ is a quasi-mixing C_0-semigroup.

Results and discussion

This paper introduces the quasi-mixing C_0-semigroups. This set of C_0-semigroups is a subset of recurrent C_0-semigroups and contains the set of mixing C_0-semigroups.

Theorem 5 proves that the quasi-mixing of a C_0-semigroup $(T_s)_{s\geq 0}$ implies that any T_s is quasi-mixing that is similar to this result for hypercyclic C_0-semigroups that the authors have stated in Theorem 2.3 of.[6] Some sufficient conditions are proved in this paper, which are in accordance with some vectors called, the quasi-mixing vectors and open sets.

Surprisingly, in light of Theorem 7, the recurrence of any discretization of a C_0-semigroup $(T_t)_{t\geq 0}$ and the quasi-mixing of any discretization of $(T_t)_{t\geq 0}$ are equivalent conditions to the quasi-mixing of $(T_t)_{t\geq 0}$.

In addition, Theorems 8 and 9 prove that the quasi-mixing of two C_0-semigroups $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ is equivalent to the quasi-mixing of $(T_t \oplus S_t)_{t\geq 0}$. Also, Theorem 10 proves that the direct sum of a quasi-mixing and a recurrent C_0-semigroup is a recurrent C_0-semigroup which is an extension of Theorem 12 in.[14]

Conclusions

Quasi-mixing for C_0-semigroups is a concept between mixing and recurrence. This concept extends our knowledge on the theory of dynamical systems about C_0-semigroups. In this paper, their properties, and their direct sum are investigated. It is suggested that this structure be studied further and that more criteria and sufficient conditions for the quasi-mixing of C_0-semigroups be stated. Also, some relations between the quasi-mixing and recurrence are studied. It is suggested that the relations between the quasi-mixing C_0-semigroups and other types of C_0-semigroups, like supercyclicity, are studied.

Authors' declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
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Authors' contributions statement

The idea for the article and the writing of the first draft of the cases were done by M. M., and the proofreading and editing were done by O. B.

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شبه-الخلط (Quasi-mixing) لأنصاف الزمر C_0 وعلاقته بأنصاف الزمر C_0 الراجعة

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الخلاصة

تقدم هذه المقالة مفهوم شبه-الخلط (Quasi-mixing) لأنصاف الزمر C_0 وتبحث في خصائصه. وقد أُثبت أن شبه-الخلط لنصف زمرة C_0 القابلة للعكس يكافئ شبه-الخلط لمقلوبها. كما تم تعريف متجهات شبه-الخلط واستخدامها لإثبات شرط كافٍ لتحقيق شبه-الخلط. وتبين أن شبه-الخلط لنصف زمرة C_0 يكافئ شبه-الخلط لأيٍّ من تقريباتها المتقطعة (Discretizations). كما أظهرت الدراسة أن شبه-الخلط لنصف الزمرة (T_t)_(t≥0) يستلزم شبه-الخلط لكل مؤثر T_t حيث t≥0. وأُثبت كذلك أن شبه-الخلط للمجموع المباشر (Direct Sum) لنصفي زمرة C_0 يكافئ شبه-الخلط لكل واحد منهما على حدة. علاوة على ذلك، تستكشف المقالة بعض العلاقات بين شبه-الخلط وخاصية الرجوع (Recurrence). فقد تبين أن مؤثرات نصف الزمرة C_0 شبه المختلطة هي مؤثرات راجعة. كما أظهرت الدراسة أن رجوع كل تقريب متقطع لنصف زمرة C_0 يكافئ شبه-الخلط لنصف الزمرة نفسها.

الكلمات المفتاحية: المجموع المباشر، الخلط، أنصاف الزمر شبه المختلط، أنصاف الزمر الراجعة، أنصاف الزمر.